Modes of Bar Recursion

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(based on joint work with Martín Escardó and Paulo Oliva)
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Outline

1. Introduction
   - Bar recursion
   - Overview of talk

2. Modes of bar recursion
   - PS / Finite bar recursion
   - EPS / Spector’s bar recursion
   - IPS / Modified bar recursion
   - UR / Berardi-Bezem-Coquand functional

3. Interdefinability results
   - Relationship between EPS and IPS
   - Relationship between IPS and UR
**Bar recursion**

**Primitive recursion:** Recursion over the natural numbers. For \( n \in \mathbb{N} \):

\[
R(n) := \begin{cases} 
  y & \text{if } n = 0 \\
  z_{n-1}(R(n-1)) & \text{otherwise}
\end{cases}
\]
**Bar recursion**

**Primitive recursion:** Recursion over the natural numbers. For $n \in \mathbb{N}$:

$$R(n) := \begin{cases} y & \text{if } n = 0 \\ z_{n-1}(R(n - 1)) & \text{otherwise} \end{cases}$$

**Bar recursion:** Recursion over well-founded trees. For $s \in T$:

$$B(s^{\times^*}) := \begin{cases} Y_s & \text{if } s \text{ is a leaf} \\ Z_s(\lambda x . B(s \ast x)) & \text{otherwise} \end{cases}$$

Bar recursion is the wrong way round: $B(s)$ looks at the values of $B(s \ast x)$ for extensions of $s$!
• **Gödel 1958** Dialectica interpretation of arithmetic

  Arithmetic (induction) $\leftrightarrow$ System T (primitive recursion)

**Spector 1962** Dialectica interpretation of analysis

  Arithmetic $+$ Countable choice $\leftrightarrow$ System T $+$ Bar recursion
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  \text{Arithmetic } + \text{ Countable choice} \leftrightarrow \text{System T } + \text{Bar recursion}
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• **Escardó/Oliva 2010**- Product of selection functions. Interdefinability results. Links with game theory made precise.
Theme of talk

Computational aspects of bar recursion

1. Key computational features of different modes of bar recursion.
2. The relative strength of these modes of bar recursion.

To a lesser extent: The semantics of bar recursion (links with language of sequential games).

Why is this important?
Theme of talk

Computational aspects of bar recursion

1. Key computational features of different modes of bar recursion.
2. The relative strength of these modes of bar recursion.

To a lesser extent: The semantics of bar recursion (links with language of sequential games).

Why is this important?

- Open questions about an important class of non-primitive recursive functionals.
- Better understand computational content of classical proofs.
- Interesting mathematical problem.
Modes of bar recursion

UR: symmetric, implicit

store memory via $X^\omega$

IPS: sequential, implicit

T defines $\mu_{\text{Spector}}$

IPS not S1-S9 definable

EPS: sequential, $|s| \geq \varphi(\hat{s})$

$\varphi$ constant

$S^\omega \not\models$ EPS

PS: sequential, $|s| \geq n$

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The finite product of selection functions


- $q : X^n \rightarrow R$ determines the outcome of a play of type $X^n$ (by instead considering $q : X^\omega \rightarrow R$ type independent of $n$).
- $\varepsilon_s : (X \rightarrow R) \rightarrow X$ dictates a strategy for $|s|$th round given a partial play $s^{X^*}$.
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- $\varepsilon_s: (X \rightarrow R) \rightarrow X$ dictates a strategy for $|s|$th round given a partial play $s^X$.

$$\text{PS}^{\varepsilon,q,n}(s^X) := \begin{cases} 
\langle \rangle & \text{if } |s| \geq n \\
 a_s \ast \text{PS}(s \ast a_s) & \text{otherwise}
\end{cases}$$

where $a_s := \varepsilon_s(\lambda x . q(s \ast x \ast \text{PS}(s \ast x)))$.

- For $|s| < n$, PS$(s)$ is the optimal continuation of (of length $n - |s|$) of the play $s$.
- PS$(\langle \rangle)$ is an optimal strategy for the game.
Example 1

\[ X = [2] \; ; \; R = \mathbb{N} \; ; \; n = 3 \; ; \; q: [2]^3 \to \mathbb{N}^3 \; ; \]
\[ \varepsilon_i(p^{[2] \to \mathbb{N}}) = x \text{ maximising } p(x)_i \]

\[ q(x_0, y_0, z_0) = (0,1,2) \]
\[ q(x_0, y_0, z_1) = (2,1,1) \]
\[ q(x_0, y_1, z_0) = (3,0,2) \]
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Example 1

\[ PS(x_1, y_0) = \langle \varepsilon_2(z_0 \mapsto 0, z_1 \mapsto 1) \rangle \ast PS(x_1, y_0, z_i) = \langle z_1 \rangle * \langle \rangle = \langle z_1 \rangle \]

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\[ PS(x_1) = \langle y_0 \rangle \ast PS(x_1, y_0) = \langle y_0, z_1 \rangle \]
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PS(\langle \rangle) = \langle x_1 \rangle \ast PS(x_1) = \langle x_1, y_0, z_1 \rangle
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Example 2 (Noughts and Crosses)

\[ X = \{0, \ldots, 8\} \text{ and } R = \{1, 0, -1\}. \]
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- $X = \{0, \ldots, 8\}$ and $R = \{1, 0, -1\}$.
- $X^{(9)}$ encodes a game (only part of this may be relevant).

$$q(s^{X^{(9)}}) := \begin{cases} 1 & \text{if first player wins} \\ 0 & \text{if players draw} \\ -1 & \text{if second player wins} \end{cases}$$
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- $\varepsilon_{2i}(p^{X \rightarrow R})/\varepsilon_{2i+1}(p)$ selects $x$ maximising/minimising $p(x)$.
- $\text{PS}^{\varepsilon,q,n}(\langle \rangle)$ returns an ‘optimal’ play, resulting in a draw.
Is PS well-defined?

PS well-defined (i.e. defining equations have a unique solution) in any model of system T:
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Remark. PS is equivalent over a weak base theory to Gödel's primitive recursion in all finite types.
Properties of PS

- **Order.**
  - Computation carried out sequentially: value of $\text{PS}(\langle \rangle)_1$ depends on the value of $\text{PS}(\langle \rangle)_0$ and so on.

- **Well-foundedness.**
  - Underlying tree given explicitly: $s$ a leaf $\iff |s| = n$.

- **Models.**
  - Exists in any model of (higher-type) primitive recursion i.e. standard set theoretic model, total continuous functionals...

- **Semantics.**
  - Operation that computes optimal strategies in finite sequential games.
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Models. Exists in any model of (higher-type) primitive recursion i.e. standard set theoretic model, total continuous functionals...

Semantics. Operation that computes optimal strategies in finite sequential games.
Modes of bar recursion

PS sequential, $|s| \geq n \iff$ Gödel’s primitive recursion
Explicitly iterated product of selection functions (EPS)

Idea: Sequential game with unbounded number of rounds.

- $q : X^\omega \to R$ determines outcome of each infinite play $X^\omega$.
- $\varepsilon_s : (X \to R) \to X$ dictates a strategy for $|s|$th round given any partial play $s^X^*$. 
- $\varphi : X^\omega \to \mathbb{N}$ gives ‘relevant’ part of infinite play.
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\text{EPS}^{\varepsilon, q, \varphi}(s^X) := \begin{cases} 
0^{X^\omega} & \text{if } |s| \geq \varphi(\hat{s}) \\
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where $a_s := \varepsilon_{|s|}(\lambda x . q(s \ast x \ast \text{EPS}(s \ast x)))$ (and $\hat{s} := s \ast 0$).
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where \( a_s := \varepsilon_{|s|}(\lambda x . q(s \ast x \ast \text{EPS}(s \ast x))) \) (and \( \hat{s} := s \ast 0 \)).

Stopping condition now depends on \( s \)!
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For underlying tree to be well-founded, need property that for all infinite sequences $\alpha^\omega$ there must exists $n$ such that $n \geq \varphi([\alpha](n))$. 
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Fails for e.g.

$$\varphi(\alpha) := i + 1 \text{ for least } i (\alpha_i = 0), \text{ 0 otherwise.}$$

If $\alpha = \lambda i.1$ then for arbitrary $n$
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If $\alpha = \lambda i.1$ then for arbitrary $n$

$$\varphi([\lambda i.1](n)) = \varphi(1, \ldots, 1, 0, 0, \ldots)$$

$n$ times

$$= n + 1 > n.$$
**Theorem.** EPS exists in the total continuous functionals $C^\omega$. 
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\[
\text{CONT : } \forall \varphi^{X^\omega \rightarrow N} \forall \alpha^{X^\omega} \exists N \forall \beta ([\alpha](N) \overset{X^*}{=} [\beta](N) \rightarrow \varphi \alpha = \varphi \beta)
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\text{CONT : } \forall \varphi^X_\omega \rightarrow \mathbb{N} \forall \alpha^X_\omega \exists N \forall \beta ([\alpha](N) \overset{X^*}{=} [\beta](N) \rightarrow \varphi\alpha = \varphi\beta)
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By CONT, $\varphi([\alpha](n)) = \varphi\alpha$ for all $n \geq N$, so for $n = \max\{N, \varphi\alpha\}$ we have $n \geq \varphi\alpha = \varphi([\alpha](n))$. 

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For all $\alpha$ there exists some $n$ such that $\text{EPS}([\alpha](n)) = 0$ and therefore $\text{EPS}([\alpha](n))$ well-defined.
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If $\text{EPS}(s \ast x)$ well-defined for all extensions $s \ast x$ of $s$, then by definition so is $\text{EPS}(s)$. 
Theorem. EPS exists in the total continuous functionals $C^\omega$.

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If $\text{EPS}(s \ast x)$ well-defined for all extensions $s \ast x$ of $s$, then by definition so is $\text{EPS}(s)$.

By the principle of bar induction $\text{EPS}(\langle \rangle)$ is well-defined.
**Properties of EPS**

**Order.** Like PS, computation carried out sequentially: value of \( \text{EPS}(\langle \rangle)_1 \) depends on the value of \( \text{EPS}(\langle \rangle)_1 \) and so on.
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Order. Like PS, computation carried out sequentially: value of $\text{EPS}(\langle \rangle)_1$ depends on the value of $\text{EPS}(\langle \rangle)_1$ and so on.

Well-foundedness. Like PS, underlying tree given *explicitly*.

$$s \text{ a leaf } \iff |s| \geq \varphi(\hat{s}) \land \forall t < s(|t| < \varphi(\hat{t})).$$
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**Well-foundedness.** Like PS, underlying tree given *explicitly*.

$$s \text{ a leaf } \iff |s| \geq \varphi(\hat{s}) \land \forall t \prec s(|t| < \varphi(\hat{t})).$$

**Models.** Unlike PS, well-foundedness of recursion not provable in $T$. EPS exists in $\mathcal{C}^\omega$ where CONT holds, but not in the standard model $S^\omega$. 
Properties of EPS

**Order.** Like PS, computation carried out sequentially: value of $\text{EPS}(_{1})$ depends on the value of $\text{EPS}(_{1})$ and so on.

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**Models.** Unlike PS, well-foundedness of recursion not provable in T. EPS exists in $\mathcal{C}^{\omega}$ where CONT holds, but not in the standard model $\mathcal{S}^{\omega}$.

**Semantics.** Operation that computes optimal strategies in unbounded sequential games, relevant part of play $\alpha$ given by $\varphi(\alpha)$. 
Modes of bar recursion

EPS sequential, $|s| \geq \varphi(\hat{s})$ ← Spector's bar recursion

$\varphi$ constant $S^\omega \not| EPS$

PS sequential, $|s| \geq n$ ← ← ← → Gödel's primitive recursion

_Dialectica interpretation (1962)_
Implicitly iterated product of selection functions

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\text{IPS}^{\varepsilon \cdot q}(s^{X^*}) := a_s \ast \text{IPS}(s \ast a_s)
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where $a_s := \varepsilon_s(\lambda x . q(s \ast x \ast \text{IPS}(s \ast x)))$. 
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Idea: Sequential game with unbounded number of rounds, but now we forget about the control functional $\varphi : X^\omega \to \mathbb{N}$.

$$\text{IPS}^{\varepsilon,q}(s^{X^*}) \overset{X^\omega}{=} a_s \ast \text{IPS}(s \ast a_s)$$

where $a_s := \varepsilon_s(\lambda x . q(s \ast x \ast \text{IPS}(s \ast x)))$.

No longer a stopping condition!
Why is IPS well defined?

Even in $C^\omega$ there are obvious instances where IPS is not computable.
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Even in $C^\omega$ there are obvious instances where IPS is not computable.

Let $X = N$, $R = N^\omega$, $q = \text{id}: N^\omega \to N^\omega$ and
\[\varepsilon_s(p^{N \to N^\omega}) = p(0)_{|s|+1} + 1.\]
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Let $X = \mathbb{N}$, $R = \mathbb{N}^\omega$, $q = \text{id}: \mathbb{N}^\omega \to \mathbb{N}^\omega$ and

$$\varepsilon_s(p^{\mathbb{N} \to \mathbb{N}^\omega}) = p(0)|s|+1 + 1.$$

$$\text{IPS}(\langle \rangle)_0 \overset{\mathbb{N}}{=} \varepsilon_{\langle \rangle}(\lambda x . x * \text{IPS}(x))$$
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$$\text{IPS(⟨⟩)}_0 \overset{\mathbb{N}}{=} \varepsilon_{⟨⟩}(\lambda x . x \ast \text{IPS}(x))$$

$$= \text{IPS(0)}_0 + 1$$

$$= \varepsilon_{⟨0⟩}(\lambda x . 0 \ast x \ast \text{IPS}(0 \ast x)) + 1$$

$$= \text{IPS}(0, 0)_0 + 2$$
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\[
\text{IPS}(\langle \rangle)_0 = \varepsilon_{\langle \rangle}(\lambda x . x * \text{IPS}(x)) \\
= \text{IPS}(0)_0 + 1 \\
= \varepsilon_{\langle 0 \rangle}(\lambda x . 0 * x * \text{IPS}(0 * x)) + 1 \\
= \text{IPS}(0, 0)_0 + 2 \\
\vdots \\
= \text{IPS}(0, \ldots, 0)_0 + n \\
\quad n \text{ times} \\
\vdots
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**Theorem.** IPS exists in $C^\omega$ whenever the outcome type $R$ in $q: X^\omega \to R$ has type level 0 (more generally open, discrete...).
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\text{CONT : } \forall q^X \forall \alpha^X \exists N \forall \beta (\alpha(N) X^* = \beta(N) \to q\alpha = q\beta)
\]

Therefore $\text{IPS}([\alpha](N))_0 = \varepsilon_{[\alpha]}(N)(\lambda x . q(\alpha))$ and by induction

\[
\text{IPS}([\alpha](N)) = \lambda k . \varepsilon_{[\alpha]}(N)^* t_k (\lambda x . q\alpha).
\]

where $t_k = [\text{IPS}([\alpha](N))](k)$, so $\text{IPS}([\alpha](N))$ well-defined.
Theorem. IPS exists in $C^\omega$ whenever the outcome type $R$ in $q: X^\omega \to R$ has type level 0 (more generally open, discrete...).

$$\text{CONT : } \forall q X^\omega \to R \forall \alpha X^\omega \exists N \forall \beta ([\alpha](N) \overset{X^*}{=} [\beta](N) \to q\alpha = q\beta)$$

Therefore $\text{IPS}([\alpha](N))_0 = \varepsilon_{[\alpha](N)}(\lambda x. q(\alpha))$ and by induction

$$\text{IPS}([\alpha](N)) = \lambda k. \varepsilon_{[\alpha](N) \ast t_k}(\lambda x. q\alpha).$$

where $t_k = [\text{IPS}([\alpha](N))](k)$, so $\text{IPS}([\alpha](N))$ well-defined.

If $\text{IPS}(s \ast x)$ is well-defined for all extension $s \ast x$ of $s$, then by definition $\text{IPS}(s)$ is also well-defined.
**Theorem.** IPS exists in $C^\omega$ whenever the outcome type $R$ in $q : X^\omega \to R$ has type level 0 (more generally open, discrete...).

$$\text{CONT : } \forall q : X^\omega \to R \forall \alpha : X^\omega \exists N \forall \beta : (\alpha(N) \ x^* \alpha = \beta(N) \to q\alpha = q\beta)$$

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If $\text{IPS}(s * x)$ is well-defined for all extension $s * x$ of $s$, then by definition $\text{IPS}(s)$ is also well-defined.

By bar induction $\text{IPS}(\langle \rangle)$ is well-defined.
Properties of IPS

**Order.** Like EPS, computation carried out sequentially: value of IPS(⟨⟩)_1 depends on the value of IPS(⟨⟩)_0 and so on.
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Well-foundedness. Unlike EPS, underlying tree exists implicitly, and cannot be written down in system T.
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**Models.** Like EPS, well-foundedness of recursion not provable in T. IPS exists in $\mathcal{C}^\omega$, where we require additional condition that $R$ has level 0.

**Semantics.** Operation that computes optimal strategies in unbounded sequential games, only finite part of a play considered by continuity of $q$. 
Modes of bar recursion

- IPS: sequential, implicit
- EPS: sequential, $|s| \geq \varphi(\hat{s})$
- PS: sequential, $|s| \geq n$
- Modified bar recursion
  - Modified realizability (2003)

- Spector's bar recursion
  - Dialectica interpretation (1962)
  - Gödel's primitive recursion
Update recursion (UR)

Idea: Compute a sequence, but not sequentially...

\[ IPS'(s) := s \ast \lambda k . \varepsilon_{s \ast t_k}(\lambda x . q(IPS'(s \ast t_k \ast x))) \]

where \( t_k := [IPS'(s)](l) \).
Update recursion (UR)

Idea: Compute a sequence, but not sequentially...

\[ \text{IPS}'(s) := s * \lambda k . \varepsilon_{s*t_k}(\lambda x . q(\text{IPS}'(s * t_k * x))) \]

where \( t_k := [\text{IPS}'(s)](l) \).

- Suppose that \( u : X_\perp^\omega \) is a partial function.
- Let \( u^x_k \) denote update of \( u \) with \( x \) where \( k \) not in domain of \( u \).
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- Suppose that \( u : X^\omega_\bot \) is a partial function.
- Let \( u^x_k \) denote update of \( u \) with \( x \) where \( k \) not in domain of \( u \).

\[
\text{UR}(u) := u @ \lambda k . \varepsilon_k (\lambda x . q(\text{UR}(u^x_k))).
\]
Update recursion (UR)

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- Let $u^x_k$ denote update of $u$ with $x$ where $k$ not in domain of $u$.

$$UR(u) := u @ \lambda k . \varepsilon_k(\lambda x . q(UR(u^x_k))).$$

Recursion is no longer sequential!
Why is UR well defined?

UR($u$) requires us to know value of UR($u^x_k$) for all *updates* of $u$ (not just extension), so not clear how we can use bar induction to show that UR is total...
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UR($u$) requires us to know value of UR($u^x_k$) for all *updates* of $u$ (not just extension), so not clear how we can use bar induction to show that UR is total...

An *open predicate* on sequences $X^\omega$ is one of the form

$$A(\alpha) = \exists N B([\alpha](N)).$$

**Definition.** Update induction is given by the schema

$$\forall u (\forall n \notin \text{dom}(u), x A(u^x_n) \rightarrow A(u)) \rightarrow \forall u A(u).$$

Update induction follows from dependent choice.
Theorem. UR exists in $C^\omega$ for $R$ of type level 0.
**Theorem.** UR exists in $C^\omega$ for $R$ of type level 0.

By CONT, the predicate ‘$q(\text{UR}(u))$ is total’ is equivalent to an open predicate on partial sequences $u : X^\omega_\bot$, because if $q(\text{UR}(u))$ is total, it must only look at a finite part of $u$. 
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If $q(\text{UR}(u^x_n))$ is total for all updates of $u$, then UR($u$) and hence $q(\text{UR}(u))$ is total.
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By CONT, the predicate ‘$q(\text{UR}(u))$ is total’ is equivalent to an open predicate on partial sequences $u : X_\omega$, because if $q(\text{UR}(u))$ is total, it must only look at a finite part of $u$.

If $q(\text{UR}(u_n^x))$ is total for all updates of $u$, then UR($u$) and hence $q(\text{UR}(u))$ is total.

By update induction $q(\text{UR}(u))$ total for all $u$, and therefore UR($u$) is total for all $u$. 
Properties of UR

**Order.** Unlike IPS, computation of individual entries carried out independently. Value of \( \text{UR}(s)_0 \) does not affect value of \( \text{UR}(s)_1 \) and so on.
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**Semantics.** Can be viewed as computing an optimal strategy in games where players ‘ignore’ the others. *Game theoretic semantics not properly formalised for UR!*
Modes of bar recursion

UR symmetric, implicit

Berardi/Bezem/Coquand functional realizability (1998)

IPS sequential, implicit

modified bar recursion
modified realizability (2003)

EPS sequential, $|s| \geq \varphi(\hat{s})$

Spector’s bar recursion
Dialectica interpretation (1962)

PS sequential, $|s| \geq n$ → Gödel’s primitive recursion

\[ \varphi \text{ constant } S^\omega \nmid EPS \]
Outline

1 Introduction
   - Bar recursion
   - Overview of talk

2 Modes of bar recursion
   - PS / Finite bar recursion
   - EPS / Spector’s bar recursion
   - IPS / Modified bar recursion
   - UR / Berardi-Bezem-Coquand functional

3 Interdefinability results
   - Relationship between EPS and IPS
   - Relationship between IPS and UR
Introduction

Modes of bar recursion

Interdefinability results

T-definablility

**Definition** A functional $\Psi$ is $T$-definable from a functional $\Phi$ over a theory $S$ (we write $S \vdash \Phi \geq_T \Psi$) if there exists a term $t$ in system $T$ such that $t(\Phi)$ satisfies the defining equation of $\Psi$ provably in $S$. 
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In general our theory $S$ will be something like $\text{HA}^\omega + \text{CONT} + \text{BI}$.
IPS T-defines EPS (Oliva/Escardo)

Key observation: The so-called Spector's search functional

\[ \mu_{Sp}(\phi_{X\omega}\to\mathbb{N})(\alpha_{X\omega}) := \text{least } n (n \geq \phi(\hat{\alpha}(n))) \]

is definable in system T (even if T cannot prove it exists)!

Define \( \alpha_{\phi}(i) := \begin{cases} 0 & \text{if } \exists k \leq i + 1 (k \geq \phi(\hat{\alpha}(k))) \\ \alpha(i) & \text{otherwise} \end{cases} \)

If \( n = \mu_{Sp}(\phi_{\alpha}) \) then \( \alpha_{\phi} = \hat{\alpha}(n-1) \).

Because \( n \) is the least we have \( \phi(\hat{\alpha}(n)) > n - 1 \) and so \( n \leq \phi(\hat{\alpha}(n)) \).

Can encode stopping condition \( |s| \geq \phi(\hat{s}) \) into \( \tilde{\epsilon}, \tilde{q} \) such that

\( IPS_{\tilde{\epsilon}, \tilde{q}} T\text{-defines } EPS \).
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Kleene (1959). Schemes S1-S9 of computations in higher types.
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Key observations:
- Spector’s bar recursion/EPS is S1-S9 computable in $C^\omega$.
- The fan functional FAN exists in $C^\omega$ but is not S1-S9 computable in $C^\omega$. 
EPS does not T-define IPS (Oliva/Escardo)

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FAN is S1-S9 + IPS computable in $\mathcal{C}^\omega$

$\Rightarrow$ IPS is not S1-S9 computable in $\mathcal{C}^\omega$

$\Rightarrow$ IPS is not T-definable from EPS in any theory that has a model in $\mathcal{C}^\omega$. 
Modes of bar recursion

UR (symmetric, implicit) → Berardi/Bezem/Coquand functional realizability (1998)

IPS (sequential, implicit) → modified bar recursion
modified realizability (2003)

T defines $\mu_{\text{Spector}}$ → IPS not S1-S9 definable

EPS (sequential, $|s| \geq \varphi(\hat{s})$) → Spector's bar recursion
Dialectica interpretation (1962)

$\varphi$ constant → $S^\omega \nvdash$ EPS

PS (sequential, $|s| \geq n$) → Gödel's primitive recursion

Gödel's primitive recursion
UR T-defines IPS (unpublished)

How do we simulate a sequential algorithm like IPS with a non-sequential algorithm like UR?
UR T-defines IPS (unpublished)

How do we simulate a sequential algorithm like IPS with a non-sequential algorithm like UR?

**Key idea:** Use UR to compute a sequence of *sequences*, i.e. moves of type $X^\omega$. Entries may be computed independently, but using sequence types allows us to store recursive calls.
How do we simulate a sequential algorithm like IPS with a non-sequential algorithm like UR?

**Key idea:** Use UR to compute a sequence of *sequences*, i.e. moves of type $X^\omega$. Entries may be computed independently, but using sequence types allows us to store recursive calls.

*A fairly complex construction with a long and tedious verification. Won’t go into any more detail!*
Does IPS T-define UR?

Can we simulate a non-sequential algorithm like UI with a sequential algorithm like IPS?
Does IPS T-define UR?

Can we simulate a non-sequential algorithm like UI with a sequential algorithm like IPS?

Unknown!
Modes of bar recursion

**UR** symmetric, implicit

Berardi/Bezem/Coquand functional realizability (1998)

store memory via $X^\omega$

**IPS** sequential, implicit

modified bar recursion

modified realizability (2003)

T defines $\mu_{Spector}$

IPS not S1-S9 definable

**EPS** sequential, $|s| \geq \varphi(\hat{s})$

Spector's bar recursion

Dialectica interpretation (1962)

$\varphi$ constant

$S^\omega \not\models$ EPS

**PS** sequential, $|s| \geq n$

Gödel's primitive recursion
The key difference between UR and IPS...

IPS $\sim$ usual order $<$ on $\mathbb{N}$

UI $\sim$ discrete order $\mathbb{N}$
Further questions

The key difference between UR and IPS...

\[ \text{IPS} \sim \text{usual order} < \text{on } \mathbb{N} \]
\[ \text{UI} \sim \text{discrete order } \mathbb{N} \]

Can we generalise this to associate a form of bar recursion to an arbitrary tree \(<\)?

How is this family of bar recursion functionals related?

New realisers for program extraction?
Direction for future work

Complete interdefinability question for main known modes of bar recursion.

Formulate a uniform framework in which they can be compared, to better understand their behaviour and semantics.

Look at new modes of bar recursion. New realizers for proof interpretations? How do they fit into current picture?

Develop some new results and machinery in theory of higher-type computability.
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References

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