Proof theoretic approaches to rewriting

Thomas Powell

Institut des Hautes Études Scientifiques

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Outline of talk

Part I. Looking to the past

- The computational content of proofs - classical proof theoretic work of the 20th century.
- Link to rewriting: Recursive path orders and fragments of arithmetic.

Part II. Looking towards the future

- Modern proof theory as a tool for establishing complexity bounds on programs.
- Some concrete examples of how this technique could be applied

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multiset path ordering $\Rightarrow$ primitive recursive d. c.

lexicographic path ordering $\Rightarrow$ multiply recursive d. c.

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Techniques such as proof interpretations - which analyse the logical structure of proofs - provide us with a powerful tool for tackling this problem, and have great potential for both:

- Obtaining new complexity results;
- Understanding *why* termination techniques yield the complexity bounds that they do.

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The computational complexity of f_ε is determined by the proof-theoretic complexity of A.

Classic results of 20th century proof theory (Gödel 1958, Parsons 1972):

$$\begin{aligned}
\mathsf{PA} \vdash \forall n \exists m A_0(n,m) \Rightarrow & \exists f \in \mathcal{F}_{\varepsilon_0} \forall n A_0(n, f(n)) \\
& f \textit{ in system } \mathsf{T} \\
& \vdots \\
\mathsf{I}\Sigma_2 \vdash \forall n \exists m A_0(n,m) \Rightarrow & \exists f \in \mathcal{F}_{\omega^\omega} \forall n A_0(n, f(n)) \\
& f \textit{ multiply recursive} \\
\mathsf{I}\Sigma_1 \vdash \forall n \exists m A_0(n,m) \Rightarrow & \exists f \in \mathcal{F}_{\omega} \forall n A_0(n, f(n)) \\
& f \textit{ primitive recursive}
\end{aligned}$$

Application to program complexity (Buchholz 1994):

Step 1. Any finite rewrite system $\mathcal{R}$ reducing under $\prec_{\text{mpo}}$ or $\prec_{\text{lpo}}$ uses only a (finitely branching) approximation $\prec^k_{\text{mpo}}$, $\prec^k_{\text{lpo}}$ of these orderings, where k depends on $\mathcal{R}$.

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Result.

$$\text{multiset path ordering } (\mathsf{I}\Sigma_1) \Rightarrow N(t) \leq f(|t|) \text{ for } f \in \mathcal{F}_\omega$$
$$\text{lexicographic path ordering } (\mathsf{I}\Sigma_2) \Rightarrow N(t) \leq f(|t|) \text{ for } f \in \mathcal{F}_{\omega^\omega}$$

Very rough explanation...

Want to prove $t_1, \ldots, t_n$ well-founded w.r.t. $\prec$ then $(t_1, \ldots, t_n)$ well-founded w.r.t. $\prec^{\text{mul}}/\prec^{\text{lex}}$.

$$(t_1, \ldots, t_{j-1}, t'_1, \ldots, t'_m, t_{j+1}, \ldots, t_n) \prec^{\text{mul}} (t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n)$$

$$(t_1, \ldots, t_{j-1}, t'_j, \underbrace{s_{j+1}, \ldots, s_n}_{\text{'uncontrolled'}}) \prec^{\text{lex}} (t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n)$$

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Need a universal quantification to deal with uncontrolled terms in $\prec^{\text{lex}}$:

$$B(t_{j-1}) :\equiv \forall s_{j+1}, \ldots, s_n[(t_1, \ldots, t_{j-1}, t'_j, s_{j+1}, \ldots, s_n) \text{ w.f.}]$$

Causes a one-step shift up the arithmetic hierarchy!

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- Great potential for new applications in the analysis of a wider range of termination methods, building on substantial amount of work done in last 20 years.
- Can simultaneously work towards new complexity bounds while developing broadly applicable results in proof theory.

The last few decades has seen great advances in proof theoretic techniques and their application in mathematics and computer science.

In particular, meta-theorems of the form

$$\mathcal{T} \vdash A \Rightarrow \exists t \in \mathcal{F}(t \text{ interprets } A)$$

have become increasingly refined and sophisticated, and oriented towards practical as opposed to foundational results.

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QUESTION:

Can we develop new and interesting meta-theorems (along the lines of Buchholz '94, but not necessarily restricted to path orders!) which capture families of termination proofs and can be used to derive corresponding complexity bounds in a uniform way?

Example

$$(t_1, \ldots, t_{j-1}, t'_1, \ldots, t'_m, t_{j+1}, \ldots, t_n) \prec^{\text{mul}} (t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n)$$

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Could be used to derive primitive recursive closure properties e.g. unnested multiple recursion:

$$f(x+1, y+1) = h(x, y, f(x, p(x, y)), f(x+1, y))$$

Typically, path orders developed to incorporate such schemata resort to a complex ordinal analysis to verify complexity bound.

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QUESTION:

By focusing on restricted forms of these principles can we extract useful numerical information from termination proofs which use them?

Example

A form of Ramsey's theorem for pairs has been used to prove program termination via *disjuctively well-founded transition invariants* (Podelski/Rybalchenko 2004): If a program P has a transition invariant of the form

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Current work by Berardi, Oliva and Steila is focusing on the formal analysis of this result and aims towards producing upper bounds on the complexity of programs shown to terminate using this method.

Proof theoretic machinery designed to give computational interpretations to proofs often finds applications in other areas, some quite unexpected!

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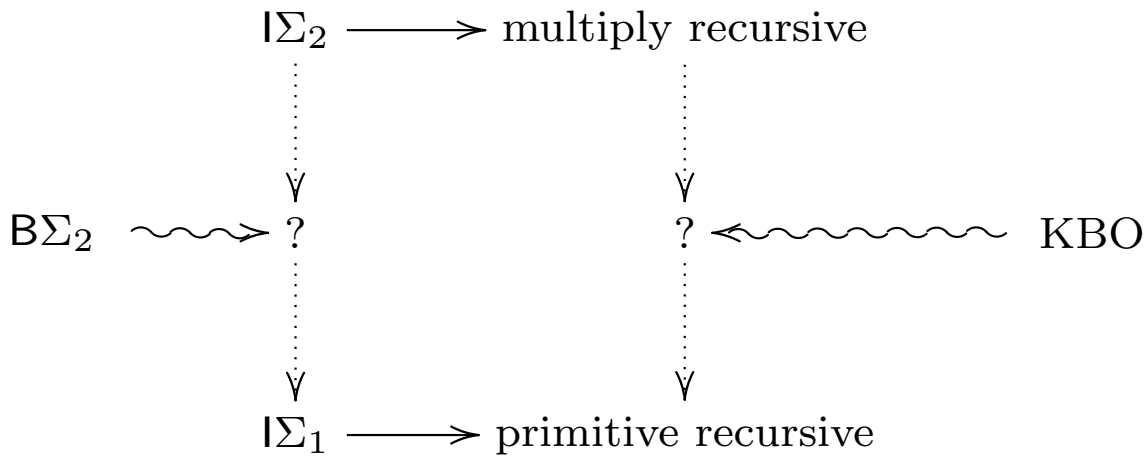
QUESTION

Will developing new theoretical results tailored to analysing termination proofs have an impact in other areas of logic or computer science?

Example

Most existing applications of proof theory focus on obtaining primitive recursive bounds. On the other hand, many termination methods induce multiply recursive complexity.



Can Parsons' result be extended to systems between $\mathsf{I}\Sigma_1$ and $\mathsf{I}\Sigma_2$?

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- In particular this approach has been used to obtain bounds on the derivational complexity of path orderings, but arguably has not been used to anywhere near its full potential.
- Given the amount of progress made in applied proof theory in recent years, there is a great scope for new applications in complexity analysis
- Can be used not only to directly obtain new results, but to provide a proof-theoretic framework with which to better understand existing work