Prop: S regular $\Rightarrow$ pull(S,m) regular

proof: ω tight, $\omega'(m) = \omega(m) - \varepsilon$, $\omega'(a) = \omega(a)$ for $a \in A \setminus \{m\}$

F is a face of pull(S,m) if

(1) F is a face of S not containing m

(2) $F = \mathrm{conv}(m, F')$ for $F' \subseteq G \in S$ with $m \in G \setminus F'$

(1) ✓

(2): There are $u_{F'}, \beta_{F'}$ and u_G, β_G normals at F' and G

$\to$ for ε small enough

$$\langle u_{F'}, m \rangle + \omega'(m) - \beta_{F'} > 0$$

$$\langle u_G, m \rangle + \omega'(m) - \beta_G < 0$$

$\Rightarrow$ There are $\lambda, \mu > 0$, $\lambda + \mu = 1$:

$$\lambda(\langle u_{F'}, m \rangle + \omega'(m) - \beta_{F'}) + \mu(\langle u_G, m \rangle + \omega'(m) - \beta_G) = 0$$

$\Rightarrow$ $(\lambda u_{F'} + \mu u_G), (\lambda \beta_{F'} + \mu \beta_G)$ defines the face $F = \mathrm{conv}(F', m)$:

$$\lambda(\langle u_{F'}, a \rangle + \omega'(a) - \beta_{F'}) + \mu(\langle u_G, a \rangle + \omega'(a) - \beta_G)$$

is 0 for $a = m$ and $a \in F'$, > 0 for $a \notin F$

We still need to check that pull(S, m) is a subdivision of P

Let $v \in P$ and let G be a face of S containing v.
if $m \notin G$, then $G \in$ pull(S, m)

Otherwise: the ray from m through v hits some face F' of G in its relative interior

$\Rightarrow v \in \mathrm{conv}(F', m) \in$ pull(S, m). ☒

Cor (1) Pulling all pts in A gives a triangulation of P with vertex set A

(2) If only vertices are pulled, then every maximal cell of S is the join of the first pulled vertex v with a maximal cell in the pulling subdivisions of the facets not containing v.

proof (1) A face of pull(S, m) containing m is a pyramid with apex m.

if $F \in S$ has some $a \in A$ as apex, then

$a \in F' \leq F \Rightarrow a$ is apex of F'

$\Rightarrow$ any face of pull(S, m) contained in F and containing v has v as apex.

So after all $a \in \mathcal{A}$ are pulled they are vertices of S and apex in all faces they are contained in

$\Rightarrow$ all faces are simplices.

(2) Start with the trivial subdivision P and use the argument above. ☒

Prop: C rational cone $\Rightarrow$ C has a regular <u>unimodular</u> subdivision.

proof: Using pulling we find a reg. subdivision S into simplicial cones.

Let D be the max determinant of a cone in S

$D = 1 \Rightarrow S$ unimodular

$D > 1$: pick inclusion minimal cone $F \in S$ with det D, $F = \text{cone}(a_1, \dots, a_k)$

$\Rightarrow \exists\, u \in \Pi(F) : u = \sum \lambda_i a_i$ for $0 < \lambda_i < 1$

pulling at u replaces F with cones G_j of det $\lambda_j \det F < \det F$ ☒

22. Compressed Polytopes

24.4

Def: P lattice polytope.

P <u>compressed</u> iff $V(P) = |P \cap \Lambda|$ and all pulling triangulations are unimodular.

<u>Lemma</u> P lattice polytope

$\mathrm{width}_u P = 1$ for facet normals u of P

$\Rightarrow$ $\mathrm{width}_u F = 1$ for all faces F of P and facet normals u of F.

<u>proof</u>: By induction only for facets of P.

G facet of F $\Rightarrow$ there is F' facet of P: $G = F \cap F'$

Let a be a normal for F', $\langle a, F' \rangle = \beta$

then $\langle a, F \rangle - \beta \in [0,1]$

$\Rightarrow$ F has facet width 1 wrt to G. $\boxtimes$

<u>Thm</u> P lattice polytope. Then the following are equivalent:

(1) P compressed

(2) P has width 1 wrt all facets

(3) $P \cong [0,1]^m \cap H$ for an affine space H.

proof: (2) $\Rightarrow$ (1) :

let $V = (v_1, \dots, v_k) = P \cap \Lambda$ ordered.

S pulling triangulation obtained by pulling $v_1, \dots, v_k$

The restriction of S to a face of P is the pulling triangulation obtained from the order restricted to the face

$\to$ by induction the triangulation in the faces is unimodular.

By the previous prop. the cells of S are the pyramid of a face in the subdivision of a facet with v_1

This has height 1 by assumption, so S is unimodular.

(1) $\Rightarrow$ (2) : if F is a facet with width 2, then pick order $v_1, \dots, v_k$ of vertices with $v_1 \in P \setminus F$

$\Rightarrow$ If S' is induced subdivision on F, then for $G \in S'$ $\operatorname{conv}(G, v_1)$ is a face of the subdivision of P that is not unimodular.

(2) $\Leftrightarrow$ (3) ✓

$\boxtimes$