Last time: P d-dimensional lattice polytope

$\deg P = s$

$d > f(h_1^*, s) = h_1^*(2s+1) + 4s - 2$

$\Rightarrow P$ is a lattice pyramid over a lattice polytope Q of dimension at most $f(h_1^*, s)$

Cor These are up to taking lattice pyramids only finitely many lattice polytopes with a given h^*-vector

proof: f depends on h_1^* and $\deg h^*$

beyond that only lattice pyramids with the same h^*

$\rightsquigarrow$ need only consider polytopes up to dim $f(h_1^*, s)$

Now $h^*(1) = \operatorname{vol} P$

and for given volume and dimension there are only finitely many lattice polytopes

$\boxtimes$

14. (Some) Further Classification Results

There are many more results similar to the finiteness for given h^*-vectors.

We collect some of them here. Proofs are deferred to the end of the course if there is still time.

One of the most famous and important finiteness results is the following:

Thm Given $d, i \in \mathbb{Z}_{\geq 1}$, there is a bound $V(d,i)$ so that every lattice polytope with exactly i interior lattice points has volume at most $V(d,i)$

One bound is

$$\operatorname{vol} P \leq (8d)^d \, 15^{d 2^{2d+1}} \cdot i$$

(Hensley, Lagarias & Ziegler, Pikhurko)

- recall classification of Scott:
  $i \geq 1$, $P \neq 3\Delta_2 \Rightarrow a \leq 2(i+1)$
- no such bound can exist if $i = 0$ (Reeve!)

This Theorem implies:

<u>Cor</u>: Given $d, i \in \mathbb{Z}_{\geq 1}$ there are, up to lattice equivalence, only finitely many lattice polytopes with exactly i interior lattice points.

We have seen that fixing the h^*-polynomial leaves only finite choices for the lattice polytope, up to lattice pyramid constructions.

We know more in the cases $\deg h^* \leq 2$

The only polytope of degree 0 is the unimodular simplex

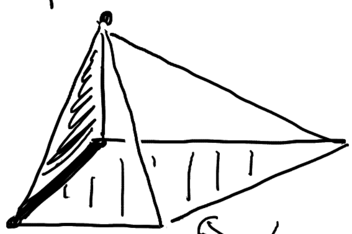
A <u>Lawrence prism</u> for nonnegative integers $h_0, \dots, h_{d-1}$ is the convex hull

$$L(h_0, \dots, h_{d-1}) = \mathrm{conv}(0, e_1, \dots, e_{d-1}, h_0 e_d, e_1 + h_1 e_d, \dots, e_{d-1} + h_{d-1} e_d)$$

Equivalently, this is a d-dimensional polytope that projects onto the $(d-1)$-dimensional unimodular simplex

<u>Note</u>: • $h_i = 0$ is allowed
so the simplex is a Lawrence prism
$\to$ we require $\sum h_i \geq 2$ to exclude this

- A lattice pyramid over a Lawrence prism is a Lawrence prism (with new height 0):



We compute the degree of a Lawrence prism:

Let k be the minimal k s.th. kL contains an interior lattice pt.

Let Δ be the basis of L

$\Rightarrow$ $k\Delta$ contains an interior lattice pt

$\Rightarrow$ $k \geq (d-1)+1 \Rightarrow \deg L \leq d+1-k = 1$

Now $\deg L = 0 \Leftrightarrow L = \Delta_d$

$\Rightarrow$ $\deg L = 1$ if $\sum h_i \geq 2$.

We know one more polytope of degree 1:

The h^*-polynomial of $2\Delta_2$ is $1+3t$

$\rightsquigarrow$ also any lattice pyramid over $2\Delta_2$ has degree 1

Note: $2\Delta_2$ is <u>not</u> a Lawrence prism

There is the following theorem:

<u>Thm</u> (Batyrev / Hill) This list of lattice polytopes of degree 1 is complete. ☒

As we can construct Lawrence prisms of any normalized volume $v \geq 1$, any polynomial

$$h^*(t) = 1 + kt \quad \text{for } k \in \mathbb{Z}_{\geq 1}$$

is an h^*-polynomial of a lattice polytope.

For lattice polytopes of degree 2 we don't anymore know a classification of all polytopes, but we do know something about the h^*-polynomials:

Thm (Treutlein)
P d-dimensional lattice polytope of degree 2

If P is a $(d-2)$-fold lattice pyramid over $3\Delta_2$, then

$\operatorname{nvol} P = 9$, $|P \cap \mathbb{Z}^d| = d+8$ $|\operatorname{int} P \cap \mathbb{Z}^d| = 1$

Otherwise, let $l := |P \cap \mathbb{Z}^d|$, $i := |\operatorname{int}(d-1)P \cap \mathbb{Z}^d|$

Then:
(1) $\operatorname{nvol} P \leq 4(i+1)$
(2) $l \leq 3i + d + 4$
(3) $l \leq \frac{3}{4} \operatorname{nvol} P + d + 1$

These 3 conditions are equivalent

→ observe similarity to Scott!

This result has been extended by Balletti and Higashitani:

Essentially the same Thm already holds if just $h_3^* = 0$
(only the exceptional case of pyramids over $3\Delta_2$ changes)

→ This is now independent of the dimension and the degree!

We can remove intermediate zeros in the coefficients by passing to the "spanning polytope" (Hofscheier, Katthän, Nill)

→ this might be the right class of polytopes to look at.

We can further characterize lattice polytopes of small degree

Def For $P_0, \ldots, P_t \subseteq \mathbb{R}^k$ the Cayley sum is

$P_0 * P_1 * \cdots * P_t := \mathrm{conv}(P_0 \times \{0\}, P_1 \times \{e_1\}, \ldots, P_t \times \{e_t\}) \subseteq \mathbb{R}^k \times \mathbb{R}^t$

P is a Cayley polytope if there is an (affine) lattice basis of $\mathbb{Z}^d = \mathbb{Z}^k \times \mathbb{Z}^t$ identifying P with a Cayley sum

The dimension of the Cayley sum is

$$\dim P_0 * \dots * P_t = \dim \operatorname{aff}(P_0, \dots, P_t) + (t-1)$$

There are different characterizations of these polytopes:

<u>Lemma</u> P lattice polytope $C(P) = \operatorname{cone}(\{1\} \times P)$

The following are equivalent:

(1) P is a Cayley polytope of length $t+1$

(2) There is a lattice projection that maps P onto a unimodular t-simplex

(3) There are nonzero $x_0^*, \dots, x_t^* \in (\mathbb{Z}^{d+1})^*$ with

$$x_0^* + \dots + x_t^* = e_0^*$$

The lattice vectors $x_0^*, \dots, x_t^*$ are part of a basis of $(\mathbb{Z}^{d+1})^*$ and the Cayley structure of P is uniquely determined by

$$P_j := \{ v \in P : \langle v, x_k^* \rangle = 0 \text{ for } k \neq j \}$$

Then $\langle P_j, x_j^* \rangle = 1$ and $P = \operatorname{conv}(P_0, \dots, P_t)$

proof: Ex ☒

Lemma: For a Cayley polytope P of length $t+1$:

$$\operatorname{codeg} P \geq t+1$$

proof: Ex ☐

The following thm shows that polytopes of small degree are always Cayley polytopes:

Thm (Haase, Nill, Payne)

A lattice polytope of degree s is a Cayley polytope of length at least $d+1-\frac{1}{2}(s^2+19s-4)$

($\Leftrightarrow$ P is a Cayley polytopes of lattice polytopes in dim at most $\frac{1}{2}(s^2+19s-4)$)

Conj: The bound is linear.