Learning procedures arising from Gödel's functional interpretation

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Mathematical Logic Seminar
LMU Munich, Germany
4 November 2015

Outline

Notation. Σ^0_2 formulas written as $P \equiv \exists x \forall y |P|^x_y$, where $|P|^x_y$ is always decidable.

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A SIMPLE NON-CONSTRUCTIVE THEOREM: $\exists n \forall m (|P|^m \to |P|^n)$

"In a pub there is always a person such that if anyone is drinking, then that person in drinking"

$$\dfrac{\exists k |P|^k \vee \forall k \neg |P|^k \qquad \dfrac{\dfrac{|P|^k \to \forall m (|P|^m \to |P|^k)}{|P|^k \to \exists n \forall m (|P|^m \to |P|^n)}}{\exists k |P|^k \to \exists n \forall m (|P|^m \to |P|^n)} \qquad \dfrac{\dfrac{\forall k \neg |P|^k \to |P|^m \to |P|^0}{\forall k \neg |P|^k \to \forall m (|P|^m \to |P|^0)}}{\forall k \neg |P|^k \to \exists n \forall m (|P|^m \to |P|^n)}}{\exists n \forall m (|P|^m \to |P|^n)}$$

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There is no *effective* way of realizing $\exists n$. So what is the constructive interpretation of the drinkers paradox?

We give three answers...

Method I: Hilbert's ϵ-calculus

Idea: Replace quantifiers by 'magic' ϵ-terms:

$$\exists k A(k) \rightsquigarrow A(\epsilon_k A),$$

and quantifier axioms by critical formulas:

$$A(t) \to A(\epsilon_k A).$$

1. *Translation.* Convert proofs in predicate logic to proofs in epsilon calculus. Instances of quantifier axioms replaced by critical formulas.

2. *Epsilon elimination.* Suppose we only use a finite set of critical formulas. Interpret all ϵ-terms by 0. If we find a mistake i.e. $A(t) \wedge \neg A(0)$, 'learn' from this mistake and update $\epsilon_k A \mapsto t$.

Interpreted proof:

$$
\dfrac{
\exists k |P|^k \vee \forall k \neg |P|^k
\qquad
\dfrac{\dfrac{|P|^k \to \forall m (|P|^m \to |P|^k)}{|P|^k \to \exists n \forall m (|P|^m \to |P|^n)}}{\exists k |P|^k \to \exists n \forall m (|P|^m \to |P|^n)}
\qquad
\dfrac{\dfrac{\dfrac{\neg |P|^m \to |P|^m \to |P|^0}{\forall k \neg |P|^k \to |P|^m \to |P|^0}}{\forall k \neg |P|^k \to \forall m (|P|^m \to |P|^0)}}{\forall k \neg |P|^k \to \exists n \forall m (|P|^m \to |P|^n)}
}{
\exists n \forall m (|P|^m \to |P|^n)
}
$$

Critical formulas:

ϵ-elimination:

Interpreted proof:

$$\dfrac{|P|^{\epsilon_k} \vee \neg|P|^{\epsilon_k} \qquad \dfrac{|P|^{\epsilon_k} \to \forall m(|P|^m \to |P|^{\epsilon_k})}{|P|^{\epsilon_k} \to \exists n \forall m(|P|^m \to |P|^n)} \qquad \dfrac{\dfrac{\dfrac{\neg|P|^m \to |P|^m \to |P|^0}{\neg|P|^{\epsilon_k} \to |P|^m \to |P|^0}}{\neg|P|^{\epsilon_k} \to \forall m(|P|^m \to |P|^0)}}{\neg|P|^{\epsilon_k} \to \exists n \forall m(|P|^m \to |P|^n)}}{\exists n \forall m(|P|^m \to |P|^n)}$$

Critical formulas:

$$|P|^m \to |P|^{\epsilon_k}$$

ϵ-elimination:

Interpreted proof:

$$\frac{|P|^{\epsilon k} \vee \neg|P|^{\epsilon k} \qquad \dfrac{|P|^{\epsilon k} \to |P|^{\epsilon m \epsilon k} \to |P|^{\epsilon k}}{|P|^{\epsilon k} \to \exists n(|P|^{\epsilon m n} \to |P|^{n})} \qquad \dfrac{\dfrac{\neg|P|^{\varepsilon m 0} \to |P|^{\varepsilon m 0} \to |P|^{0}}{\neg|P|^{\epsilon k} \to |P|^{\epsilon m 0} \to |P|^{0}}}{\neg|P|^{\epsilon k} \to \exists n(|P|^{\epsilon m n} \to |P|^{n})}}{\exists n(|P|^{\epsilon m n} \to |P|^{n})}$$

Critical formulas:

$$|P|^{\epsilon_m 0} \to |P|^{\epsilon k}$$

ϵ-elimination:

Interpreted proof:

$$\dfrac{|P|^{\epsilon k} \vee \neg |P|^{\epsilon k} \qquad \dfrac{|P|^{\epsilon k} \to |P|^{\epsilon m \, \epsilon k} \to |P|^{\epsilon k}}{|P|^{\epsilon k} \to (|P|^{\epsilon m \, \epsilon n} \to |P|^{\epsilon n})} \qquad \dfrac{\dfrac{\neg |P|^{\varepsilon m \, 0} \to |P|^{\varepsilon m \, 0} \to |P|^{0}}{\neg |P|^{\epsilon k} \to |P|^{\epsilon m \, 0} \to |P|^{0}}}{\neg |P|^{\epsilon k} \to (|P|^{\epsilon m \, \epsilon n} \to |P|^{\epsilon n})}}{|P|^{\epsilon m \, \epsilon n} \to |P|^{\epsilon n}}$$

Critical formulas:

$$|P|^{\epsilon_m 0} \to |P|^{\epsilon_k}$$
$$(|P|^{\epsilon_m \epsilon_k} \to |P|^{\epsilon_k}) \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})$$
$$(|P|^{\epsilon_m 0} \to |P|^{0}) \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})$$

ϵ-elimination:

Interpreted proof:

$$
\dfrac{|P|^{\epsilon_k} \vee \neg|P|^{\epsilon_k} \qquad \dfrac{|P|^{\epsilon_k} \to |P|^{\epsilon_m \epsilon_k} \to |P|^{\epsilon_k}}{|P|^{\epsilon_k} \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})} \qquad \dfrac{\dfrac{\neg|P|^{\varepsilon_m 0} \to |P|^{\varepsilon_m 0} \to |P|^{0}}{\neg|P|^{\epsilon_k} \to |P|^{\epsilon_m 0} \to |P|^{0}}}{\neg|P|^{\epsilon_k} \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})}}{|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n}}
$$

Critical formulas:

$$
\begin{array}{rl}
|P|^{\epsilon_m 0} \to |P|^{0} & ? \\
(|P|^{\epsilon_m 0} \to |P|^{0}) \to (|P|^{\epsilon_m 0} \to |P|^{0}) & \checkmark \\
(|P|^{\epsilon_m 0} \to |P|^{0}) \to (|P|^{\epsilon_m 0} \to |P|^{0}) & \checkmark
\end{array}
$$

ϵ-elimination:

- Try $\epsilon_k = \epsilon_n = 0$. Works unless $|P|^{\epsilon_m 0} \wedge \neg|P|^{0}$.

Interpreted proof:

$$\dfrac{|P|^{\epsilon_k} \vee \neg|P|^{\epsilon_k} \qquad \dfrac{|P|^{\epsilon_k} \to |P|^{\epsilon_m \epsilon_k} \to |P|^{\epsilon_k}}{|P|^{\epsilon_k} \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})} \qquad \dfrac{\dfrac{\neg|P|^{\varepsilon_m 0} \to |P|^{\varepsilon_m 0} \to |P|^{0}}{\neg|P|^{\epsilon_k} \to |P|^{\epsilon_m 0} \to |P|^{0}}}{\neg|P|^{\epsilon_k} \to (|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n})}}{|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n}}$$

Critical formulas:

$$|P|^{\epsilon_m 0} \to |P|^{\epsilon_m 0} \qquad \checkmark$$

$$(|P|^{\epsilon_m \epsilon_m 0} \to |P|^{\epsilon_m 0}) \to (|P|^{\epsilon_m \epsilon_m 0} \to |P|^{\epsilon_m 0}) \qquad \checkmark$$

$$(|P|^{\epsilon_m 0} \to |P|^{0}) \to (|P|^{\epsilon_m \epsilon_m 0} \to |P|^{\epsilon_m 0}) \qquad \checkmark$$

ϵ-elimination:

- Try $\epsilon_k = \epsilon_n = 0$. Works unless $|P|^{\epsilon_m 0} \wedge \neg|P|^{0}$.
- But now we have a witness for $\exists k |P|^{k}$, so set $\epsilon_k = \epsilon_m = \epsilon_m 0$.

Finitary drinker's paradox I: For an arbitrary ϵ-term $\epsilon_m(\cdot)$ there exists some ϵ_n satisfying

$$|P|^{\epsilon_m \epsilon_n} \to |P|^{\epsilon_n}.$$

This can be computed by the algorithm

- `Set` $\epsilon_n := 0$.
- `Check` $|P|^{\epsilon_m 0} \to |P|^0$. `If true, END.`
- `Else` $\epsilon_n := \epsilon_m 0$.

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The term $\epsilon_m(\cdot)$ represents the *proof theoretic environment*, a measure of how we might *use* the drinkers paradox as a lemma. More specifically, exactly when we need the $\forall$-axiom

$$\exists n \forall m(|P|^m \to |P|^n) \to \exists n(|P|^t \to |P|^n).$$

Method II: Realizability $\sim$ Backtracking Games

Idea: Two stage translation: *Negative translation + Realizability.*

1. Eliminate classical reasoning by applying negative translation.

2. Extract realizing terms for some variant of the BHK interpretation of this formula i.e.

$$(A \to B) \rightsquigarrow \exists f \forall x (x \text{ realizes } A \to f(x) \text{ realizes } B)$$

There is a deep link with certain notions of game semantics. In our example

correctness of realizer $\sim$ $\exists$loise has winning strategy in 1-backtracking game.

Original formula: $\exists n \forall m(|P|^m \to |P|^n)$.

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Negated formula: $(\exists n \forall m[|P|^m \to (|P|^n \to \bot) \to \bot] \to \bot) \to \bot$

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This is interpreted by a term Φ satisfying

$$\forall F (F \text{ rz } \exists n \forall m [|P|^m \to (|P|^n \to \bot) \to \bot] \to \bot) \to \Phi F \text{ rz } \bot)$$
$$\forall F (\forall n, g (g \text{ rz } \forall m [|P|^m \to (|P|^n \to \bot) \to \bot] \to Fng \text{ rz } \bot) \to \Phi F \text{ rz } \bot)$$
$$\forall F (\forall n, g (\forall m, a [|P|^m \to (|P|^n \to a \text{ rz } \bot) \to gma \text{ rz } \bot] \to Fng \text{ rz } \bot) \to \Phi F \text{ rz } \bot)$$

Original formula: $\exists n \forall m (|P|^m \to |P|^n)$.

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Omitting details, can solve this by

$$\Phi F := F \left(0, \lambda m, a \, . \, \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases} \right)$$

Assuming

$$\forall n, g(\forall m, a[|P|^m \to (|P|^n \to a \text{ rz } \bot) \to gma \text{ rz } \bot] \to Fng \text{ rz } \bot)$$

must show that

$$F\left(0, \lambda m, a \;.\; \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right) \text{ rz } \bot.$$

Assuming

$$\forall m, a[|P|^m \to (|P|^0 \to a \text{ rz } \bot) \to g_\exists ma \text{ rz } \bot] \to F0g_\exists \text{ rz } \bot$$

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- $\exists$ ∃loise tries to realize $\exists n \forall m(|P|^m \to |P|^n)$ by setting $n := 0$.

Assuming

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- $\exists$ $\exists$loise tries to realize $\exists n \forall m(|P|^m \to |P|^n)$ by setting $n := 0$.
- $\forall$ $\forall$belard refutes $\forall m(|P|^m \to |P|^0)$ by selecting some counterexample m.

Assuming

$$|P|^m \to (|P|^0 \to a \text{ rz } \bot) \to a \text{ rz } \bot$$

must show that

$$F\left(0, \lambda m, a\ .\ \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right) \text{ rz } \bot.$$

- ∃ ∃loise tries to realize $\exists n \forall m(|P|^m \to |P|^n)$ by setting $n := 0$.
- ∀ ∀belard refutes $\forall m(|P|^m \to |P|^0)$ by selecting some counterexample m.
- ∃ If $|P|^m \to |P|^0$ then $g_\exists ma = a$ rz $\bot$, so ∃loise wins. If not, she changes her mind, stealing ∀belard$'s$ witness $n := m$.

Assuming

$$\forall m', a'[|P|^{m'} \to (|P|^{m} \to a' \text{ rz } \bot) \to a' \text{ rz } \bot] \to Fm(\lambda m'.a'.a') \text{ rz } \bot$$

must show that

$$F\left(0, \lambda m, a \;.\; \begin{cases} a & \text{if } |P|^{m} \to |P|^{0} \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right) \text{ rz } \bot.$$

- $\exists$ $\exists$loise tries to realize $\exists n \forall m(|P|^m \to |P|^n)$ by setting $n := 0$.
- $\forall$ $\forall$belard refutes $\forall m(|P|^m \to |P|^0)$ by selecting some counterexample m.
- $\exists$ If $|P|^m \to |P|^0$ then $g_\exists ma = a$ rz $\bot$, so $\exists$loise wins. If not, she changes her mind, stealing $\forall$belard$'s$ witness $n := m$.

Assuming

$$|P|^{m'} \to (|P|^m \to a' \text{ rz } \bot) \to a' \text{ rz } \bot$$

must show that

$$F\left(0, \lambda m, a \,.\, \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right) \text{ rz } \bot.$$

- $\exists$ $\exists$loise tries to realize $\exists n \forall m(|P|^m \to |P|^n)$ by setting $n := 0$.
- $\forall$ $\forall$belard refutes $\forall m(|P|^m \to |P|^0)$ by selecting some counterexample m.
- $\exists$ If $|P|^m \to |P|^0$ then $g_\exists ma = a$ rz $\bot$, so $\exists$loise wins. If not, she changes her mind, stealing $\forall$belard$'s$ witness $n := m$.
- $\forall$ $\forall$belard attempts to refute $\forall m'(|P|^{m'} \to |P|^m)$ by selecting some counterexample m'. But since $|P|^m$ holds, we must have a' rz $\bot$ and hence $\exists$loise wins.

FINITARY DRINKER'S PARADOX II: For an arbitrary functional F satisfying

$$\forall n, g(\forall m, a[|P|^m \to (|P|^n \to a \text{ rz } \bot) \to gma \text{ rz } \bot] \to Fng \text{ rz } \bot,$$

we have ΦF rz $\bot$, where

$$\Phi F := F\left(0, \lambda m, a\ .\ \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right)$$

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we have ΦF rz $\bot$, where

$$\Phi F := F\left(0, \lambda m, a\ .\ \begin{cases} a & \text{if } |P|^m \to |P|^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right)$$

The proof theoretic environment is represented by function abstractions $\lambda m, a$ and $\lambda m', a'$ representing ∀belard's attempts to refute ∃loise's strategy. As with the epsilon calculus, these essentially represent realizers how we want to 'use' universal quantifiers of drinkers paradox in a bigger proof.

Method III: Gödel's functional interpretation

Idea: Another two stage translation: *Negative translation + Dialectica interpretation.*

1. Eliminate classical reasoning by applying negative translation (can be more flexible here e.g. ignore atomic formulas).

2. Extract realizing terms for *Dialectica* interpretation of this formula. More complex than realizability - need to fully Skolemize implication:

$$\begin{aligned}(A \to B) &\rightsquigarrow (\exists x \forall y |A|^x_y \to \exists u \forall v |B|^u_v) \\ &\rightsquigarrow \forall x \exists u \forall v \exists y (|A|^x_y \to |B|^u_v) \\ &\rightsquigarrow \exists U, Y \forall x, v (|A|^x_{Yxv} \to |B|^{Ux}_v)\end{aligned}$$

Contraction problem: Interpretation of classical reasoning requires us to test atomic formulas and case definitions.

$$\dfrac{\exists k|P|^k \vee \forall k \neg |P|^k \qquad \begin{array}{c}[\exists k|P|^k] \\ \vdots \\ \exists n \forall m (|P|^m \to |P|^n)\end{array} \qquad \begin{array}{c}[\forall k \neg |P|^k] \\ \vdots \\ \exists n \forall m (|P|^m \to |P|^n)\end{array}}{\exists n \forall m (|P|^m \to |P|^n)}$$

$$\dfrac{\exists k|P|^k \vee \forall k\neg|P|^k \qquad \begin{array}{c}[\exists k|P|^k]\\ \vdots \\ \neg\neg\exists n\forall m(|P|^m \to |P|^n)\end{array} \qquad \begin{array}{c}[\forall k\neg|P|^k]\\ \vdots \\ \neg\neg\exists n\forall m(|P|^m \to |P|^n)\end{array}}{\neg\neg\exists n\forall m(|P|^m \to |P|^n)}$$

$$\dfrac{\exists k|P|^k \vee \forall k\neg|P|^k \qquad \begin{matrix}[|P|^k] \\ \vdots \\ |P|^{gk} \to |P|^k\end{matrix} \qquad \begin{matrix}[\forall k\neg|P|^k] \\ \vdots \\ \neg\neg\exists n\forall m(|P|^m \to |P|^n)\end{matrix}}{\neg\neg\exists n\forall m(|P|^m \to |P|^n)}$$

First branch:

$$\exists k|P|^k \to \neg\neg\exists n\forall m(|P|^m \to |P|^n)$$
$$\rightsquigarrow \exists k|P|^k \to \forall g^{\mathbb{N}\to\mathbb{N}}\exists n(|P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g,k\exists n(|P|^k \to |P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g,k(|P|^k \to |P|^{gk} \to |P|^k)$$

$$\dfrac{\exists k|P|^k \vee \forall k\neg|P|^k \qquad \begin{matrix}[|P|^k]\\ \vdots \\ |P|^{gk} \to |P|^k\end{matrix} \qquad \begin{matrix}[\neg|P|^{g0}]\\ \vdots \\ |P|^{g0} \to |P|^0\end{matrix}}{\neg\neg\exists n\forall m(|P|^m \to |P|^n)}$$

First branch:

$$\exists k|P|^k \to \neg\neg\exists n\forall m(|P|^m \to |P|^n)$$
$$\rightsquigarrow \exists k|P|^k \to \forall g^{\mathbb{N}\to\mathbb{N}}\exists n(|P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g, k\exists n(|P|^k \to |P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g, k(|P|^k \to |P|^{gk} \to |P|^k)$$

Second branch:

$$\forall k\neg|P|^k \to \neg\neg\exists n\forall m(|P|^m \to |P|^n)$$
$$\rightsquigarrow \forall k\neg|P|^k \to \forall g\exists n(|P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g\exists k, n(\neg|P|^k \to |P|^{gn} \to |P|^n)$$
$$\rightsquigarrow \forall g(\neg|P|^{g0} \to |P|^{g0} \to |P|^0)$$

$$\dfrac{|P|^{g0} \vee \neg |P|^{g0} \qquad \begin{matrix}[|P|^{g0}] \\ \vdots \\ |P|^{g(g0)} \to |P|^{g0}\end{matrix} \qquad \begin{matrix}[\neg |P|^{g0}] \\ \vdots \\ |P|^{g0} \to |P|^{0}\end{matrix}}{\neg\neg \exists n \forall m (|P|^m \to |P|^n)}$$

First branch:

$$\begin{aligned} & \exists k |P|^k \to \neg\neg \exists n \forall m (|P|^m \to |P|^n) \\ \rightsquigarrow & \exists k |P|^k \to \forall g^{\mathbb{N}\to\mathbb{N}} \exists n (|P|^{gn} \to |P|^n) \\ \rightsquigarrow & \forall g, k \exists n (|P|^k \to |P|^{gn} \to |P|^n) \\ \rightsquigarrow & \forall g, k (|P|^k \to |P|^{gk} \to |P|^k) \end{aligned}$$

Second branch:

$$\begin{aligned} & \forall k \neg |P|^k \to \neg\neg \exists n \forall m (|P|^m \to |P|^n) \\ \rightsquigarrow & \forall k \neg |P|^k \to \forall g \exists n (|P|^{gn} \to |P|^n) \\ \rightsquigarrow & \forall g \exists k, n (\neg |P|^k \to |P|^{gn} \to |P|^n) \\ \rightsquigarrow & \forall g (\neg |P|^{g0} \to |P|^{g0} \to |P|^0) \end{aligned}$$

$$\dfrac{|P|^{g0} \vee \neg |P|^{g0} \qquad \begin{matrix}[|P|^{g0}] \\ \vdots \\ |P|^{g(g0)} \to |P|^{g0}\end{matrix} \qquad \begin{matrix}[\neg |P|^{g0}] \\ \vdots \\ |P|^{g0} \to |P|^{0}\end{matrix}}{|P|^{g(Ng)} \to |P|^{Ng}}$$

Solved by

$$Ng := \begin{cases} 0 & \text{if } \neg |P|^{g0} \\ g0 & \text{if } |P|^{g0} \end{cases}$$

FINITARY DRINKER'S PARADOX III: For an arbitrary function $g \colon \mathbb{N} \to \mathbb{N}$ there exists some $N \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ satisfying

$$|P|^{g(Ng)} \to |P|^{Ng}.$$

This can be defined as

$$Ng := \begin{cases} 0 & \text{if } \neg |P|^{g0} \\ g0 & \text{otherwise.} \end{cases}$$

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This can be defined as

$$Ng := \begin{cases} 0 & \text{if } \neg|P|^{g0} \\ g0 & \text{otherwise.} \end{cases}$$

The proof theoretic environment is represented by an explicit 'counterexample function' g. Any instance of the drinkers paradox in a bigger proof will involve a concrete instantiation g_v of g:

$$\exists n \forall m (|P|^m \to |P|^n) \to B$$

$$\exists U, g \forall n, v((|P|^{gn} \to |P|^n) \to |B|_v^{Un})$$

and hence $\forall v |B|_v^{U(Ng)v}$ holds.

General finitary drinker's paradox: There exists an approximate witness $\mathcal{N}$ to $\exists n \forall m(|P|^n \to |P|^m)$, that works relative to any proof theoretic environment $\mathcal{M}$ representing $\forall m$.

Technique	$\mathcal{N}$	$\mathcal{M}$
ϵ-calculus	$\left[\begin{array}{l} \epsilon_n := 0 \\ \texttt{Check } \lvert\texttt{P}\rvert^{\epsilon_m 0} \to \lvert\texttt{P}\rvert^{0}\texttt{. If true, END.} \\ \texttt{Else } \epsilon_n := \epsilon_m 0 \end{array}\right.$	$\epsilon_m(\cdot)$
Realizability	$\Phi F := F\left(0, \lambda m, a \,.\, \begin{cases} a & \text{if } \lvert P\rvert^m \to \lvert P\rvert^0 \\ Fm(\lambda m', a'.a') & \text{otherwise} \end{cases}\right)$	$\lambda m, a$
Dialectica	$Ng := \begin{cases} 0 & \text{if } \neg\lvert P\rvert^{g0} \\ g0 & \text{otherwise} \end{cases}$	$g\colon \mathbb{N} \to \mathbb{N}$

They all carry out learning, but in completely different frameworks!

Outline

ND INTERPRETATION SOUNDNESS THEOREM. If $P_1, \ldots, P_n \vdash Q$ in $\mathcal{T}$ then

$$|P_1^N|^{x_1}_{F_1 \underline{x} v}, \ldots, |P_n^N|^{x_n}_{F_n \underline{x} v} \vdash |Q^N|^{f \underline{x}}_{v}$$

where f, $F_1, \ldots, F_n$ are closed terms of T which can be extracted from the derivation of Q.

Note. We will write this is proof tree notation:

$$\begin{array}{c}[P_1], \ldots, [P_n] \\ \vdots \\ Q\end{array}$$

and

$$\begin{array}{c}[|P_1^N|^{x_1}_{F_1 \underline{x} v}], \ldots, [|P_n^N|^{x_n}_{F_n \underline{x} v}] \\ \vdots \\ [|Q^N|^{f \underline{x}}_{v}]\end{array}$$

$$\dfrac{\exists x|A|^x \vee \forall x \neg |A|^x \qquad \begin{matrix} [\exists x|A|^x] \\ \vdots \\ \forall v \exists u|B|_v^u \end{matrix} \qquad \begin{matrix} [\forall x \neg |A|^x] \\ \vdots \\ \forall v \exists u|B|_v^u \end{matrix}}{\forall v \exists u|B|_v^u}$$

$$\dfrac{\exists x|A|^x \vee \forall x \neg |A|^x \qquad \begin{matrix}[|A|^{x_1}], \ldots, [|A|^{x_i}] \\ \vdots \\ |B^N|_v^{F_1 x_1 \ldots x_n v}\end{matrix} \qquad \begin{matrix}[\forall x \neg |A|^x] \\ \vdots \\ \forall v \exists u |B|_v^u\end{matrix}}{\forall v \exists u |B|_v^u}$$

$$
\dfrac{\exists x|A|^x \vee \forall x\neg|A|^x \qquad \begin{matrix}[|A|^{x_1}],\ldots,[|A|^{x_i}] \\ \vdots \\ |B^N|_v^{F_1 x_1 \ldots x_n v}\end{matrix} \qquad \begin{matrix}[\neg|A|^{X_1 v}],\ldots,[\neg|A|^{X_j v}] \\ \vdots \\ |B^N|_v^{U_1 v}\end{matrix}}{\forall v\exists u|B|_v^u}
$$

$$\dfrac{|A|^{Tv} \vee \neg|A|^{Tv} \qquad \begin{matrix}[|A|^{Tv}],\ldots,[|A|^{Tv}] \\ \vdots \\ |B^N|_v^{F_1(\underline{Tv})v}\end{matrix} \qquad \begin{matrix}[\neg|A|^{X_1 v}],\ldots,[\neg|A|^{X_j v}] \\ \vdots \\ |B^N|_v^{U_1 v}\end{matrix}}{|B^N|_v^{Uv}}$$

where

$$Tv := \begin{cases} X_1 v & \text{if } |A|^{X_1 v} \\ \ldots & \ldots \\ X_{j-1}v & \text{if } |A|^{X_{j-1}v} \\ X_j v & \text{otherwise} \end{cases} \quad \text{and} \quad Uv := \begin{cases} F_1(\underline{X_1 v})v & \text{if } |A|^{X_1 v} \\ \ldots & \ldots \\ F_{j-1}(\underline{X_j v})v & \text{if } |A|^{X_j v} \\ U_1 v & \text{otherwise} \end{cases}$$

Only need finitely many witnesses $X_1 v, \ldots, X_j v$ for $\forall x \neg |A|^x$.

$$\dfrac{|A|^{Tv} \vee \neg|A|^{Tv} \qquad \begin{array}{c}[|A|^{Tv}],\ldots,[|A|^{Tv}]\\ \vdots \\ |B^N|_v^{F_1(\underline{Tv})v}\end{array} \qquad \begin{array}{c}[\neg|A|^{X_1 v}],\ldots,[\neg|A|^{X_j v}] \\ \vdots \\ |B^N|_v^{U_1 v}\end{array}}{|B^N|_v^{Uv}}$$

Any instance of Σ_1-LEM initiates a learning algorithm. However, the parameters $X_1, \ldots, X_j$, F_1 and U_1 may also contain nested learning algorithms:

Run $X_1 v$. [Test $|A|_v^{X_1}$.] If true, run $F_1(X_1 v)v$.

$\vdots$

Else run $X_j v$. [Test $|A|_v^{X_j}$.] If true, run $F_1(X_j v)v$.

Else run $U_1 v$.

THEOREM. For any function $f\colon \mathbb{N} \to \mathbb{N}$ there exists some k such that $\forall i \,.\, f(k) \leq f(i)$.

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$$\dfrac{f(N) > f(gN) \vee f(N) \leq f(gN) \qquad \begin{matrix}[f(N) > f(gN)] \\ \vdots \ \text{(IND)} \\ f(K_{g,gN}) \leq f(g(K_{g,gN}))\end{matrix} \qquad \begin{matrix}[f(N) \leq f(gN)] \\ \vdots \\ f(N) \leq f(gN)\end{matrix}}{\forall g \exists k (f(k) \leq f(gk))}$$

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For

$$K_{g,N} := \begin{cases} N & \text{if } f(N) \leq f(gN) \\ \boxed{K_{g,gN}} & \text{if } f(N) > f(gN) \end{cases}$$

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For

$$K_{g,N} := \begin{cases} N & \text{if } f(N) \leq f(gN) \\ gN & \text{if } f(gN) \leq f(g^{(2)}N) \\ \boxed{K_{g,g^{(2)}N}} & \text{if } f(gN) > f(g^{(2)}N) \end{cases}$$

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$$K_{g,N} := \begin{cases} N & \text{if } f(N) \leq f(gN) \\ gN & \text{if } f(gN) \leq f(g^{(2)}N) \\ g^{(2)}N & \text{if } f(g^{(2)}N) \leq f(g^{(3)}N) \\ \vdots & \vdots \end{cases}$$

For any $g\colon \mathbb{N} \to \mathbb{N}$ the formula $f(k) \leq f(gk)$ is realized by $K_{g,0}$.

Outline

By Σ_1-LEM^- we mean law of excluded-middle for Σ^0_1 formulas with only number parameters:

$$\forall n \exists x \forall y (|P_n|^x \vee \neg |P_n|^y).$$

Much work has been done recently on extracting programs from proofs relative to non-computable Skolem functions f_P for Σ_1-LEM^- i.e. satisfying

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In particular this forms the backbone of recent work by Aschieri, Berardi et al. on 'interactive learning-based realizability':

1. If $\mathsf{HA} + \Sigma_1$-$\mathsf{LEM}^- \vdash A$ then there is a realizing term t for A that is primitive recursive in Skolem functions f_P.

2. When A is e.g. Π_2 the term t only requires *approximations* to f_P. Therefore an approximation $t_0 \sqsubset t_1 \sqsubset \ldots \sqsubset t$ to t can be built by 'learning' a finite amount of information about f_P.

Formally, the existence of Skolem functions f_P requires Σ_1-comprehension. But let's try to prove their existence using induction!

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Suppose we already have some finite information about f, which we encode as a partial function u satisfying

$$\mathrm{App}(u) :\equiv \forall n \in \mathrm{dom}(u) \;.\; |P_n|^{un}$$

We want to prove 'extension lemma'

$$B_u :\equiv \forall u(\mathrm{App}(u) \to \exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^{y})).$$

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Let's now use law of excluded middle on the formula

$$|A|^{n,y} :\equiv \exists n, y(n \notin \mathrm{dom}(u) \wedge |P_n|^y)$$

$$\dfrac{\exists n, y|A|^{n,y} \vee \forall n, y \neg |A|^{n,y} \qquad \begin{array}{c}[\exists n, y|A|^{n,y}] \quad [\forall n, y(B_{u \oplus (n,y)})] \\ \vdots \ (\mathsf{IND}) \\ \exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^y)\end{array} \qquad \begin{array}{c}[\forall n, y \neg |A|^{n,y}] \\ \vdots \ (f := \hat{u}) \\ \exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^y\end{array}}{B_u}$$

The Dialectica interpretation of B_u is given by

$$\exists F \forall u, \varphi, \phi (\mathrm{App}(u) \to |P_{\varphi F_u^{\varphi,\phi}}|^{F_u^{\varphi,\phi}(\varphi F_u^{\varphi,\phi})} \vee \neg |P_{\varphi F_u^{\varphi,\phi}}|^{\phi F_u^{\varphi,\phi}})$$

Let's interpret the tree (ignoring the decidable premise $\mathrm{App}(u)$ and parameters φ, ϕ):

$$\dfrac{\vdots \qquad \dfrac{\dfrac{[\exists n, y|A|^{n,y}] \quad [\forall n, y(B_{u\oplus(n,y)})]}{\vdots\ (\mathsf{IND})}}{\exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^y)} \qquad \dfrac{\dfrac{[\forall n, y \neg |A|^{n,y}]}{\vdots\ (f := \hat{u})}}{\exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^y)}}{\exists f \forall n, y(|P_n|^{fn} \vee \neg |P_n|^y)}$$

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$$\dfrac{\vdots \qquad \dfrac{\begin{matrix}[|A|^{n,y}] \quad [B_{u\oplus(n,y)}] \\ \vdots\ \text{(IND)}\end{matrix}}{|P_{\varphi F_{u\oplus(n,y)}}|^{F_{u\oplus(n,y)}(\varphi F_{u\oplus(n,y)})} \vee \neg |P_{\varphi F_{u\oplus(n,y)}}|^{\phi F_{u\oplus(n,y)}}} \qquad \dfrac{\begin{matrix}[\neg |A|^{\varphi \hat{u}, \phi \hat{u}}] \\ \vdots\end{matrix}}{|P_{\varphi \hat{u}}|^{\hat{u}(\varphi \hat{u})} \vee \neg |P_{\varphi \hat{u}}|^{\phi \hat{u}}}}{\forall \varphi, \phi \exists f (|P_{\varphi f}|^{f(\varphi f)} \vee \neg |P_{\varphi f}|^{\phi f})}$$

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Let's interpret the tree (ignoring the decidable premise $\mathrm{App}(u)$ and parameters φ, ϕ):

$$[|A|^{\varphi\hat{u},\phi\hat{u}}] \qquad [B_{\underbrace{u \oplus (\varphi\hat{u},\phi\hat{u})}_{v_u}}] \qquad [\neg|A|^{\varphi\hat{u},\phi\hat{u}}]$$

$$\vdots \ (\mathsf{IND}) \qquad\qquad \vdots$$

$$\frac{|A|^{\varphi\hat{u},\phi\hat{u}} \vee \neg|A|^{\varphi\hat{u},\phi\hat{u}} \qquad |P_{\varphi F_{v_u}}|^{F_{v_u}(\varphi F_{v_u})} \vee \neg|P_{\varphi F_{v_u}}|^{\phi F_{v_u}} \qquad |P_{\varphi\hat{u}}|^{\hat{u}(\varphi\hat{u})} \vee \neg|P_{\varphi\hat{u}}|^{\phi\hat{u}}}{|P_{\varphi F_u}|^{F_u(\varphi F_u)} \vee \neg|P_{\varphi F_u}|^{\phi F_u}}$$

where

$$F_u^{\varphi,\phi} := \begin{cases} \hat{u} & \text{if } \neg|A|^{\varphi\hat{u},\phi\hat{u}} \\ F_{u\oplus(\varphi\hat{u},\phi\hat{u})}^{\varphi,\phi} & \text{otherwise.} \end{cases}$$

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where

$$F_u^{\varphi,\phi} := \begin{cases} \hat{u} & \text{if } \varphi\hat{u} \in \mathrm{dom}(u) \vee \neg|P_{\varphi\hat{u}}|^{\phi\hat{u}} \\ F_{u\oplus(\varphi\hat{u},\phi\hat{u})}^{\varphi,\phi} & \text{otherwise.} \end{cases}$$

Start with $u_0 := \emptyset$ and let $n_0 := \varphi\hat{u}_0$:

$$\hat{u}_0 = 0, 0, 0, \ldots$$

If $n_0 \in \emptyset$ or $\neg|P_{n_0}|^{q\hat{u}_0}$ then we're done. Otherwise update as $u_1 := (n_0, q\hat{u}_0)$:

$$\hat{u}_1 = 0, 0, \ldots, 0, \underbrace{q\hat{u}_0}_{n_0}, 0, \ldots$$

If $n_1 := \varphi\hat{u}_1 \in \{n_0\}$ or $\neg|P_{n_1}|^{q\hat{u}_1}$ we're done. Otherwise update as $u_2 := (n_0, a_0) \oplus (n_1, q\hat{u}_1)$:

$$\hat{u}_2 = 0, 0, \ldots, 0, \underbrace{q\hat{u}_0}_{n_0}, 0, \ldots, 0, \underbrace{q\hat{u}_1}_{n_1}, 0, \ldots$$

If $n_2 := \varphi\hat{u}_2 \in \{n_0, n_1\}$ or $\neg|P_{n_2}|^{q\hat{u}_2}$ we're done. Otherwise update again...

$$\hat{u}_3 = 0, 0, \ldots, 0, \underbrace{q\hat{u}_2}_{n_2}, 0, \ldots, 0, \underbrace{q\hat{u}_0}_{n_0}, 0, \ldots, 0, \underbrace{q\hat{u}_1}_{n_1}, 0, \ldots$$

$$\vdots$$

Theorem. Define the sequence of partial functions u_i by

$$u_0 := \emptyset \quad \text{and} \quad u_{i+1} := \begin{cases} u_i & \text{if } \varphi\hat{u}_i \in \mathrm{dom}(u_i) \vee \neg|P_{\varphi\hat{u}_i}|^{\phi\hat{u}_i} \\ u_i \oplus (\varphi\hat{u}_i, \phi\hat{u}_i) & \text{otherwise.} \end{cases}$$

Then $F^{\varphi,\phi} := \hat{v}$ realizes the Dialectica interpretation of arithmetical comprehension, where v is the limit of the (u_i).

Idea. Start with an empty approximation $u_0 = \emptyset$ to f_P and successively build better approximations $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset \ldots$ until we reach one which is sufficient.

At each stage, either u_i is sufficient, or $|P_{\varphi\hat{u}_i}|^{\phi\hat{u}_i}$ holds, so we can improve the current approximation.

At some point we reach M such that either $\varphi\hat{u}_M \notin \mathrm{dom}(u_M)$ and

$$\neg|P_{\varphi\hat{u}_M}|^{\phi\hat{u}_M},$$

or $\varphi\hat{u}_M \in \mathrm{dom}(u_M)$ and thus

$$|P_{\varphi\hat{u}_M}|^{u_M(\varphi\hat{u}_M)}.$$

Why does this learning process terminate, and what sort of recursion is going on?

Suppose that $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset \ldots$ does not converge to a limit. Take the *domain-theoretic* of the (u_i):

$$w(n) := \begin{cases} u_i(n) & \text{if } n \in \mathrm{dom}(u_i) \text{ for some } i \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now suppose that φ is continuous, so in particular depends only on a finite amount of information about $\hat{w}$:

$$\exists N \forall v (\forall j < N \, . \, \hat{w}(j) = v(j) \to \varphi \hat{w} = \varphi v).$$

Let I be the least index such that $\forall j < \max\{N, \varphi \hat{w} + 1\} \, . \, u_I(j) = w$. We must have $\varphi \hat{u}_I \notin \mathrm{dom}(u_I)$, and since $\varphi \hat{u}_I = \varphi \hat{w}$, $u_I \notin \mathrm{dom}(w)$. But since $u_{I+1} = u_I \oplus (\varphi \hat{u}_i, \phi \hat{u}_i)$, this contradicts construction of w.

Skolem functions $\rightsquigarrow$ continuous, well-founded backward recursion over $u \sqsubset v$

How does this compare to the traditional solution using Spector's bar recursion? Define

$$\mathsf{BR}_s^{\varphi,\phi} := \begin{cases} \hat{s} & \text{if } \varphi\hat{s} < \mathrm{len}(s) \\ \varepsilon_{\mathrm{len}(s)}(\lambda x \,.\, \phi(\mathsf{BR}_{s*x})) & \text{otherwise} \end{cases}$$

where

$$\varepsilon_n(p) := \begin{cases} 0 & \text{if } \neg|P_n|^{p0} \\ p0 & \text{otherwise.} \end{cases}$$

Then $G^{\varphi,\phi} := \mathsf{BR}_{[]}^{\varphi,\phi}$ satisfies

$$|P_{\varphi G}|^{G(\varphi F)} \vee \neg|P_{\varphi G}|^{\phi G}$$

How does this compare to the traditional solution using Spector's bar recursion? Define

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$$\varepsilon_n(p) := \begin{cases} 0 & \text{if } \neg|P_n|^{p0} \\ p0 & \text{otherwise.} \end{cases}$$

Then $G^{\varphi,\phi} := \mathsf{BR}_{[]}^{\varphi,\phi}$ satisfies

$$|P_{\varphi G}|^{G(\varphi F)} \vee \neg|P_{\varphi G}|^{\phi G}$$

How does this compare to the traditional solution using Spector's bar recursion? Define

$$\mathsf{BR}^{\varphi,\phi}_s := \begin{cases} \hat{s} & \text{if } \varphi\hat{s} < \text{len}(s) \\ \mathsf{BR}_{s*0} & \text{if } \neg|P_{\text{len}(s)}|^{\phi(\mathsf{BR}(s*0))} \\ \mathsf{BR}_{s*\phi(\mathsf{BR}(s*0))} & \text{otherwise} \end{cases}$$

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$$\bar{\mathsf{BR}}_u^{\varphi,\phi} := \begin{cases} \hat{u} & \text{if } \varphi\hat{u} \in \mathrm{dom}(u) \\ \bar{\mathsf{BR}}_{u\oplus(\varphi\hat{u},0)} & \text{if } \neg|P_{\mathrm{len}(s)}|^{\phi(\bar{\mathsf{BR}}(u\oplus(\varphi\hat{u},0)))} \\ \bar{\mathsf{BR}}_{u\oplus(\varphi\hat{u},\phi(\mathsf{BR}(u\oplus(\varphi\hat{u},0)))} & \text{otherwise} \end{cases}$$

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Since $\varphi(\widehat{u \oplus (\varphi\hat{u}, 0)}) = \varphi\hat{u} \in \mathrm{dom}(u \oplus (\varphi\hat{u}, 0))$ then

$$\bar{\mathsf{BR}}_{u\oplus(\varphi\hat{u},0)} = \hat{u},$$

hence

$$\bar{\mathsf{BR}}^{\varphi,\phi}_u := \begin{cases} \hat{u} & \text{if } \varphi\hat{u} \in \mathrm{dom}(u) \\ \hat{u} & \text{if } \neg|P_{\mathrm{len}(s)}|^{\phi\hat{u}} \\ \bar{\mathsf{BR}}_{u\oplus(\varphi\hat{u},\phi\hat{u})} & \text{otherwise} \end{cases}$$

CLAIM:

Bar recursion over partial functions $\approx$ learning

Spector's bar recursion over sequences $\approx\approx\approx$ 'forgetful' learning

Start with $s_0 := []$:

$$\hat{s}_0 = 0, 0, 0, \ldots$$

Search for the least $n_0 \leq \varphi\hat{s}_0$ such that $\neg|P_{n_0}|^{q\hat{s}_0}$ otherwise we're done. Else, update as $s_1 := [0, 0, \ldots, q(\hat{s}_0)]$:

$$\hat{s}_1 = 0, 0, \ldots, 0, \underbrace{q\hat{s}_0}_{n_0}, 0, \ldots$$

Search for the least $n_1 \leq \max\{n_0, \varphi\hat{s}_1\}$ with $n_1 \leq n_0$ satisfying $\neg|P_{n_1}|^{q\hat{s}_1}$. If $n_1 > n_0$ set $s_2 := [0, 0, \ldots, 0, q\hat{s}_0, 0, \ldots, 0, q\hat{s}_1]$:

$$\hat{s}_2 = 0, 0, \ldots, 0, \underbrace{q\hat{s}_0}_{n_0}, 0, \ldots, 0, \underbrace{q\hat{s}_1}_{n_1}, 0, \ldots$$

else if $n_1 < n_0$ set $s_2 := [0, 0, \ldots, q\hat{s}_1]$:

$$\hat{s}_2 = 0, 0, \ldots, 0, \underbrace{q\hat{s}_1}_{n_1}, 0, \ldots$$

The witness $q\hat{s}_0$ for $\exists x|P_{n_0}|^x$ is erased!

Outline

1. Explain what's happening underneath

The learning procedures associated with Σ_1-LEM essentially a form of *update recursion*:

$$\forall u(\forall n \notin \mathrm{dom}(u), y\ B(u \oplus (n, y)) \to B(u)) \to B(u)$$

for $B(u)$ open. This was studied by (Berger 2004), although in the context of realizability.

What happens in the case of the Dialectica intepretation?

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What happens in the case of the Dialectica intepretation?

update induction	$\rightsquigarrow$	update recursion	(Berger 2004)
$\Downarrow$		$\Downarrow$	
Σ_1-LEM$^-$	$\rightsquigarrow$	BBC functional	(Berardi et al. 1998)
		$\downarrow$	
		winning strategy for $\exists$loise	(Berardi et al. 1998)

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What happens in the case of the Dialectica intepretation?

update induction	$\rightsquigarrow$	update 'bar recursion'	???
$\Downarrow$		$\Downarrow$	
Σ_1-LEM$^-$	$\rightsquigarrow$	symmetric bar recursion	(Oliva/P. 2015)
		$\downarrow$	
		learning procedure	today

2. Use the Dialectica interpretation together with learning realizers to give new constructive interpretations of theorems

THEOREM. In PA we can derive the fact that there is no injection $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$.

$$\forall H^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}[\Sigma^1_1\text{-}\mathsf{LEM}^-(P_F) \to \exists \alpha^{\mathbb{N}\to\mathbb{N}}, \beta^{\mathbb{N}\to\mathbb{N}}, i^{\mathbb{N}}(\alpha(i) \neq \beta(i) \wedge H\alpha = H\beta)].$$

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PROOF. Take some arbitrary $H\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. Then each n is either in, or not in, the range of H:

$$\forall n\ .\ \exists \alpha(H(\alpha) = n) \vee \forall \beta(H(\beta) \neq n)$$

There is an associated Skolem function f_H satisfying

$$\forall n, \beta^1(H(f_H n) = n \vee H(\beta) \neq n).$$

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There is an associated Skolem function f_H satisfying

$$\forall n, \beta^1(H(f_H n) = n \vee H(\beta) \neq n).$$

Define $\alpha_H := \lambda n\ .(f_H n)(n) + 1$, $i_H := H(\alpha_H)$ and $\beta_H := f_H i_H$.

- Since $H(\alpha_H) = i_H$, we must have $H(\beta_H) = H(f_H i_H)) = i_H = H(\alpha_H)$.
- $\alpha(i_H) = (f_H i_H)(i_H) + 1 = \beta_H(i_H) + 1 \neq \beta_H(i_H)$.

A constructive technique for finding α, β and i can be formally extracted, which uses the following learning procedure.

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Define the sequence of functions $\gamma_i \colon \mathbb{N} \to \mathbb{N}$ together with subsets $D_i \subset \mathbb{N}$ as:

$$\gamma_i := \lambda k \, . \begin{cases} 1 & \text{if } k \in D_i \\ 0 & \text{otherwise,} \end{cases}$$

where

$$D_0 := \emptyset \qquad D_{i+1} := D_i \cup \{H(\gamma_i)\}.$$

A constructive technique for finding α, β and i can be formally extracted, which uses the following learning procedure.

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There is some least M such that $H(\gamma_M) \in D_M$. Let $\alpha_H := \gamma_M$, $i_H := H(\alpha_H)$.

- There must be some $j < M$ such that $H(\gamma_j) = i_H \in D_M$. Set $\beta_H := \gamma_j$, so we have $H(\alpha_H) = H(\gamma_j)$.
- Moreover, by minimality of M we must have $H(\gamma_j) = i_H \notin D_j$. Hence $H(\alpha_H) = 1 \neq 0 = H(\beta_H)$.

Infinite Ramsey theorem. For any colouring $c\colon \mathbb{N}^{(2)} \to \{0,1\}$ there exists an infinite set $X \subset \mathbb{N}$ which is pairwise monochromatic.

Infinite Ramsey theorem. For any colouring $c\colon \mathbb{N}^{(2)} \to \{0,1\}$ there exists an infinite set $X \subset \mathbb{N}$ which is pairwise monochromatic.

Proof (Erdös/Rado 1984). Organise $\mathbb{N}$ into a tree $\prec$ whose branches are min-monochromatic i.e.

$$i \prec j \prec k \to c(i,j) = c(i,k).$$

This is finitely branching, and hence contains an infinite branch (i_n). Applying the infinite pigeonhole principle to the colouring

$$\bar{c}(n) := c(i_n, i_{n+1})$$

there exists a colour b and a sequence (n_k) such that $\bar{c}(n_k) = b$ for all k. Set $X := \{n_{k_j} \mid j \in \mathbb{N}\}$.

INFINITE RAMSEY THEOREM. For any colouring $c\colon \mathbb{N}^{(2)} \to \{0,1\}$ there exists an infinite set $X \subset \mathbb{N}$ which is pairwise monochromatic.

PROOF (ERDÖS/RADO 1984). Organise $\mathbb{N}$ into a tree $\prec$ whose branches are min-monochromatic i.e.

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KOHLENBACH/KREUZER 2009. The Erdös/Rado proof can be formalised using a single instance of Π^0_1-CA i.e. a Skolem function for Σ^0_1-LEM.

3. Understand link to ϵ-calculus, learning realizability.

COROLLARY. Suppose that in PA we can derive

$$\forall x[\mathsf{CA}(P_x) \to \exists y A_0(x, y)].$$

Then there is some learning procedure $\mathcal{L}_{\varphi,q,P_x}$ and a primitive recursive function g such that

$$\forall x A_0(x, g(\mathcal{L}_{\varphi,q,P_x}, x))$$

- How does this learning procedure relate to that extracted using Aschieri-Berardi interactive realizability, or even ϵ-calculus?
- Can we obtain corresponding complexity results? Learning $\approx$ bar recursion, and the primitive recursive functionals are closed under the rule of bar recursion of low types (Schwichtenberg 1979).

Thank you!