We know that for each convex body K we can find an ellipsoid E centered at the origin s.th.

$$a + E \subseteq K \subseteq a + dE$$

However, the proof was not constructive.

→ construction is as for ellipsoid method
→ in general: LP / SDP - problem
→ we only need approximate version
→ for $K \subseteq a + 2dE$ instead of $a + dE$

Here is a sketch:

Find some ball E_0 containing P
by translation assume E_0 centered at origin

- if $\frac{1}{2d} E_0 \subseteq P$ we are done
- otherwise: there is $x \in \frac{1}{2d} E_0 \setminus P$ and a hyperplane H separating x from P.

→ for a polytope P, x and H can be found using a linear program.

H splits E_0 into two almost equal parts
let E_1' be the part containing P

let E_1 be an ellipsoid containing E_1' (and thus P)

it can be shown that E_1 is smaller than E_0 by a factor $(1 - \frac{1}{d})$

Now apply an affine transformation s.th. E_1 is a ball centered at the origin and repeat

$\to$ this terminates in polynomial time with an ellipsoid E s.th.

$$a + \frac{1}{2d} E \subseteq P \subseteq a + E \quad \text{for some } a.$$

This gives an algorithm for integer programming in fixed dimension:

Input: $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, $A \in \mathbb{Q}^{m \times d}$, $b \in \mathbb{Q}^m$

Output: Either $u \in P \cap \mathbb{Z}^d$ or decision that no such pt exists.

(1) Compute ellipsoid E : $z + E \subseteq P \subseteq z + dE$

(2) Use LLL to obtain lattice basis $V = (v_1, \ldots, v_d)$ s.th. orth. defect wrt the norm $\|\cdot\|_E$ is bounded and $\|v_1\|_E \leq \ldots \leq \|v_d\|_E$

(3) Determine λ_i : $z = \sum \lambda_i v_i$, let $u := \sum \lfloor \lambda_i \rfloor v_i$

(4) If $u \in P$ return u

(5) Set $c := w_n$

(6) for $\beta \in \{\lceil \min(c^t x, x \in P)\rceil, \ldots, \lfloor \max c^t x, x \in P \rfloor\}$ do

(6a) use HNF to find lattice basis $V' \subseteq \mathbb{Q}^n$ and $d \in \mathbb{R}^n$ st

$$d + \Lambda(V') = \{x \in \mathbb{Z}^d \mid c^t x = \beta\}$$

(6b) Call this algorithm on $\{x' \in \mathbb{R}^{n-1} \mid A(d + \sum_{j=1}^{d-1} x_j' v_j') \le b\}$

(6c) if there is a lattice pt u found, return u.

(7) return : no lattice pt in P.

Let $T(d)$ be the number of recursive calls

$\rightarrow$ Total running time is $T(d) \cdot \text{poly}(\text{input size})$

By the previous corollary

$$T(d) \le T(d-1)\Big(\underbrace{d}_{\text{flatness}} \cdot \underbrace{2^{\frac{d(d-1)}{2}}}_{\text{bound for ellipsoid}} + 1\Big)$$

$$\Rightarrow T(d) \le \prod_{k=1}^{d} \left(k\, 2^{\frac{k(k-1)}{2}} + 1\right) \le 2^{O(d^3)}$$

This proves:

Thm For a polytope $P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ we can find $u \in P \cap \Lambda$ or decide that there is none in time $2^{O(d^3)}$ poly (input size of A, b) $\boxtimes$

This is due to H.W. Lenstra '83

Note: Kannan '83: can be done in $d^{O(d)}$ poly (input)

Open: can this be done in $2^{O(d)}$?

Note: This is the first known algorithm for integer programming in polynomial time in fixed dimension

We can also use Barvinok's algorithm to solve the feasibility problem

$\to$ we find the optimal value by binary search.

19. Reflexive Polytopes

P lattice polytope $\rightarrow P = \{x \mid \langle a_i, x\rangle \le b_i\}$

for primitive $a_i \in \Lambda^*$, $b_i \in \mathbb{Z}$
irredundant.

Dual polytope if $0 \in \operatorname{int} P$

$$P^\vee := \{y \mid \langle x, y\rangle \le 1 \ \forall x \in P\} = \{y \mid \langle x, y\rangle \le 1 \ \forall x \in V(P)\}$$

hyperplane: $H = \{x \mid \langle a, x\rangle = b\}$, a primitive

lattice distance of $y \in \Lambda$ for H: $|b - \langle a, y\rangle|$

<u>Def</u> P is <u>reflexive</u> if there is $w \in \operatorname{int} P \cap \Lambda$ s.th. all facets have distance 1 for w

$\iff \langle a_i, w\rangle = b_i - 1 \quad \forall i$

$\iff P - w = \{x \mid \langle a_i, x\rangle \le 1\}$

<u>Prop</u> P reflexive $\Rightarrow |\operatorname{int} P \cap \Lambda| = 1$

"$\Leftarrow$" only in dimension 2

proof: Ex

$\boxed{E}$

Prop P lattice polytope, $0 \in \operatorname{int} P$

Thm P reflexive $\iff P^\vee$ lattice polytope $\iff P^\vee$ reflexive

proof: Ex.

Cor In each dimension $\{P \text{ reflexive}\}/_{\sim}$ is finite

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Prop Any lattice polytope P is the face of an reflexive one