25.1

Def $A = \{a_1, \dots, a_r\} \subset (\mathbb{Z}^d)^*$ with $\operatorname{lin} A = (\mathbb{R}^d)^*$ is <u>unimodular</u> if

$$\det A' \in \{0, \pm 1\} \quad \forall (d\times d)\text{-minors of } A.$$

$\to A$ induces a subdivision D of $\mathbb{R}^d$ via the hyperplanes

$$H_{a_i, k} := \{x \in \mathbb{R}^d \mid \langle a_i, v\rangle = k\} \text{ for } 1 \le i \le r,\ k \in \mathbb{Z}$$

$\to$ all cells are lattice polytopes

Def such a subdivision is a <u>lattice dicing</u>

Def P is <u>facet unimodular</u> if the set of primitive normals is unimodular.

$P \cap D$ is the <u>canonical subdivision</u>

Canonical subdivisions are regular and induce canonical subdivisions of the faces.

<u>**Thm**</u> P facet unimodular.

Then P has a regular unimodular triangulation.

<u>proof</u> The cells of the dicing have width 1 wrt to all facets.

$\Rightarrow$ any pulling refinement of this will be unimodular $\square$

24. Unimodality of the h^*-Vector

Goal of this section:

Thm P Gorenstein with regular unimodular triangulation.
Then h^* is unimodal, i.e.

$$1 = h_0^* \le h_1^* \le \dots \le h^*_{\lfloor s/2 \rfloor} \ge \dots \ge h_s^* = 1$$

$s = \deg P$

(Bruns, Römer '07)

A polytope has the integer decomposition property (IDP) if for all $k \in \mathbb{Z}_{\geq 1}$

$$\otimes_k \ |P \cap \mathbb{Z}^d| \twoheadrightarrow |kP \cap \mathbb{Z}^d| \quad \text{surjective}$$

Conj: In the theorem we can replace the requirement of a unimodular triangulation by IDP.

Line of proof:

- h-vector of a simplicial polytope is unimodal by the g-Theorem
- h^* of a polytope coincides with h-vector of a unimodular triangulation.

- identify a subcomplex Γ of P as a "complement" of a "special simplex"
- show that Γ has the same h^*-vector and an induced regular triangulation T
- show that T is isomorphic to the boundary complex of a polytope.

<u>h-vector</u> Combinatorial invariant of a shellable simplicial complex

<u>Shelling</u> of a pure simplicial complex S

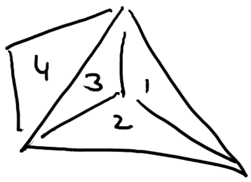
order $C_1, \dots, C_k$ of the max. faces s.th.

$$B_j := \bigcup_{i<j} C_i \cap C_j \quad \text{is pure } (d-1)\text{-dimensional}$$

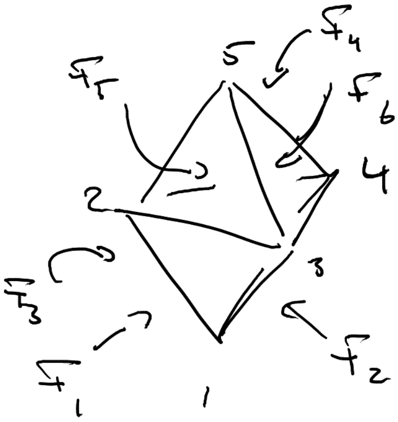
$\rightsquigarrow$ $C_j \setminus B_j$ has a unique minimal element R_j with r_j vertices, $0 \leq r_j \leq d$

The h-vector is the vector with entries

$$h_k := |\{ i \mid r_i = k \}|$$



$r_1 = 0, \; r_2 = 1$
$r_3 = 2 \quad r_4 = 1$ $\Big\}$ $h = (1, 2, 1)$



$R_1 = \emptyset \quad R_2 = \{4\} \quad R_3 = \{2, 4\}$

$R_4 = \{5\} \quad R_5 = \{3, 5\}, \; R_6 = \{3, 4, 5\}$

$\to h = (1, 2, 2, 1)$

<u>Thm</u> (g-Thm, McMullen, Stanley ~1970's)

S boundary complex of a simplicial polytope
Then

$$g := (1, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$$

is an M-sequence. In particular, it is non-negative
(The g-Thm claims more)