21.1

Prop Any lattice polytope P is the face of an reflexive one

proof:

no interior lattice pt:

let $r = \operatorname{codeg} P$, then

$P' := \operatorname{conv}(P \times \{0\}, 2rP \times \{2\})$

contains a lattice pt and P is a face.

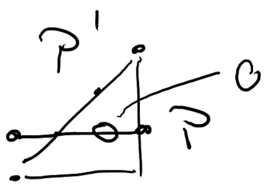
So assume $0 \in \operatorname{int} P$

$\Rightarrow P = \{x \mid \langle a_i, x\rangle \le b_i\}$ for $b_i \geq 1$

If $b_1 = 2$, let

$P' := \{(x_0, x) \mid x_0 \geq -1, \langle a_i, x\rangle \le b_i$ for $i \geq 2$

$\langle a_1 x\rangle + x_0 \le b_1 - 1\}$

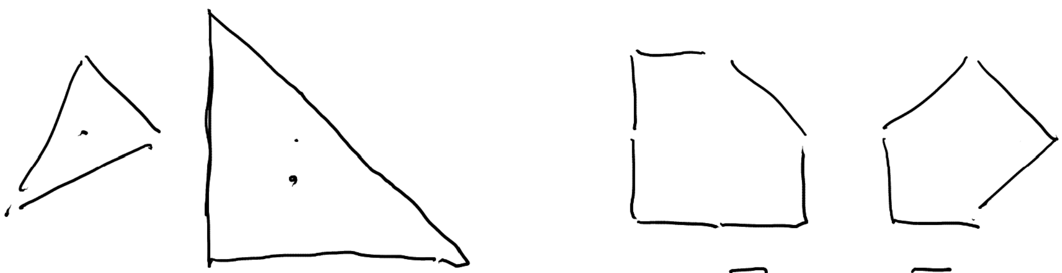


"shifted" wedge

$\boxed{E}$

Thm P reflexive lattice polygon

Then $|\partial P \cap \Lambda| + |\partial P^\vee \cap \Lambda^\vee| = 12$

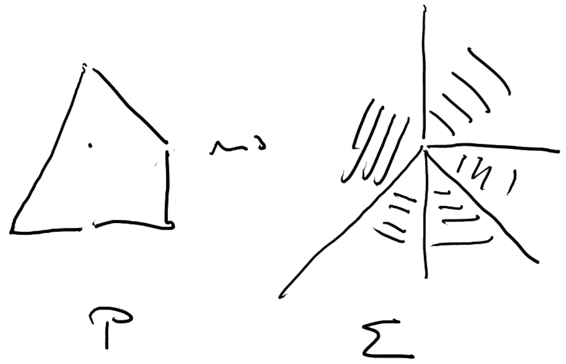


3 + 9

7 + 5

→ this will follow from a Thm about fans

P reflexive lattice polygon $\rightsquigarrow$ Σ complete fan with maximal cones $\mathrm{cone}(u,v)$ for consecutive lattice pts on bdry of P



Pick $\Rightarrow$ fan is unimodular $\Leftrightarrow$ generators of cones are lattice basis

v, u, v' consecutive

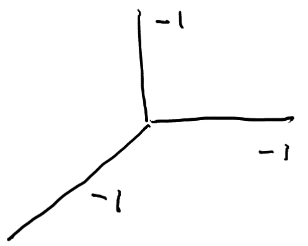
$$\Rightarrow \quad v' = \mu u + \eta v = \mu u + \eta(\alpha u + \beta v')$$

$$\Rightarrow \quad \eta = \beta = -1 \quad \Rightarrow \quad v + v' = a(u)\,u \quad \text{for some } a(u) \in \mathbb{Z}$$

By convexity $a(u) \leq 2$

<u>Thm F</u> $\Sigma \subseteq \mathbb{R}^2$ complete unimodular fan

Then
$$\sum_{u \text{ ray gen}} (3 - a(u)) = 12$$



$\rightsquigarrow 4+4+4 = 12$

This implies Thm P via

21.3

Prop

$$2-a(u) = \begin{cases} u \text{ not a vertex of } P \\ \text{length of the dual edge otherwise} \end{cases}$$

proof: e, e' edges for u with v, v'

≙ primal facet normals
evaluate at
v, v', u

$\Rightarrow$ dual vertices are $v^* + u^*$ and $u^* + (a-1)v^*$

$\Rightarrow$ length of dual edge is $2-a$ ☒

Proof of Thm ?:

$$12 = \sum_u 3-a(u) = \sum_{u \text{ vertex}} 2-a(u) + \sum_{u \in \partial P \cap \Lambda} 1$$

$$= |\partial P^\vee \cap \Lambda^*| + |\partial P \cap \Lambda|$$

☒

Def <u>smooth blowup</u> of max cone $\sigma = \mathrm{cone}(u,v) \in \Sigma$:
Let $w := u+v$, then

$$bl(\Sigma, \sigma) := \Sigma \setminus \{\sigma\} \cup \{\mathrm{cone}\, w, \mathrm{cone}(u,w), \mathrm{cone}(v,w)\}$$

We have $a(w) = 1$ and $a(u), a(v)$ increase by 1, so a blowup does not change

$$\sum_{u \text{ ray gen}} 3-a(u)$$

So Thm F follows for our example and

Thm: Σ, Σ' complete unimodular

Then there is Σ'' complete unimodular s.th.
Σ'' can be obtained from Σ, Σ' by smooth blowups.

$\rightarrow$ For general dim: Smooth Oda Conjecture

We need 4 steps:

(1) σ pointed 2-cone, then there is a subdivision into unimodular cones:

Take cones over finite faces of $P := \mathrm{conv}(\sigma \cap \mathbb{Z}^2 \setminus \{0\})$

(2) v, u, v' neighboring rays
with $v + v' = au$
$P := \mathrm{conv}(0, v, v', u)$

Then 0 is a vertex iff $a \geq 1$
u — " — $a \leq 1$

21.5

0 is vertex $\iff$

in $0 = \frac{1}{2-a}(v+v'-au)$ we have $a>0$

u is vertex $\iff$

in $u = \frac{1}{a}(v+v') + \frac{2-a}{a}\,0$ $\quad \frac{2-a}{a} < 0$

$\iff a \geq 1$

(3) Σ, Σ unimodular and Σ refines Σ'

$\Rightarrow$ Σ can be obtained from Σ' by smooth blowups.

By induction on r := rays in Σ not in Σ'

$r=0$: $\Sigma = \Sigma'$

$r \geq 1$: u gen in $\Sigma(1) \setminus \Sigma'(1)$

$\Rightarrow a(u) \geq 1$

if $\sigma = \mathrm{cone}(v,v') \in \Sigma'$ contains u, then

$u \notin \mathrm{conv}(0,v,v') \Rightarrow a \leq 1$

$\Rightarrow a = 1$

(4) $\Sigma'' := \{\sigma \cap \sigma' \mid \sigma \in \Sigma, \sigma \in \Sigma'\}$

and Σ'' unimodular refinement of Σ''

$\square$

(Thm ?)