Prop: $P$ lattice polytope of degree $r$. Tfae:

(1) $P$ Gorenstein

(2) $\forall\, k \geq r$: $\operatorname{int} kP \cap \Lambda = \omega + (k-r)P \cap \Lambda$

(3) $C_P^\vee$ is a Gorenstein cone.

(1) $\Rightarrow$ (2) already done

(2) $\Rightarrow$ (3) show $\underbrace{\langle u_F', \omega' \rangle}_{\text{considered in } C_P} = 1$ for facets $F$ of $P$

Let $C := \operatorname{cone}(F', \omega')$

$\rightarrow$ choose lattice basis $a_1, \dots, a_{d+1} \in \Lambda'$ with
$a_1, \dots, a_d \in F'$, $a_{d+1} \notin F'$.

$\rightsquigarrow$ can assume $a_{d+1} \in C$ (translate by $\mu \sum_{i=1}^{d} a_i$, $\mu \in \mathbb{Z}$)

$\Rightarrow$ $a_{d+1} \in \operatorname{int}((k, kP))$ for some $k \geq r$

so by assumption $a_{d+1} = \omega' + m'$ for $m \in (k-r)P$

The dual $a_{d+1}^*$ to $a_{d+1}$ is primitive and $0$ on $F$

$\Rightarrow$ $a_{d+1}^* = u_F$

$\Rightarrow$ $1 = \langle u_F', a_{d+1} \rangle = \underbrace{\langle u_F', \omega' \rangle}_{\geq 1} + \underbrace{\langle u_F', \omega \rangle}_{\geq 0}$

$\Rightarrow$ $\langle u_F', \omega' \rangle = 1$

So $C_P^\vee \cap \{x \mid \langle \omega', x \rangle = 1\} = \operatorname{conv}(u_F')$ is a lattice poly.

(3) ⇒ (1) There is $w' \in C_P \cap \Lambda'$:

$$1 = \langle u'_F, w' \rangle \quad \forall F$$

⇒ $w \in \operatorname{int} kP \cap \Lambda$ for some $k \geq r$

and $kP$ is reflexive

If $k > r$, then

$$|\operatorname{int} kP \cap \Lambda| \geq |w + (k-r)P \cap \Lambda| = |(k-r)P \cap \Lambda| \geq |P \cap \Lambda| > 1$$ ☒

<u>Cor</u>: $P$ Gorenstein iff $e_P(-k) = (-1)^d e_P(k-r) \quad \forall k \geq r$.

proof: Ex. ☒

$P$ Gorenstein of index $r$:

$$\sum_{k\geq 1} |\operatorname{int} kP \cap \Lambda| t^k = \sum_{k\geq 1} (-1)^d e_P(-k) t^k = \sum_{k\geq 1} e_P(k-r) t^k = t^r \sum_{k\geq 0} e_P(k) t^k$$

So by Stanley's reciprocity:

$$\mathrm{Ehr}_P\left(\frac{1}{t}\right) \underset{\text{Stanley}}{=} (-1)^{d+1} \mathrm{Ehr}_{\operatorname{int} P}(t) = (-1)^{d+1} t^r \mathrm{Ehr}_P(t)$$

⇝ on the level of rational functions

$$\frac{\sum_{i=0}^{s} h_i^* t^i}{(1-t)^{d+1}} = (-1) \frac{1}{t^r} \frac{\sum_{i=0}^{s} h_i^* \left(\frac{1}{t}\right)^i}{\left(1-\frac{1}{t}\right)^{d+1}} = \frac{\sum_{i=0} h_i^* t^{s-i}}{(1-t)^{d+1}} = \frac{\sum_{i=0}^{s} h_{s-i}^* t^i}{(1-t)^{d+1}}$$

⇒ by comparing coefficients: $h_i^* = h_{s-i}^*$ for $0 \leq i \leq s$

Conversely, if $h_i^* = h_{s-i}^*$, then

$$(-1)^{d+1} \operatorname{Ehr}_P\left(\tfrac{1}{t}\right) = t^r \operatorname{Ehr}_P(t)$$

So

$$\sum_{k \geq r} e_P(k-r)\, t^k = t^r \sum_{k \geq 0} e_P(k)\, t^k = t^r \operatorname{Ehr}_P(t) = (-1)^{d+1} \operatorname{Ehr}_P\left(\tfrac{1}{t}\right)$$

$$\underset{\text{Stanley}}{=} \operatorname{Ehr}_{\operatorname{int} P}(t) = (-1)^d \sum_{k \geq 1} e_P(-k)\, t^k$$

→ comparing coefficients: $e_P(k-r) = (-1)^d e_P(-k)$

⇒ $P$ Gorenstein.

<u>Thm</u>: $P$ Gorenstein of index $r = d+1-s$ iff $h_i^* = h_{s-i}^*$, $0 \leq i \leq s$ ∎

# 21. Regular Triangulations



Recall: A subdivision $S$ of a point configuration $A$ is regular if there is a height function

$$\omega : A \longrightarrow \mathbb{R}$$

s.th. $S$ is the projection of the lower hull of

$$L_\omega(A) := \operatorname{conv}\left( (\omega(a), a) \mid a \in A \right)$$

We denote this by $S_\omega(A)$, or $S_\omega(P)$ for $P = \operatorname{conv}(A)$.

Faces of $S$ are the domains of linearity of

$$\begin{aligned} \psi_\omega : P &\longrightarrow \mathbb{R} \\ v &\longmapsto \min\left( h \mid (h,v) \in L_\omega(A) \right) \end{aligned}$$

The height function is tight if all $(\omega(a), a)$ are on the lower hull of $L_\omega(A)$

We can replace a height function with a tight one without changing $S$.

$F \subseteq A$ is face of the lower hull of $S_\omega(A)$ iff there is $u_F \in (\mathbb{R}^d)^*$, $\beta_F \in \mathbb{R}^d$ s.th.

$$\langle u_F, a\rangle + \omega(a) \geq \beta_F \quad \text{for all } a \in A.$$

and $F = \mathrm{conv}(a \in A,\ \langle u_F, a\rangle + \omega(a) = \beta_F)$

Def: $A$ point configuration, $S$ subdivision of $P = \mathrm{conv}(A)$

The pulling refinement $\mathrm{pull}(S, m)$ of $S$ for $m \in A$ is the subdivision obtained by replacing each $F$ that contains $m$ by

$\mathrm{conv}(F', m)$ for faces $F'$ of $F$ not containing $m$.

Prop: $S$ regular $\Rightarrow$ $\mathrm{pull}(S, m)$ regular

proof $S$ regular subdivision of $P = \mathrm{conv}(A)$ with height $\omega$.

Define

$$\omega'(m) = \omega(m) - \varepsilon$$
$$\omega'(a) = \omega(a) \quad \text{for } a \in A \setminus \{m\}$$

Claim: For $\varepsilon$ small enough $\omega'$ induces $\mathrm{pull}(S, m)$

$F$ is a face of pull$(S, u)$ if

(1) $F$ is a face of $S$ not containing $u$

(2) $F = \mathrm{conv}(u, F')$ for $F' \leq G \in S$ with $u \in G \setminus F'$

(1): $\omega|_F = \omega'|_F \Rightarrow L_\omega(F) = L_{\omega'}(F)$

and this is still a face of $L_{\omega'}(A)$ for $\varepsilon$ small