22.1

(3) Σ, Σ' unimodular and Σ refines Σ'

$\Rightarrow \Sigma$ can be obtained from Σ' by smooth blowups.

By induction on $r :=$ rays in Σ not in Σ'

$r = 0$: $\Sigma = \Sigma'$

$r \geq 1$: u gen in $\Sigma(1) \setminus \Sigma'(1)$

$\Rightarrow a(u) \geq 1$

if $\sigma = \mathrm{cone}(v, v') \in \Sigma'$ contains u, then

$u \notin \mathrm{conv}(0, v, v') \Rightarrow a \leq 1$

$\Rightarrow a = 1$

(4) $\bar{\Sigma}'' := \{\sigma \cap \sigma' \mid \sigma \in \Sigma, \sigma \in \Sigma'\}$

and Σ'' unimodular refinement of $\bar{\Sigma}''$

$\boxed{\mathrm{Z}}$

(Thm $\mathcal{F}$)

A similar result is true in dimension 3:

<u>Thm</u> P 3-dimensional reflexive. Then

$$\sum_{e \text{ edge of } P} (\text{length } e)\cdot(\text{length } e^*) = 24$$

(no proof) $\boxtimes$

From now on: P <u>simplicial</u> and reflexive

<u>Prop</u> $x, y \in \partial P \cap \Lambda$, $x \neq y$. Then either

(1) x, y are in the same facet

(2) $x + y = 0$

(3) $x + y \in \partial P \cap \Lambda$.

Proof: <u>Ex</u> $\boxtimes$

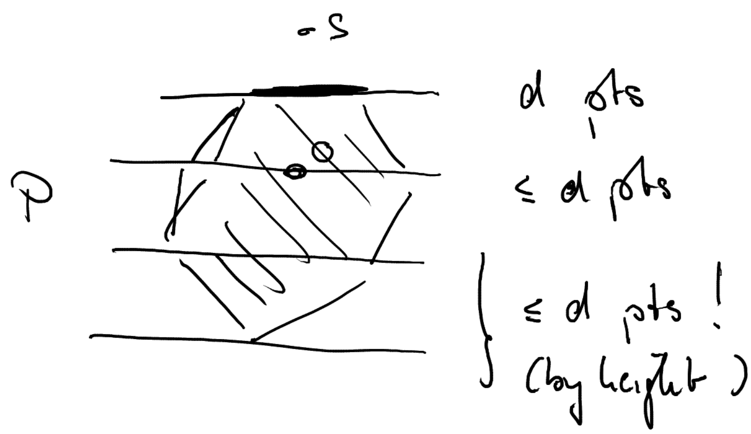
<u>Prop</u> F facet with normal a,
Then
$$|\{v \in V(P) \mid \langle a, v\rangle = 0\}| \leq d$$

(no proof)

This bounds the number of vertices:

let $s := \sum_{v \text{ vertex}} v$

Then there is a cone $C(u,v)$ for vertices u,v that contains s (maybe not unique)



<u>Thm</u> P simplicial reflexive. Then $|V(P)| \le 3d$. ☒

The equality case is known:

d even and P is a $d/2$-fold free sum of 

<u>Conj</u> This is true for <u>all</u> reflexive polytopes.

3C Gorenstein Polytopes

Def P is a Gorenstein polytope of index r if rP is a reflexive polytope.

Def A cone C is a Gorenstein cone if C is equivalent to a cone over a polytope

Prop $C = \mathrm{cone}(u_1, \dots, u_d)$ for primitive u_i is Gorenstein iff there is $w \in C^\vee \cap \Lambda^*$ s.th.

$$\langle u_i, w \rangle = 1$$

$\boxtimes$

Prop P lattice polytope of codegree r. Then the following is equivalent:

(1) P is Gorenstein of index r

(2) $\forall k \geq r$: $\mathrm{int}\, kP \cap \Lambda = w + (k-r)P \cap \Lambda$ for some $w \in \mathrm{int}\, rP \cap \Lambda$

(3) $C_P^\vee$ is a Gorenstein cone.

proof: Let $x' = (k, x)$ for $x \in kP$,

$u_F' = (b_F, u_F)$ for a facet F: $\langle u_F, \cdot \rangle = b_F$ of P

$\Lambda' := \mathbb{Z}e_0 \times \Lambda$

22.5

(1) $\Rightarrow$ (2) show both inclusions in (2)

„$\supseteq$" ✓

„$\subseteq$" let $w \in rP$ be the unique interior pt

$$\Rightarrow \langle u_F', w' \rangle = 1 \quad \forall F$$

Let $x \in$ int $kP \cap \Lambda$. Then

$$\langle u_F', x' - w' \rangle = \underbrace{\langle u_F', x' \rangle}_{\geq 1} - \underbrace{\langle u_F, w' \rangle}_{= 1} \geq 0$$

$\Rightarrow x' - w' \in C_P$ at height $k - r$ $\Rightarrow x - w \in (k-r)P \cap \Lambda$.