Thm (g-Thm, McMullen, Stanley ~1970's)

S boundary complex of a simplicial polytope

Then

$$g := (1, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$$

is an M-sequence. In particular, it is non-negative (The g-Thm claims more)

h vs h^*

Thm (Betke, McMullen 1985)

P lattice polytope with triangulation T

h^*: h^*-vector of P, h: h-vector of T

Then

$h_k \leq h_k^*$ for $0 \leq k \leq d$,

equality iff T is unimodular.

proof: Take half-open decomposition of T

This satisfies the shelling condition.

Let $C := \left\{ \sum \lambda_i v_i \;\middle|\; \begin{array}{ll} 0 < \lambda_i \leq 1 & i \in I \\ 0 \leq \lambda_i < 1 & i \notin I \end{array} \right\}$

be a cone over a half-open simplex

where $I = \{ i \mid \mu_i < 0 \}$ if $\xi = \sum \mu_i v_i$ generic.

recall:

$h_k^* :=$ # lattice pts at height k in half-open fundamental parallelepipeds

let $k := |I|$

Then C contributes at least 1 to h_k^*, exactly 1 iff C is unimodular.

The half-open decomposition satisfies the shelling condition:

minimal new face is $\{v_i \mid i \in I\}$

$\longrightarrow$ contributes 1 to h_k $\boxtimes$

<u>Lemma</u> T reg. triang. s.th. all max simplices in T have a common vertex $v \in \text{int } P$

Then T has the same h-vector as a simplicial polytope. $\boxtimes$

$\longrightarrow$ extends to join with a simplex.

26.3

Def $S \subseteq P$ simplex is <u>special</u> if

$S \cap F$ is a facet of F for all facets F of P

Prop P Gorenstein, $r = \operatorname{codeg} P$

P integrally closed $\Rightarrow$ P has special simplex of dim $r-1$

P with special simplex S of dim $r-1$

$\Rightarrow$ vertices of S have lattice distance 0 or 1 for all facets of P

proof: $m \in \operatorname{int} rP \cap \mathbb{Z}^d$. Then

$m = v_1 + \dots v_r$ for $v_i \in P \cap \mathbb{Z}^d$

Let $S := \operatorname{conv}(v_1, \dots, v_r)$

<u>Claim</u> S is special.

Let u be primitive ray generator of $C_P^\vee$

Then

$\langle u, v_i \rangle \geq 0$ for all i

$1 = \langle u, m \rangle = \sum \langle u, v_i \rangle$

$\Rightarrow \langle u, v_j \rangle = 1$ for exactly one $1 \leq j \leq r$

$\Rightarrow$ For the facet F of P defined by u:

$v_i \in F \quad \forall i \neq j, \quad v_j \notin F$

and this v_j has distance 1 from F $\qquad \boxtimes$

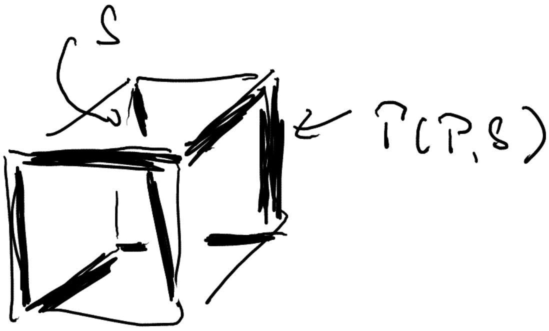
Def $S := \mathrm{conv}(v_1, \dots, v_r) \subseteq 1 \times P$ special simplex

The

$\Gamma(P,S) :=$ subcomplex of boundary of P generated by faces of the form

$$F_1 \cap F_2 \cap \dots \cap F_r$$

where F_i is a facet of C_P not containing v_i



In the following we consider C_P and S embedded at height 1.

$A_i := \{ a \text{ facet normal} \mid \langle a, v_i \rangle = 1 \}$

$\longrightarrow$ partition $A_1 \,\dot\cup\, A_2 \,\dot\cup \dots \dot\cup\, A_r$ of facets of C_P

Thm T restriction of a regular unimodular triangulation of P to $\Gamma(P,S)$

Then $T * S$ is a regular unimodular triangulation of P.

proof: Define:

$$\omega(x) := \min \Big\{ \sum_{i=1}^{r} \langle a_i, x \rangle \;\Big|\; a_i \in A_i,\ 1 \le i \le r \Big\}.$$

Claim: ω is piecewise linear, and linear on $\mathrm{cone}(F \cup S)$ for $F \in \Gamma(P,S)$

26.5

$\ulcorner$ $x \in P$,

choose $a_i \in \mathcal{H}_i$ minimizing $\langle a_i, x \rangle$ for $1 \le i \le r$

$(a_1, \dots, a_r)$ determines a face of Γ

For $u \in \mathcal{H}_i$:

for $y \in S$: $\langle u, y \rangle = \langle a_i, y \rangle$

(interpolates between 0 and 1)

for $y \in F$: $\langle u, y \rangle \ge \langle a_i, y \rangle = 0$

$\Rightarrow$ $\omega(x) = \sum \langle a_i, x \rangle$ along $\operatorname{cone}(F \cup S)$

Conversely: let $x \in 1 \times P$, $\omega(x) = \sum \langle a_i, x \rangle$

want to show that $x \in \operatorname{cone}(F \cup S)$

Let $y := x - \sum \langle a_i, x \rangle v_i$

For $u \in \mathcal{H}_i$:

$\langle u, y \rangle = \langle u, x \rangle - \sum \langle a_i, x \rangle \langle u, v_i \rangle$

$= \langle u, x \rangle - \langle a_i, x \rangle \ge 0$

$\Rightarrow y \in C$. and $\langle a_i, y \rangle = 0$ for $1 \le i \le r$

$\Rightarrow y \in \operatorname{cone} F$ $\Rightarrow$ $x \in \operatorname{cone}(F \cup S)$

So $\{\operatorname{cone}(F \cup S) \mid F \in \Gamma\}$ is a reg. subdivision $\lrcorner$

Let $\omega_1, \dots, \omega_s$ be a simplex in Γ

Claim: $\omega_1, \dots, \omega_s, v_1, \dots, v_r$ generate the lattice

$\Big\lceil$ Let $x \in \mathbb{Z}^{d+1}$

$\rightarrow \quad x = y + \sum_i \lambda_i v_i \qquad y \in \tau, \ \lambda_i \geq 0$

$\omega_1, \dots, \omega_s$ are in a face of Γ

$\rightarrow$ there are $a_i \in H_i$: $\langle a_i, \omega_j \rangle = 0$

$\langle a_i, v_j \rangle = \delta_{ij}$

$\Rightarrow \quad \lambda_i = \langle a_i, x \rangle \in \mathbb{Z}$

$\Rightarrow \quad y = x - \sum \lambda_i v_i \quad \in \tau \cap \mathbb{Z}^{d+1}$

and

$y = \sum \mu_i \omega_i$ for integral μ_i as τ is unimodular! $\Big\rfloor$

Let ω' be a triangulation of P that restricts to τ on Γ

$\rightsquigarrow \quad \omega + \varepsilon \omega'$ for ε small induces $\tau * S$

$\Big\lceil$ it is a refinement of $\Gamma * S$ induced by ω so that every conv$(F \cup S) = F * S$ is subdivided according to ω'.

But S is a simplex, so ω' induces the subdivision induced on F joined with S $\Big\rfloor$

☒

Now we can finally prove the main thm of this section:

proof:

P Gorenstein with reg triang

$\Rightarrow$ P IDP

$\Rightarrow$ P has special simplex of dim $r-1$

$\Rightarrow$ can define $\Gamma(P,S)$

$\Rightarrow$ get reg. unimod triang of P of the form $\Gamma * S$

$\Rightarrow h^*(P) = h^*(\Gamma) = h(\Gamma)$.

Claim: Γ is comb. isom to boundary of a simplicial polytope.

Let φ be strictly convex piecewise linear on $\Gamma * S$.

S is a face of the triangulation

$\Rightarrow$ there is functional u:

$\langle u, v\rangle = \varphi(u)$ for $v \in S$

$< \varphi(u)$ for $v \notin S$

Let L be a subspace that meets S transversally

$\Rightarrow$ for ε small: $Q := \{x \in L \mid \varphi(x) - \langle u, x\rangle \leq \varepsilon\}$

is a polytope whose boundary complex is comb. isom. to Γ

Now apply the g-Theorem.

☒

26.8

Corollary P, Q reflexive, then

$$h^*_{P\oplus Q} = h^*_P \cdot h^*_Q$$

proof: this is true for the join of P and Q
the origins of P, Q are a special simplex S

projecting along S does not change h^*
and the projection is $P \oplus Q$ $\boxtimes$