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zum Thema

Degenerate Diffusion Equations with Gradient Terms

von

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Introduction

During the last 15 years degenerate diffusion equations without gradient terms have been studied intensively. In particular for the following degenerate parabolic problem many results have been found:

$$\begin{aligned} u_t &= u^p \Delta u + u^q \quad \text{in} \quad \Omega \times (0,T) \\ u|_{\partial\Omega} &= 0 \\ u|_{t=0} &= u_0 \end{aligned}$$

where $p > 0$ and $q > 0$ are given parameters, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $u_0 \in C^0(\bar{\Omega})$ is a given nonnegative function that is positive in Ω and vanishes on $\partial\Omega$.

In this diploma thesis we now study the following degenerate parabolic problem with gradient terms:

$$\begin{aligned} u_t &= u^p u_{xx} + u^q + \kappa u^r u_x^2 \quad \text{in} \quad \Omega \times (0,T) \\ u|_{\partial\Omega} &= 0 \\ u|_{t=0} &= u_0 \end{aligned} \tag{1}$$

where

$$p \geq 1, \; q \geq 1, \; r \geq 0 \quad \text{and} \quad \kappa > 0 \tag{2}$$

are fixed parameters,

$$\Omega \subset \mathbb{R} \qquad \text{is a bounded open interval} \tag{3}$$

and u_0 is a given function that fulfills

$$u_0 \in C^3(\bar{\Omega}) \quad \text{with} \quad u_0 > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0|_{\partial\Omega} = 0 \tag{4}$$

Degenerate parabolic differential equations of the types that are presented above arise in hydromagnetics, where they describe the resistive diffusion process of a force-free magnetic field in a passive medium in one space dimension ($n = 1$) under certain geometrical circumstances, as well as in differential geometry, where they determine (for $n = 1$) the curvature u of a plane immersed curve which evolves according to its curvature, and in population genetics, where so called "biased diffusion processes" are modelled by such equations.

In this diploma thesis we restrict to the case that Ω is a real interval and $u_0 \in C^3(\bar{\Omega})$ for simplicity. We want to distinguish if a solution u of problem (1) exists globally in $\Omega \times (0, \infty)$

or if there is a $T \in (0,\infty)$ with $\lim\limits_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$, so that T is a finite blow-up time of u. In most cases that are considered in this diploma thesis the solution u of (1) blows up in finite time. Then we analyse the size of the blow-up set

$$S := \{x \in \bar{\Omega} \quad | \quad \exists ((x_k, t_k))_{k\in\mathbb{N}} \subset \Omega \times (0,T) \quad \text{such that} \quad x_k \to x \quad \text{and} \quad u(x_k,t_k) \to \infty \quad \text{as} \quad k \to \infty\}$$

of u where T is the blow-up time of u. We distinguish if $|S| = 0$, what is called single point blow-up, or if $|S| > 0$, what is called regional blow-up, (where $|S|$ denotes the (n-dimensional) Lebesgue measure of S).

This paper is divided into five sections:
In Section 1 we present some basic results concerning parabolic differential equations. Among them there are two variants of the parabolic comparison principle, several theorems that are needed to show the existence of a solution of problem (1) and the Sturmian theorem that describes the zero set of a solution of a linear parabolic differential equation with smooth coefficients.
In Section 2 we prove the existence of a local solution of problem (1). We also show that (1) has a maximal solution, which we analyse in this diploma thesis, and prove some properties of this maximal solution.
In Section 3 we present some of the results that have been found concerning the degenerate parabolic problem without gradient terms given above. It was shown that the question if the solution of this problem exists globally or blows up in finite time depends on the relation between $p+1$ and q.
In Section 4 we study the solutions u of (1) in case of $r = p-1$. We can transform this solution to a solution of the problem that is considered in Section 3. By this we can prove that u exists globally if $p+1 > q$ or if $p+1 = q$ and $\Omega := (a,b)$ fulfills $b - a \leq \frac{\pi}{\sqrt{\kappa+1}}$ or if $p+1 < q$ and the initial data u_0 are small enough. Furthermore we show that u blows up in finite time if $p+1 = q$ and $\Omega := (a,b)$ fulfills $b - a > \frac{\pi}{\sqrt{\kappa+1}}$ and we always get regional blow-up in this case. Moreover u blows up in finite time if $p+1 < q$ and the initial data u_0 are large enough. But in this case we show in Section 5 that u only blows up in a single point if u_0 has some special properties.
In Section 5 we want to analyse the size of the blow-up set of the maximal solution u of (1) in the cases where we can show that u blows up in finite time. For that purpose we restrict to the case that $\Omega = (-a,a)$ for $a > 0$ and u_0 is a symmetric function with maximum in $x = 0$. In some cases we even need stronger assumptions on u_0. Then we can show that for $p+1 < q$ and large initial data u_0 the point $x = 0$ is the only blow-up point if $r < q-2$ and in the case $r > q-2$ we get regional blow-up. The case $r = q-2$ remains open here. Furthermore, we prove that for $p+1 = q$ and domains Ω with $a > \frac{\pi}{2}$ we always get regional blow-up for all $r \geq 0$. In case of regional blow-up we also give an estimate from below for the size of the blow-up set if the initial data u_0 have special properties.

Notation

Here we give some notation and definitions that are used in this diploma thesis.
Let $n \in \mathbb{N}$, $K, \Omega \subset \mathbb{R}^n$, $I \subset \mathbb{R}$, $T \in (0,\infty]$, $k, l \in \mathbb{N}$, $\alpha \in (0,1)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ with $|\beta| := \sum_{i=1}^{n} \beta_i$.

The set $\Sigma_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$ is the parabolic boundary of $\Omega \times (0,T)$.
The notation $K \subset\subset \Omega$ denotes that the closure $\bar{K}$ of K is a compact subset of Ω.

For the definitions of the following function spaces let $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ be open subsets or closures of open subsets.
The function space $C^0(\Omega)$ is the set of all real-valued functions that are continuous in Ω. If $\Omega \subset \mathbb{R}^n$ is open and bounded, the set $C^0(\bar{\Omega})$ becomes a Banach space with the norm $\|u\|_{C^0(\bar{\Omega})} := \sup_{x \in \bar{\Omega}} |u(x)|$.
The function space $C^k(\Omega)$ is the set of all real-valued continuous functions in Ω having continuous derivatives up to order k inclusively. If $\Omega \subset \mathbb{R}^n$ is open and bounded, the set $C^k(\bar{\Omega})$ becomes a Banach space with the norm $\|u\|_{C^k(\bar{\Omega})} := \sum_{|\beta| \le k} \|D^\beta u\|_{C^0(\bar{\Omega})}$, where $D^\beta u(x) := \frac{\partial^{|\beta|} u(x)}{\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}}$ for $x = (x_1, \dots, x_n) \in \bar{\Omega}$.
The set of all functions that belong to $C^k(\Omega)$ for all $k \in \mathbb{N}$ is $C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega)$.
The function space $C^{k,l}(\Omega \times I)$ is the set of all real-valued continuous functions in $\Omega \times I$ having continuous derivatives up to order k with respect to the variables x_i, $i = 1, ..., n$, and continuous derivatives up to order l with respect to t where $(x_1, ..., x_n, t) \in \Omega \times I$ denote the variables. If $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ are open and bounded, the set $C^{2,1}(\bar{\Omega} \times \bar{I})$ becomes a Banach space with the norm
$\|u\|_{C^{2,1}(\bar{\Omega}\times\bar{I})} := \|u\|_{C^0(\bar{\Omega}\times\bar{I})} + \|u_t\|_{C^0(\bar{\Omega}\times\bar{I})} + \sum_{i=1}^{n} \|u_{x_i}\|_{C^0(\bar{\Omega}\times\bar{I})} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{C^0(\bar{\Omega}\times\bar{I})}$.

If $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ are open and bounded, the function space $C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times \bar{I})$ is the set of all continuous functions u in $\bar{\Omega} \times \bar{I}$ so that the Hölder norm

$$\begin{aligned} \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} \quad &:= \quad \sup_{(x,t)\in\bar{\Omega}\times\bar{I}} |u(x,t)| + \sup_{x,y\in\bar{\Omega},x\neq y,t\in\bar{I}} \frac{|u(x,t)-u(y,t)|}{|x-y|^\alpha} \\ &\quad + \sup_{x\in\bar{\Omega},s,t\in\bar{I},s\neq t} \frac{|u(x,t)-u(x,s)|}{|t-s|^{\frac{\alpha}{2}}} \end{aligned}$$

is finite. In this case we say that u is Hölder continuous with exponent α with respect to x and with exponent $\frac{\alpha}{2}$ with respect to t.
For general $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ the set $C^{\alpha,\frac{\alpha}{2}}(\Omega \times I)$ is defined as the intersection of $C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}' \times \bar{I}')$, where $\Omega' \subset \mathbb{R}^n$ and $I' \subset \mathbb{R}$ run through the open bounded subsets that fulfill $\Omega' \subset\subset \Omega$ and $I' \subset\subset I$.

If $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ are open and bounded, the function space $C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega} \times \bar{I})$ is the set of all functions $u \in C^{2,1}(\bar{\Omega} \times \bar{I})$ so that u and its derivatives u_{x_i}, $u_{x_i x_j}$ and u_t for $i, j = 1, ..., n$ are elements of $C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times \bar{I})$ with the norm

$$\begin{aligned} \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} := & \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} + \|u_t\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} + \sum_{i=1}^{n} \|u_{x_i}\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} \\ & + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\bar{I})} \end{aligned}$$

For general $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ the set $C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times I)$ is defined as the intersection of $C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}' \times \bar{I}')$, where $\Omega' \subset \mathbb{R}^n$ and $I' \subset \mathbb{R}$ run through the open bounded subsets that fulfill $\Omega' \subset\subset \Omega$ and $I' \subset\subset I$.

The function space $L^\infty(\Omega)$ for $\Omega \subset \mathbb{R}^n$ measurable denotes the Lebesgue space which is the set of all functions that are measurable in Ω so that the norm
$\|u\|_{L^\infty(\Omega)} := \text{ess sup}\{|u(x)| \;|x \in \Omega\}$ is finite.

A domain Ω is smoothly bounded if for all $x_0 \in \partial\Omega$ there exist a neighbourhood U_{x_0} of x_0 and a C^∞- diffeomorphism $\Phi : U_{x_0} \to (-1, 1)^n$ (that means Φ is bijective with
$\Phi, \Phi^{-1} \in C^\infty$) with $\Phi(U_{x_0} \cap \partial\Omega) = (-1, 1)^{n-1} \times \{0\}$, $\Phi(x_0) = 0$ and
$\Phi(U_{x_0} \cap \Omega) = (-1, 1)^{n-1} \times (0, 1)$.

For $u \in C^0(\Omega)$ and $\rho > 0$ the modulus of continuity $\omega(\rho)$ of u in Ω is defined by
$\omega(\rho) := \sup\{|u(x) - u(y)| \mid x, y \in \Omega;\ |x - y| \le \rho\}$.

1 Some basic results concerning parabolic differential equations

In this section we present some basic results concerning parabolic differential equations. First we show two variants of the comparison principle and then we give several theorems that we use to prove the existence of a solution of problem (1). Finally we present the Sturmian theorem which describes the zero set of a linear parabolic differential equation with smooth coefficients.
First we prove the comparison principle for parabolic differential equations. It is used to show $u_1 \geq u_2$ in $\Omega \times (0,T)$ if we already know that $u_1 \geq u_2$ holds on the parabolic boundary Σ_T of $\Omega \times (0,T)$, where u_1 and u_2 are solutions (or sub solutions or super solutions) of a parabolic differential equation in $\Omega \times (0,T)$.

Theorem 1.1 (Comparison principle)
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and

$$F(u) \quad := \quad u_t - \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u) u_{x_i x_j} - f(x,t,u,\nabla u)$$

with continuous functions f and a_{ij} with $\sum\limits_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^n$. Assume that $u_l \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T])$, $l = 1,2$, $F(u_l)$ is defined with $F(u_1) \leq F(u_2)$ in $\Omega \times (0,T]$ and $u_1 \leq u_2$ on Σ_T.
Then $u_1 \leq u_2$ in $\bar{\Omega} \times [0,T]$, provided that at least for one $l \in \{1,2\}$ we have in addition

(a) $|u_l| + |\nabla u_l| + |D^2 u_l| \leq C_1$ *in $\Omega \times (0,T]$*

(b) *$a_{ij}(x,t,u,p)$ and $f(x,t,u,p)$ are Lipschitz continuous with respect to u and p in a neighbourhood of $K := u_l(\bar{\Omega} \times [0,T]) \times \nabla u_l(\Omega \times (0,T])$.*

Proof. We assume in the proof that (a) and (b) hold for u_2. The proof of the other case is analogous.
We define $d(x,t) := u(x,t) - w(x,t) - \varepsilon e^{\gamma t}$ for $\varepsilon > 0$ and $\gamma > 0$ with $\gamma > L_1 C_1 + L_2$, where L_1 and L_2 are the Lipschitz-constants of a_{ij} and f in K, $u := u_1$ and $w := u_2$.
Then we have $d < 0$ on Σ_T.
We will show that $d \leq 0$ holds in $\bar{\Omega} \times [0,T]$.
If this was false, there would be $t_0 > 0$ and $x_0 \in \bar{\Omega}$ with $d(x_0,t_0) = \max\limits_{x \in \bar{\Omega}} d(x,t_0) = 0$ and $\max\limits_{x \in \bar{\Omega}} d(x,t) < 0 \ \forall t \in (0,t_0)$ because d is continuous.
So we have $d(x,t_0) \leq 0 \ \forall x \in \bar{\Omega}$ and therefore $x_0 \in \Omega$ because $d < 0$ on Σ_T.
Hence it follows $d_t(x_0,t_0) \geq 0$, $\nabla d(x_0,t_0) = 0$ and the matrix $-D := -(d_{x_i x_j}(x_0,t_0))_{ij}$ is positive semi-definite and symmetric. Because the matrix
$A := (a_{ij}(x_0,t_0,u(x_0,t_0),\nabla u(x_0,t_0)))_{ij}$ is positive semi-definite, the matrix $-AD$ is also positive semi-definite.

Therefore we get:

$$\begin{aligned}
0 &\geq F(u(x_0,t_0)) - F(w(x_0,t_0)) \\
&= d_t(x_0,t_0) + \gamma\varepsilon e^{\gamma t_0} - \sum_{i,j=1}^{n}[a_{ij}(x_0,t_0,u(x_0,t_0),\nabla u(x_0,t_0)) \\
&\quad -a_{ij}(x_0,t_0,w(x_0,t_0),\nabla w(x_0,t_0))]w_{x_ix_j}(x_0,t_0) + Trace(-AD) \\
&\quad -f(x_0,t_0,u(x_0,t_0),\nabla u(x_0,t_0)) + f(x_0,t_0,w(x_0,t_0),\nabla w(x_0,t_0)) \\
&\overset{-AD\geq 0}{\geq} d_t(x_0,t_0) + \gamma\varepsilon e^{\gamma t_0} - \sum_{i,j=1}^{n}[a_{ij}(x_0,t_0,u(x_0,t_0),\nabla w(x_0,t_0)) \\
&\quad -a_{ij}(x_0,t_0,w(x_0,t_0),\nabla w(x_0,t_0))]w_{x_ix_j}(x_0,t_0) \\
&\quad -f(x_0,t_0,u(x_0,t_0),\nabla w(x_0,t_0)) + f(x_0,t_0,w(x_0,t_0),\nabla w(x_0,t_0)) \\
&\geq d_t(x_0,t_0) + \gamma\varepsilon e^{\gamma t_0} - (L_1C_1+L_2)|u(x_0,t_0)-w(x_0,t_0)| \\
&\overset{d_t(x_0,t_0)\geq 0}{\geq} \varepsilon e^{\gamma t_0}(\gamma - L_1C_1 - L_2) \\
&> 0
\end{aligned}$$

which is a contradiction. Hence $d \leq 0$ in $\bar{\Omega}\times[0,T]$ and the assertion follows with $\varepsilon \searrow 0$. ■

In the next corollary we prove a variant of the comparison principle. There condition (a) of Theorem 1.1 does not have to be fulfilled if we have instead $u_1 < u_2$ on the parabolic boundary Σ_T.

Corollary 1.2
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and

$$F(u) := u_t - \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)u_{x_ix_j} - f(x,t,u,\nabla u)$$

with continuous functions f and a_{ij} with $\sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^n$. Assume that $u_l \in C^0(\bar{\Omega}\times[0,T]) \cap C^{2,1}(\Omega\times(0,T])$, $l = 1,2$, $F(u_l)$ is defined with $F(u_1) \leq F(u_2)$ in $\Omega\times(0,T]$ and $u_1 < u_2$ on Σ_T.
Then $u_1 \leq u_2$ in $\bar{\Omega}\times[0,T]$, provided that at least for one $l \in \{1,2\}$ we have in addition that $a_{ij}(x,t,u,p)$ and $f(x,t,u,p)$ are Lipschitz continuous with respect to u and p on every compact subset of $K := u_l(\bar{\Omega}\times[0,T]) \times \nabla u_l(\Omega\times(0,T])$.

Proof. We can find $M \subset\subset \Omega\times(0,T]$, M open, with $u_1 < u_2$ in $(\bar{\Omega}\times[0,T]) \setminus M$ because $u_1 < u_2$ on Σ_T and $u_l \in C^0(\bar{\Omega}\times[0,T])$ for $l = 1,2$.
As $M \subset\subset \Omega\times(0,T]$ we have $u_l \in C^{2,1}(\bar{M})$ for $l = 1,2$ and therefore there exists $C_1 > 0$ with $|u_l| + |\nabla u_l| + |D^2 u_l| \leq C_1$ in $\bar{M}$ for $l = 1,2$.

We define $d(x,t) := u_1(x,t) - u_2(x,t) - \varepsilon e^{\gamma t}$ for $\varepsilon > 0$ and $\gamma > 0$ with $\gamma > L_1 C_1 + L_2$, where L_1 and L_2 are the Lipschitz-constants of a_{ij} and f on $u_l(\bar{M}) \times \nabla u_l(\bar{M})$(where $l \in \{1,2\}$ is chosen so that the last condition of the corollary holds) which is a compact subset of K. Then we have $d < 0$ on ∂M.
So we can prove $d \leq 0$ in M just like in the proof of Theorem 1.1. With $\varepsilon \searrow 0$ we get $u_1 \leq u_2$ in M and therefore the assertion holds due to the choice of M. ■

We now give some theorems concerning parabolic problems that we need to show the local existence of a solution of problem (1).
We consider the following parabolic problem:

$$\begin{aligned} u_t &= a_{11}(x,t,u,u_x)u_{xx} - a(x,t,u,u_x) \quad \text{in} \quad \Omega \times (0,T) \\ u|_{\Sigma_T} &= \Psi|_{\Sigma_T} \end{aligned} \tag{5}$$

where $\Omega \subset \mathbb{R}$ is a bounded interval, $T > 0$ and $a_{11}(x,t,u,p)$, $a(x,t,u,p)$ and $\Psi(x,t)$ are given functions that are defined for $(x,t) \in \bar{\Omega} \times [0,T]$ and $u, p \in \mathbb{R}$.
We first present a theorem concerning the existence of a solution of that problem. For a proof we refer to Theorem 5.2 and Remark 5.1 in Chapter VI of [LSU].

Theorem 1.3
Suppose that the following conditions are fulfilled:

(a) *For $(x,t) \in \bar{\Omega} \times [0,\infty)$ and arbitrary u we have $a_{11}(x,t,u,0) \geq 0$ and $-a(x,t,u,0)u \leq (b_1|u| + b_2)|u|$ with some positive constants b_1 and b_2.*

(b) *Let $M > 0$ be arbitrary. Then for $(x,t) \in \bar{\Omega} \times [0,\infty)$, $|u| \leq M$ and arbitrary p the functions $a_{11}(x,t,u,p)$ and $a(x,t,u,p)$ are continuous and $a_{11}(x,t,u,p)$ is differentiable with respect to x, u and p and there exist constants $\nu, \mu > 0$ and a function $\mu_1(|u|)$, so that*

$$\begin{aligned} \nu \leq a_{11}(x,t,u,p) &\leq \mu \\ \left|\frac{\partial a_{11}}{\partial p}\right|(1+|p|)^3 + \left|\frac{\partial a_{11}}{\partial u}\right|(1+|p|)^2 + \left|\frac{\partial a_{11}}{\partial x}\right|(1+|p|) + |a| &\leq \mu_1(|u|)(1+|p|)^2 \end{aligned}$$

are fulfilled for $(x,t) \in \bar{\Omega} \times [0,T]$, $|u| \leq M$ and arbitrary p.

(c) *Let $M > 0$ and $M_1 > 0$ be arbitrary. For $(x,t) \in \bar{\Omega} \times [0,\infty)$, $|u| \leq M$ and $|p| \leq M_1$ the functions $a_{11}(x,t,u,p)$ and $a(x,t,u,p)$ are Hölder continuous in the variable t with exponent $\frac{\beta}{2}$ and in the variables x, u and p with exponent β.*

(d) *The function $\Psi(x,t) \in C^{2+\beta,1+\frac{\beta}{2}}(\bar{\Omega} \times [0,\infty))$ and*

$$[\Psi_t(x,0) - a_{11}(x,0,\Psi(x,0),\Psi_x(x,0))\Psi_{xx}(x,0) + a(x,0,\Psi(x,0),\Psi_x(x,0))]|_{x\in\partial\Omega} = 0$$

Then there exists $T > 0$ so that problem (5) *is uniquely solvable in* $C^{2+\beta,1+\frac{\beta}{2}}(\bar{\Omega} \times [0,T])$.

We give another theorem concerning the existence of solutions of (5) where we have weaker conditions on Ψ. Therefore the solution u has weaker properties, too. But this theorem will be used to show the existence of a solution of (1) because Ψ can be chosen more generally than it could be chosen for the last theorem. For a proof of the next theorem we refer to Theorem 4.4 and §5 in chapter VI of [LSU] and use that $\Omega \subset \mathbb{R}$ is an interval so that it is smoothly bounded.

Theorem 1.4
Suppose the conditions $(a)-(c)$ *of Theorem* 1.3 *are fulfilled and* $\Psi \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2+\beta,1+\frac{\beta}{2}}(\Omega \times (0,\infty))$. *Then there exists $T > 0$ so that problem* (5) *has a solution* $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2+\beta,1+\frac{\beta}{2}}(\Omega \times (0,T))$.

Next we give a short theorem to show that the solution u of (5) is smooth in $\Omega \times (0,T)$ and fulfills $u_x \in C^0(\Omega \times [0,T))$, if the functions a_{11} and a are smooth functions. This will be used to prove that some solutions of problem (1) are smooth in $\Omega \times (0,T)$ and fulfill $u_x \in C^0(\Omega \times [0,T))$. For a proof we refer to Theorem 12.1 in chapter III, Theorems 3.4 and 1.1 in chapter VI and to §1 and §5 in chapter VI of [LSU].

Theorem 1.5
Suppose that the conditions $(a)-(c)$ *of Theorem* 1.3 *are fulfilled,* $\Psi_0(x) := \Psi(x,0)$ *for* $x \in \bar{\Omega}$ *fulfills* $\Psi_0 \in C^{2+\beta}(\bar{\Omega})$, $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2+\beta,1+\frac{\beta}{2}}(\Omega \times (0,T))$ *is a solution of* (5) *for some* $T > 0$, $\Psi \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2+\beta,1+\frac{\beta}{2}}(\Omega \times (0,T))$ *and* $a_{11}(x,t,u,p), a(x,t,u,p) \in C^\infty(\bar{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R})$. *Then* $u \in C^\infty(\Omega \times (0,T))$ *and* $u_x \in C^0(\Omega \times [0,T))$.

In the next theorem we show that u_x is uniformly bounded in every subdomain of $\Omega \times (0,T)$ which has positive distance to the boundary Σ_T provided that u is uniformly bounded in $\Omega \times (0,T)$ and u is a classical solution of the differential equation in problem (5) with the properties that we get in Theorem 1.3. We formulate the theorem for solutions of this differential equation in $\bar{\Omega} \times [t_0,T]$ with $t_0 \geq 0$ because we will use this theorem for $t_0 > 0$ and not only in the case $t_0 = 0$. For a proof of the following theorem we refer to Theorem 3.4 in Chapter VI of [LSU].

Theorem 1.6
Suppose $u(x,t) \in C^{2,1}(\bar{\Omega} \times [t_0,T])$ *where* $\Omega \subset \mathbb{R}$ *is an interval and* $0 \leq t_0 < T < \infty$. *Suppose further that u satisfies the equation*

$$u_t \;=\; a_{11}(x,t,u,u_x) - a(x,t,u,u_x)$$

in $Q_T := \bar{\Omega} \times [t_0,T]$ *and that for* $(x,t) \in Q_T$ *and* $u,p \in \mathbb{R}$ *the coefficient functions* $a_{11}(x,t,u,p)$ *and* $a(x,t,u,p)$ *are differentiable functions and that there exist a positive* *constant ν and continuous monotonically increasing positive functions $\mu_0(\xi)$, $\mu(\xi)$, $\mu_1(\xi)$ and $\varepsilon(\xi)$ for $\xi \geq 0$, so that the following conditions are fulfilled:*

$$\begin{aligned}
\nu \leq a_{11}(x,t,u,p) &\leq \mu_0(|u|)\\
|a(x,t,u,p)| &\leq \mu(|u|)(1+|p|)^2\\
|\frac{\partial a_{11}}{\partial p}|(1+|p|)^2 + |\frac{\partial a}{\partial p}| &\leq \mu_1(|u|)(1+|p|)\\
|\frac{\partial a_{11}}{\partial x}|(1+|p|)^2 + |\frac{\partial a}{\partial x}| &\leq \varepsilon(|u|)(1+|p|)^3\\
|\frac{\partial a_{11}}{\partial u}| &\leq \varepsilon(|u|)\\
-\frac{\partial a(x,t,u,p)}{\partial u} &\leq \varepsilon(|u|)(1+|p|)^2
\end{aligned}$$

Then for any cylinder $Q' = \Omega' \times (t_1, T)$ with $\Omega' \subset\subset \Omega$ and $t_1 \in (t_0, T)$ the quantity $\max\limits_{(x,t)\in Q'} |u_x(x,t)|$ does not exceed a certain constant depending on the distance from Q' to $\Gamma := (\Omega \times \{t_0\}) \cup (\partial\Omega \times [t_0, T])$, $M := \max\limits_{(x,t)\in Q_T} |u(x,t)|$, ν, $\mu_0 = \mu_0(M)$, $\mu = \mu(M)$, $\mu_1 = \mu_1(M)$, $\varepsilon = \varepsilon(M)$ and the modulus of continuity $\omega(\rho)$ of $u(x,t)$ in Q_T for the number $\varepsilon(M)$.

Next we consider a parabolic differential equation in divergence form, because we need another theorem to show the existence of a solution of problem (1).
Let $u \in C^{2,1}(\Omega \times (0,T))$ be the solution of the equation

$$u_t(x,t) = \frac{d}{dx} b(x,t,u(x,t),u_x(x,t)) - c(x,t,u(x,t),u_x(x,t)) \quad \text{in} \quad \Omega \times (0,T) \tag{6}$$

where $\Omega \subset \mathbb{R}$ is a bounded interval, $T > 0$ and $b(x,t,u,p)$ and $c(x,t,u,p)$ are given functions that are defined for $(x,t) \in \bar{\Omega} \times [0,T]$ and $u, p \in \mathbb{R}$.
The following theorem gives an estimate from above for a suitable Hölder norm of u if u is a classical solution of (6) and $|u| \leq M$ in $\Omega \times (0,T)$. For a proof of the theorem we refer to Theorem 1.1 in Chapter V of [LSU] and remark that the boundary Σ_T fulfills the conditions of the cited theorem because $\Omega \subset \mathbb{R}$ is a bounded interval.

Theorem 1.7
Suppose the functions $b(x,t,u,p)$ and $c(x,t,u,p)$ are continuous with respect to u and p and there are positive continuous functions $\nu(\xi)$, $\mu(\xi)$ and $\mu_1(\xi)$ of $\xi \geq 0$, so that for $(x,t) \in \bar{\Omega} \times [0,T]$ and $u, p \in \mathbb{R}$ the following conditions are fulfilled:

$$\begin{aligned}
b(x,t,u,p)p &\geq \nu(|u|)p^2,\\
|b(x,t,u,p)| &\leq \mu(|u|)|p|,\\
|c(x,t,u,p)| &\leq \mu_1(|u|)p^2.
\end{aligned}$$

We also suppose that $u \in C^0(\bar\Omega \times [0,T]) \cap C^{2,1}(\bar\Omega \times (0,T])$ *is a solution of problem* (6) *with* $\sup_{(x,t)\in\Omega\times(0,T]} |u(x,t)| = M$ *and* u *is Hölder continuous (of exponent* β *with respect to* x *and* $\frac{\beta}{2}$ *with respect to* t*) on some part* Σ' *of the boundary* Σ_T.
Then $u \in C^{\alpha,\frac{\alpha}{2}}((\Omega \times (0,T)) \cup \Sigma')$ *with some* $\alpha > 0$ *depending only on* ν, μ, $\frac{M\mu_1}{\nu}$ *and* β. *The norm* $\|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q')}$, *for any* $Q' \subset ((\Omega \times (0,T)) \cup \Sigma')$ *separated from* $\Sigma_T \setminus \Sigma'$ *by a positive distance* d, *is estimated from above by a constant determined by* M, ν, μ, μ_1, β, $\|u\|_{C^{\beta,\frac{\beta}{2}}(\Sigma')}$, Ω, Σ', T *and the distance* d.

Furthermore we want to get an estimate for $\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(\bar\Omega\times[t_0,T])}$ from above if $|u| \le M$, $|u_x| \le M_1$ in $\bar\Omega \times [t_0,T]$ and $u \in C^{2,1}(\bar\Omega \times [t_0,T])$. We also need that this estimate only depends on M, M_1 and some other constants, so that we can get a uniform estimate of the Hölder norm. For a proof of the next theorem we refer to Theorem 5.1 in chapter VII of [LSU].

Theorem 1.8
Let $u(x,t) \in C^{2,1}(\bar\Omega \times [t_0,T])$ *be a solution of the equation*

$$u_t \;=\; a_{11}(x,t,u)u_{xx} - a(x,t,u,u_x)$$

in $Q_T := \bar\Omega \times [t_0,T]$ *where* $0 \le t_0 < T < \infty$ *and* $\Omega \subset \mathbb{R}$ *is an interval. We also suppose that there is* $\nu > 0$ *so that* $a_{11}(x,t,u(x,t)) \ge \nu$ *in* Q_T, *that the function* $a_{11}(x,t,u)$ *is differentiable with respect to* x *and* u *in the domain* $G := \{(x,t,u,p) \in \mathbb{R}^4 \mid (x,t) \in Q_T,$ $|u| \le M = \max_{(x,t)\in Q_T} |u(x,t)|, |p| \le M_1 = \max_{(x,t)\in Q_T} |u_x(x,t)|\}$ *and that this same function and its derivatives with respect to* x *and* u *and the function* $a(x,t,u,p)$ *are continuous in* G. *An upper bound of their moduli is denoted by* M_2. *If in addition to this the functions* $a_{11}(x,t,u)$ *and* $a(x,t,u,p)$ *satisfy a Hölder condition in* G *in the arguments* x, t, u, p *with exponents* β, $\frac{\beta}{2}$, β, β *respectively, then for every* $Q' \subset Q_T$ *with positive distance to* $\Gamma := (\bar\Omega \times \{t_0\}) \cup (\partial\Omega \times [t_0,T])$ *the norm* $\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q')}$ *is estimated from above in terms of* M, M_1, M_2, ν, *the Hölder norms of the functions* a_{11} *and* a *in* G *and the distance from* Q' *to* Γ.

We now formulate the Sturmian theorem. Let $u(x,t)$ be a classical solution of the parabolic equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{7}$$

in $Q_T := [x_0,x_1] \times [0,T]$, where $x_0 < x_1$, $T > 0$ and $a,b,c \in C^\infty(Q_T)$ with $a > 0$ in Q_T.
The Sturmian theorem describes the zero set of a solution of (7). In the formulation we use the following terminology. If $f(x)$ is a C^∞ function of one variable with $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 will be called a simple zero if $f'(x_0) \ne 0$. The order of the zero x_0 is the smallest integer k for which $f^{(k)}(x_0) \ne 0$; if all derivatives of f vanish at x_0, then x_0

is a zero of infinite order. The number of zeroes of f in (a, b) counted with multiplicity is, by definition, the sum of the orders of all zeroes of f in the interval (a, b).
For a proof of the Sturmian theorem we refer to Theorem 2.1 in [Ang].

Theorem 1.9 (Sturmian theorem)
If $u \in C^\infty(Q_T)$ is a solution of (7) and $u(x_j, t) \neq 0$ for $0 \leq t \leq T$ and $j = 0, 1$, then at any time $t \in (0, T]$ the zero set of $x \to u(x, t)$ will be finite, even when counted with multiplicity.
The number of zeroes $z(t)$ of $x \to u(x, t)$ counted with multiplicity is a nonincreasing function of t; at any time t when $x \to u(x, t)$ has a zero of order $k > 1$, $z(t)$ drops by at least $k - 1$.

2 Existence and some properties of solutions to $u_t = u^p u_{xx} + u^q + \kappa u^r u_x^2$

In this section we first show with the help of the results of the preceding section that a local solution of problem (1) exists if the assumptions (2), (3) and (4) are fulfilled. After we have shown the existence of a local solution, we show that there exists a maximal solution of (1) and we prove some properties of this maximal solution which we will use throughout this diploma thesis.
We now show the existence of a local solution of (1) by approximating this solution with solutions of parabolic problems that fulfill the conditions of the preceding section.

Theorem 2.1
Let $\Omega \subset \mathbb{R}$ be a bounded interval, $p, q \geq 1$, $r \geq 0$, $\kappa > 0$ and $u_0 \in C^3(\bar{\Omega})$ with $u_0 > 0$ in Ω and $u_0|_{\partial\Omega} = 0$. Then there is $T > 0$ so that problem (1) has a solution $u \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T))$.

Proof. Let $\varepsilon > 0$. We then define

$$\rho_\varepsilon(t) := \begin{cases} t^p & , \quad \text{for} \quad t \geq \varepsilon \\ (\frac{\varepsilon}{2})^p & , \quad \text{for} \quad t \leq \frac{\varepsilon}{2} \end{cases}$$

and smooth in between with $\rho_\varepsilon(t) \geq (\frac{\varepsilon}{2})^p$ in $\mathbb{R}$.
We also define for $s \geq 0$ and $M \geq \varepsilon + 1$

$$\varphi_{s,M,\varepsilon}(t) := \begin{cases} M^s + 1 & , \quad \text{for} \quad t \geq M+1 \\ t^s & , \quad \text{for} \quad t \in [\varepsilon, M] \\ 0 & , \quad \text{for} \quad t \leq 0 \end{cases}$$

and smooth in between with $\varphi_{s,M,\varepsilon}(t) \in [0, M^s + 1]$ for $t \in \mathbb{R}$ and $\varphi_{s,M,\varepsilon}(t) \leq t^s$ for $t \geq \varepsilon$.
We also assume that we have $\varphi_{s,M,\varepsilon_1}(t) = \varphi_{s,M,\varepsilon_2}(t)$ for $s \geq 0$, $0 < \varepsilon_1 < \varepsilon_2 \leq M-1$ and $t \geq \varepsilon_2$.
Furthermore we choose $M := 2\|u_0\|_{L^\infty(\Omega)} + 1$ and consider the problem

$$\begin{aligned} (u_{M,\varepsilon})_t &= \rho_\varepsilon(u_{M,\varepsilon})(u_{M,\varepsilon})_{xx} + \varphi_{q,M,\varepsilon}(u_{M,\varepsilon}) + \kappa\varphi_{r,M,\varepsilon}(u_{M,\varepsilon})(u_{M,\varepsilon})_x^2 \quad \text{in} \quad \Omega \times (0,T) \\ u_{M,\varepsilon}|_{\partial\Omega} &= \varepsilon \\ u_{M,\varepsilon}|_{t=0} &= u_0 + \varepsilon \end{aligned} \tag{8}$$

From Theorem 1.4 we get the existence of a local solution $u_{M,\varepsilon}$ on some maximal interval of existence $[0, T_\varepsilon)$ with $T_\varepsilon > 0$ and $u_{M,\varepsilon} \in C^0(\bar{\Omega} \times [0,T_\varepsilon)) \cap C^{2,1}(\Omega \times (0,T_\varepsilon))$
(with $\Psi(x,t) := u_0(x)+\varepsilon$, $a_{11}(x,t,u,\hat{p}) := \rho_\varepsilon(u)$ and $-a(x,t,u,\hat{p}) := \varphi_{q,M,\varepsilon}(u)+\kappa\varphi_{r,M,\varepsilon}(u)\hat{p}^2$ for $x \in \bar{\Omega}$, $t \geq 0$ and $u, \hat{p} \in \mathbb{R}$ in Theorem 1.4).
Since $u_{M,\varepsilon} \geq \varepsilon$ on Σ_{T_ε} we get $u_{M,\varepsilon} \geq \varepsilon$ in $\bar{\Omega} \times [0, T_\varepsilon)$ by Theorem 1.1 .
For $0 < \varepsilon_1 < \varepsilon_2$ we have $u_{M,\varepsilon_1} < u_{M,\varepsilon_2}$ on Σ_T with $0 < T \leq \min\{T_{\varepsilon_1}, T_{\varepsilon_2}\}$ and hence $0 \leq u_{M,\varepsilon_1} \leq u_{M,\varepsilon_2}$ on $\bar{\Omega} \times [0,T)$ by Corollary 1.2 because u_{M,ε_2} fulfills

$$(u_{M,\varepsilon_2})_t = \rho_{\varepsilon_1}(u_{M,\varepsilon_2})(u_{M,\varepsilon_2})_{xx} + \varphi_{q,M,\varepsilon_1}(u_{M,\varepsilon_2}) + \kappa\varphi_{r,M,\varepsilon_1}(u_{M,\varepsilon_2})(u_{M,\varepsilon_2})_x^2$$

in $\Omega \times (0,T)$ by our assumptions on ρ_ε and $\varphi_{s,M,\varepsilon}$ and since $u_{M,\varepsilon_2} \geq \varepsilon_2$ in $\Omega \times (0,T)$. Therefore we have $T_{\varepsilon_1} \geq T_{\varepsilon_2}$ and so we get $0 \leq u_{M,\varepsilon_1} \leq u_{M,\varepsilon_2}$ in $\bar{\Omega} \times [0,T_1)$ for $0 < \varepsilon_1 < \varepsilon_2 < 1$.
Since $u_{M,1} \in C^0(\bar{\Omega} \times [0,T_1))$ and $u_{M,1}(x,0) = u_0(x) + 1$ for $x \in \bar{\Omega}$ there is $T \in (0,T_1)$ with $u_{M,1} \leq M$ in $\bar{\Omega} \times [0,T]$ because M was suitably chosen. Hence we have $\varepsilon \leq u_{M,\varepsilon} \leq M$ in $\bar{\Omega} \times [0,T]$ for all $\varepsilon \in (0,1)$. So in fact $u_\varepsilon := u_{M,\varepsilon} \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T])$ solves

$$\begin{aligned} (u_\varepsilon)_t &= (u_\varepsilon)^p(u_\varepsilon)_{xx} + (u_\varepsilon)^q + \kappa(u_\varepsilon)^r(u_\varepsilon)_x^2 \quad \text{in} \quad \Omega \times (0,T) \\ u_\varepsilon|_{\partial\Omega} &= \varepsilon \\ u_\varepsilon|_{t=0} &= u_0 + \varepsilon \end{aligned} \tag{9}$$

for all $\varepsilon \in (0,1)$.
Since $T < T_1$, we get $0 \leq u_\varepsilon \leq u_1 \leq M$ in $\bar{\Omega} \times [0,T]$ $\forall \varepsilon \in (0,1)$. We remark that T only depends on M and $\|u_0\|_{L^\infty(\Omega)}$ because the functions ρ_1 and $\varphi_{s,M,1}$ only depend on M and hence we can get an a priori estimate $u_1 \leq M$ in $\Omega \times [0,T]$ for all solutions of (8) with $\varepsilon = 1$ where T only depends on M and $\|u_0\|_{L^\infty(\Omega)}$. Hence T only depends on $\|u_0\|_{L^\infty(\Omega)}$.
Let $K \subset\subset \Omega$ be a compact interval and G be a bounded domain with $K \subset\subset G \subset\subset \Omega$ and let Θ be a principal eigenfunction of $-\Delta$ in G with $\Delta\Theta(x) + \lambda_1\Theta(x) = 0$ $\forall x \in G$, $\Theta|_{\partial G} = 0$ and $0 < \Theta \leq 1$ in G where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ in G. We choose $c_0 > 0$ so that $u_0 \geq c_0$ in $\bar{G}$ which is possible because $u_0 > 0$ in Ω.
For $\gamma > \lambda_1(c_0)^p > 0$ we define $v(x,t) := c_0e^{-\gamma t}\Theta(x)$ for $(x,t) \in \bar{G} \times [0,T]$. Due to the choice of c_0 we have $v(x,0) \leq u_0(x)$ $\forall x \in G$ and $v(x,t) = 0$ for $(x,t) \in \partial G \times [0,T]$.
We also get for $(x,t) \in G \times (0,T)$

$$\begin{aligned} v^p(x,t)v_{xx}(x,t) + v^q(x,t) + \kappa v^r(x,t)v_x^2(x,t) &= -\lambda_1(c_0)^{p+1}e^{-(p+1)\gamma t}\Theta(x)^{p+1} \\ &\quad +(c_0)^q e^{-q\gamma t}\Theta(x)^q \\ &\quad +\kappa(c_0)^{r+2}e^{-(r+2)\gamma t}\Theta(x)^r\Theta_x(x)^2 \\ &\overset{0\leq\Theta\leq 1}{\geq} -\lambda_1(c_0)^{p+1}e^{-(p+1)\gamma t}\Theta(x) \\ &\overset{p\geq 0}{\geq} -\lambda_1(c_0)^{p+1}e^{-\gamma t}\Theta(x) \\ &\overset{\gamma>\lambda_1(c_0)^p}{\geq} -\gamma c_0 e^{-\gamma t}\Theta(x) \\ &= v_t(x,t) \end{aligned}$$

Since $\Theta \in C^\infty(\bar{G})$ we have $u_\varepsilon \geq v$ in $G \times [0,T]$ by Corollary 1.2 for all $\varepsilon \in (0,1)$ and there exists $c_K > 0$ with $u_\varepsilon \geq v \geq c_K > 0$ in $K \times [0,T]$, because $K \subset\subset G$ and $T < \infty$.
So we have $0 < c_K \leq u_\varepsilon \leq M$ in $K \times [0,T]$ for all $\varepsilon \in (0,1)$ (this is also valid for every $K \subset\subset \Omega$) . By Theorem 1.7 we get a Hölder estimate $\|u_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(K'\times[0,T])} \leq C_{K'}$ for some

$\beta \in (0,1)$ and all $\varepsilon \in (0,1)$ for every $K' \subset\subset K$ compact (with $\Omega := K$, $\Sigma' := K \times \{0\}$, $b(x,t,u,\hat{p}) := \rho_{c_K}(u)\hat{p}$ and $-c(x,t,u,\hat{p}) := \varphi_{q,c_K}(u) + (\kappa\varphi_{r,c_K}(u) - p\varphi_{p-1,c_K})\hat{p}^2$ for $x \in \bar{\Omega}$, $t \geq 0$ and $u, \hat{p} \in \mathbb{R}$ in Theorem 1.7) .
Now let $T_0 \in (0,T)$ and $K' \subset\subset K$ be an arbitrary interval. Then by Theorem 1.6 there is for every $K'' \subset\subset K'$ a constant $D_{K''} > 0$ so that $|(u_\varepsilon)_x| \leq D_{K''}$ in $K'' \times [T_0, T]$ for all $\varepsilon \in (0,1)$ (with $\Omega := K'$, $t_0 := \frac{T_0}{2}$, $a_{11}(x,t,u,\hat{p}) := \rho_{c_K}(u)$, $-a(x,t,u,\hat{p}) := \varphi_{q,M,c_K}(u) + \kappa\varphi_{r,M,c_K}(u)\hat{p}^2$ for $(x,t,u,\hat{p}) \in \mathbb{R}^4$, $\Omega' := K''$ and $t_1 = T_0$ in Theorem 1.6. We remark that there exists a common modulus $\omega(\rho)$ for all u_ε with $\varepsilon \in (0,1)$ because $\|u_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(K' \times [0,T])} \leq C_{K'}$ for all $\varepsilon \in (0,1)$.).
So by Theorem 1.8 we get for all $t > T_0$ and all $K'' \subset\subset K'$ a constant $B_{K''}$ so that $\|u_\varepsilon\|_{C^{2+\beta,1+\frac{\beta}{2}}(K'' \times [t,T])} \leq B_{K''}$ $\forall \varepsilon \in (0,1)$ (with $\Omega := K'$, $t_0 := T_0$, $a_{11}(x,t,u) := \rho_{c_K}(u)$ and $-a(x,t,u,\hat{p}) := \varphi_{q,M,c_K}(u) + \kappa\varphi_{r,M,c_K}(u)\hat{p}^2$ for $(x,t,u,\hat{p}) \in \mathbb{R}^4$ in Theorem 1.8).
As T_0 and K and hence K'' are arbitrary and since u_ε is monotonely decreasing for $\varepsilon \searrow 0$, there is by Arzelà-Ascoli a function $u \in C^{2,1}(\Omega \times (0,T])$ with $u_\varepsilon \to u$ in $C^{2,1}(\bar{K} \times [t_0, T])$ for $\varepsilon \searrow 0$ and for all $t_0 \in (0,T)$ and all intervals $K \subset\subset \Omega$ and hence this is also valid for every $K \subset\subset \Omega$. We also have $u \in C^0(\Omega \times [0,T])$ with $u_\varepsilon \to u$ in $C^0(\bar{K} \times [0,T])$ for $\varepsilon \searrow 0$ and for all intervals $K \subset\subset \Omega$ since $\|u_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(K' \times [0,T])} \leq C_{K'}$ for all $K' \subset\subset K$, K is arbitrary and the limit function u is unique.
In particular we have $u|_{t=0} = u_0$ and u solves $u_t = u^p u_{xx} + u^q + \kappa u^r u_x^2$ in $\Omega \times (0,T)$.
So we finally have to show $u \in C^0(\bar{\Omega} \times [0,T])$ with $u|_{\partial\Omega} = 0$. Therefore let $\delta > 0$. For $0 < \varepsilon < \frac{\delta}{2}$ we have $u_\varepsilon|_{\partial\Omega} = \varepsilon < \frac{\delta}{2}$. Since $u_{\frac{\delta}{2}} \in C^0(\bar{\Omega} \times [0,T])$, there is a neighbourhood U_δ of $\partial\Omega \times [0,T]$ with $u_{\frac{\delta}{2}} < \delta$ in U_δ. So we have $u_\varepsilon < \delta$ in U_δ for all $\varepsilon \in (0, \frac{\delta}{2})$ because $u_\varepsilon < u_{\frac{\delta}{2}}$. Hence we also have $u \leq \delta$ in U_δ because $u_\varepsilon(x,t) \to u(x,t)$ for $\varepsilon \searrow 0$ for all $(x,t) \in \Omega \times [0,T]$. Since $\delta > 0$ was arbitrary and $u_\varepsilon \geq 0$ in $\bar{\Omega} \times [0,T]$, we get $u \in C^0(\bar{\Omega} \times [0,T])$ and $u|_{\partial\Omega} = 0$. So we have proved the assertion. ■

We now show that problem (1) always has a maximal solution and that we can approximate this solution with solutions of problem (9) like in the proof of Theorem 2.1. The following lemma will often be used throughout this diploma thesis.

Lemma 2.2
Suppose that assumptions (2), (3) and (4) are fulfilled. Then there exists a solution $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$ of (1) with maximal time of existence $T \in (0,\infty]$, so that the following conditions are fulfilled:

(i) *For every $T_0 \in (0,T)$ there exist $\varepsilon_0 \in (0,1)$ and solutions $u_\varepsilon \in C^0(\bar{\Omega} \times [0,T_0]) \cap C^{2,1}(\Omega \times (0,T_0])$ of problem (9) for all $\varepsilon \in (0,\varepsilon_0)$ so that $u_\varepsilon(x,t) \searrow u(x,t)$ for $\varepsilon \searrow 0$ and all $(x,t) \in \bar{\Omega} \times [0,T_0]$.*

(ii) *u is a maximal solution of problem (1). So if $v \in C^0(\bar{\Omega} \times [0,T_1)) \cap C^{2,1}(\Omega \times (0,T_1))$ is another solution of (1) evolving from u_0, then we have $v \leq u$ in $\bar{\Omega} \times [0, \min\{T, T_1\})$.*

If $T < \infty$ is fulfilled, then we also have $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$.

Proof. From the proof of Theorem 2.1 we get $T_1 > 0$ and a solution $u \in C^0(\bar{\Omega} \times [0, T_1]) \cap C^{2,1}(\Omega \times (0, T_1])$ of (1), so that condition (i) is fulfilled for all $T_0 \in (0, T_1]$ with $\varepsilon_0 = 1$.
We assume that T_1 is not the maximal time of existence of u and that for every $\varepsilon \in (0, 1]$ there is a solution $u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_1]) \cap C^{2,1}(\Omega \times (0, T_1])$ of problem (9) so that condition (i) is fulfilled for T_1. Hence there is $C > 0$ so that $0 \le u \le u_\varepsilon \le C$ in $\bar{\Omega} \times [0, T_1]$ because $u_1 \in C^0(\bar{\Omega} \times [0, T_1])$ and $u_1 \ge u_\varepsilon \ge u \ge 0$ by Corollary 1.2 and the proof of Theorem 2.1. So with $M := 2C + 1$ we get, similar as in the proof of Theorem 2.1, a family of solutions $v_{M,\varepsilon} \in C^0(\bar{\Omega} \times [0, T_\varepsilon)) \cap C^{2,1}(\Omega \times (0, T_\varepsilon))$ of problem (8) for $\varepsilon \in (0, 1)$ where T_ε is the maximal time of existence of $v_{M,\varepsilon}$. Then $\rho_\varepsilon(u_1) = (u_1)^p$ and $\varphi_{s,M,\varepsilon}(u_1) = (u_1)^s$ in $\bar{\Omega} \times [0, T_1]$ for all $\varepsilon \in (0, 1)$ because $1 \le u_1 \le C \le M$ in $\bar{\Omega} \times [0, T_1]$. Hence we get $v_{M,\varepsilon} \le u_1 \le C$ in $\bar{\Omega} \times [0, \min\{T_1, T_\varepsilon\}]$ for all $\varepsilon \in (0, 1)$ by Corollary 1.2. So we have $T_1 < T_\varepsilon$ for all $\varepsilon \in (0, 1)$ because T_ε is the maximal existence time of $v_{M,\varepsilon}$. Hence $v_\varepsilon := v_{M,\varepsilon}$ is in fact a solution of (9) in $\Omega \times (0, T_1]$ since $\varepsilon \le v_\varepsilon \le u_1 \le C \le M$ in $\Omega \times (0, T_1]$. So for $\varepsilon_0 \in (0, 1)$ there is $\hat{T}_{\varepsilon_0} \in (T_1, T_{\varepsilon_0})$ so that $v_{\varepsilon_0} \le M$ in $\bar{\Omega} \times [0, \hat{T}_{\varepsilon_0}]$. Hence we get like in the proof of Theorem 2.1 a function $v \in C^0(\bar{\Omega} \times [0, \hat{T}_{\varepsilon_0}]) \cap C^{2,1}(\Omega \times (0, \hat{T}_{\varepsilon_0}])$ with $v_\varepsilon(x, t) \searrow v(x, t)$ for $\varepsilon \searrow 0$ and all $(x, t) \in \bar{\Omega} \times [0, \hat{T}_{\varepsilon_0}]$. Since $T_1 < \hat{T}_{\varepsilon_0}$ we have $v \le u_\varepsilon$ and $u \le v_\varepsilon$ in $\bar{\Omega} \times [0, T_1]$ for all $\varepsilon \in (0, \varepsilon_0)$ by Corollary 1.2. Hence $v \le u$ and $u \le v$ in $\bar{\Omega} \times [0, T_1]$ with $\varepsilon \searrow 0$. So we have continued u to $\bar{\Omega} \times [0, \hat{T}_{\varepsilon_0}]$ with $\hat{T}_{\varepsilon_0} > T_1$ and condition (i) is fulfilled for all $T_0 \in (0, \hat{T}_{\varepsilon_0})$. In the same way we can continue u until we reach the maximal time of existence T of u. So we have constructed a solution u of (1) that fulfills condition (i).
We now assume that the maximal time of existence T is finite and $u \le M_1$ in $\bar{\Omega} \times [0, T)$. Then there is $T_2 > 0$ so that every solution $u_{M,1}$ of problem (8) with $\varepsilon = 1$ and $M := 2M_1 + 1$ fulfills $u_{M,1} \le M$ in $\bar{\Omega} \times [0, T_2]$ whenever we have $\|u_0\|_{L^\infty(\Omega)} \le M_1$ (which is remarked in the proof of Theorem 2.1). Hence we get like in the proof of Theorem 2.1 solutions $v_{M,\varepsilon} \in C^0(\bar{\Omega} \times [0, T_2]) \cap C^{2,1}(\Omega \times (0, T_2])$ of (8) for all $\varepsilon \in (0, 1)$ with $v_0(x) = v(x, 0) = u(x, \hat{T})$ for $x \in \bar{\Omega}$ with $\hat{T} := \max\{T - \frac{T_2}{2}, \frac{T}{2}\}$. Like in the proof of Theorem 2.1 we also have a solution $v \in C^0(\bar{\Omega} \times [0, T_2]) \cap C^{2,1}(\Omega \times (0, T_2])$ of (1) so that $v_{M,\varepsilon}(x, t) \searrow v(x, t)$ for $\varepsilon \searrow 0$ and all $(x, t) \in \bar{\Omega} \times [0, T_2]$. We now choose $T_0 \in (\hat{T}, T)$ and define $w(x, t) := u(x, t + \hat{T})$ and $w_\varepsilon(x, t) := u_\varepsilon(x, t + \hat{T})$ for $(x, t) \in \bar{\Omega} \times [0, T_0 - \hat{T})$ and $\varepsilon \in (0, \varepsilon_0)$ where u_ε and ε_0 are chosen like in condition (i) which has already been proved for u. Since $T_0 - \hat{T} \le \frac{T_2}{2}$ we have $w < v_\varepsilon$ and $v < w_\varepsilon$ on the parabolic boundary $\Sigma_{T_0 - \hat{T}}$ for $\varepsilon \in (0, \varepsilon_0)$. Hence we have $w \le v_\varepsilon$ and $v \le w_\varepsilon$ in $\bar{\Omega} \times [0, T_0 - \hat{T})$ for $\varepsilon \in (0, \varepsilon_0)$ by Corollary 1.2. So we get $v = w$ in $\bar{\Omega} \times [0, T - \hat{T})$ with $\varepsilon \searrow 0$ because $T_0 \in (\hat{T}, T)$ was arbitrary. We define $\tilde{u} := u$ in $\bar{\Omega} \times [0; \hat{T}]$ and $\tilde{u}(x, t) := v(x, t - \hat{T})$ for $(x, t) \in \bar{\Omega} \times (\hat{T}, \hat{T} + T_2)$. Then $\tilde{u} \in C^0(\bar{\Omega} \times [0, \hat{T} + T_2)) \cap C^{2,1}(\Omega \times (0, \hat{T} + T_2))$ is a solution of (1) that fulfills $\tilde{u} = u$ in $\bar{\Omega} \times [0, T)$. Hence $\tilde{u}$ is a continuation of u and T is not the maximal time of existence of u because $\hat{T} + T_2 > T$. This is a contradiction and so we have proved that $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$ if T is finite.
To prove condition (ii) for this u we assume that $w \in C^0(\bar{\Omega} \times [0, T_1)) \cap C^{2,1}(\Omega \times (0, T_1))$ is another solution of (1) evolving from u_0. We choose $T_0 \in (0, \min\{T_1, T\})$. Because u fulfills condition (i), there are u_ε for all $\varepsilon \in (0, \varepsilon_0)$ that fulfill the condition (i) for T_0. Since $u_\varepsilon = v + \varepsilon$ on the parabolic boundary Σ_{T_0} and u_ε is a solution of (9) we get $v \le u_\varepsilon$ in $\bar{\Omega} \times [0, T_0]$ by Corollary 1.2 for all $\varepsilon \in (0, \varepsilon_0)$. Hence we have $v \le u$ in $\bar{\Omega} \times [0, T_0]$ for $\varepsilon \searrow 0$.

So the assertion is proved. ∎

From now on, let u be the maximal solution of problem (1) in $\Omega \times (0,T)$ that we get in Lemma 2.2 where $T>0$ is chosen so that $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$ and T is the maximal time of existence of u.

We now show with the help of Lemma 2.2 that the maximal solution of (1) is positive in $\Omega \times [0,T)$.

Corollary 2.3
Assume that assumptions (2), (3) and (4) are fulfilled and u is the maximal solution of (1) that we get in Lemma 2.2. Then for every $K \subset\subset \Omega$ and every $T_0 \in (0,T] \cap (0,\infty)$ there exists a constant $c_{K,T_0} > 0$, so that $u(x,t) \geq c_{K,T_0}$ for all $(x,t) \in K \times [0,T_0)$. In particular, we have $u>0$ in $\Omega \times [0,T)$.

Proof. Let $K \subset\subset \Omega$, $G \subset\subset \Omega$ be an open interval with $K \subset\subset G$ and $\Theta(x)$ be a principal eigenfunction of $-\Delta$ in G with $0 < \Theta \leq 1$ in G like in the proof of Theorem 2.1. Then we can choose $c_0 > 0$ with $u_0 \geq c_0$ in $\bar{G}$ since $u_0 > 0$ in Ω and $G \subset\subset \Omega$. We define $v(x,t) := c_0 e^{-\gamma t}\Theta(x)$ for $(x,t) \in \bar{G} \times [0,T)$ with γ chosen so that $\gamma > \lambda_1 (c_0)^p > 0$ where λ_1 is the principal eigenvalue of $-\Delta$ in G.
Now let $T_0 \in (0,T)$. By Lemma 2.2 there are $\varepsilon_0 > 0$ and solutions u_ε of (9) for $\varepsilon \in (0,\varepsilon_0)$ with $u_\varepsilon(x,t) \searrow u(x,t)$ for $\varepsilon \searrow 0$ and for all $(x,t) \in \bar{G} \times [0,T_0]$. We can show like in the proof of Theorem 2.1 that $u_\varepsilon \geq v$ in $\bar{G} \times [0,T_0]$ for all $\varepsilon \in (0,\varepsilon_0)$. So we get $u \geq v$ in $\bar{G} \times [0,T_0]$ with $\varepsilon \searrow 0$. Hence we have $u \geq v$ in $\bar{G} \times [0,T)$ because $T_0 \in (0,T)$ was arbitrary. Hence the assertion follows since $\Theta > 0$ in G and $K \subset\subset G$. ∎

In the next lemma we prove with the help of Theorem 1.5 that the maximal solution u of (1) is smooth in $\Omega \times (0,T)$.

Lemma 2.4
Let assumptions (2), (3) and (4) be fulfilled and $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$ be the maximal solution of problem (1) that we get in Lemma 2.2.
Then we have $u \in C^\infty(\Omega \times (0,T))$ and $u_x \in C^0(\Omega \times [0,T))$.

Proof. Let $K \subset\subset \Omega$ be a compact interval and $T_0 \in (0,T)$. Then there is $c_{(K,T_0)} > 0$ so that $u \geq c_{(K,T_0)}$ in $K \times [0,T_0)$ by Corollary 2.3.
Let $M \geq c_{(K,T_0)} + 1$ so that $u \leq M < \infty$ in $\bar{\Omega} \times [0,T_0]$. M exists because $u \in C^0(\bar{\Omega} \times [0,T_0])$.
Let $\rho_\varepsilon(t)$ and $\varphi_{s,M,\varepsilon}(t)$ for $\varepsilon > 0$ and $s \geq 0$ be defined as in the proof of Theorem 2.1. Then u fulfills

$$u_t \;=\; \rho_{c_{(K,T_0)}}(u)u_{xx} + \varphi_{q,M,c_{(K,T_0)}}(u) + \kappa\varphi_{r,M,c_{(K,T_0)}}(u)(u_x)^2$$

in $K \times (0,T_0)$.
Let $\varepsilon_0 \in (0,1)$ and u_ε for $\varepsilon \in (0,\varepsilon_0)$ be like in Lemma 2.2. For $0 < t_0 < t_1 < T_0$ there is a constant $B > 0$ so that $\|u_\varepsilon\|_{C^{2+\beta,1+\frac{\beta}{2}}(K \times [t_0,t_1])} \leq B \;\forall \varepsilon \in (0,\varepsilon_0)$ (like in the proof of

Theorem 2.1). Since $u_\varepsilon \to u$ in $C^{2,1}(K\times[t_0,t_1])$ for $\varepsilon \searrow 0$ we have $u \in C^{2+\beta,1+\frac{\beta}{2}}(K\times[t_0,t_1])$. Hence we get $u \in C^{2+\beta,1+\frac{\beta}{2}}(K \times (0,T_0))$ because t_0 and t_1 were arbitrary.
So we have $u \in C^\infty(K \times (0,T_0))$ and $u_x \in C^0(K \times [0,T_0))$ by Theorem 1.5 (with $\Omega := K$, $a_{11}(x,t,u,\hat{p}) := \rho_{c_{(K,T_0)}}(u)$, $-a(x,t,u,\hat{p}) := \varphi_{q,M,c_{(K,T_0)}}(u) + \kappa\varphi_{r,M,c_{(K,T_0)}}(u)\hat{p}^2$ and $\Psi(x,t) := u(x,t)$ for $(x,t) \in K \times [0,T)$ and $u,\hat{p} \in \mathbb{R}$ in Theorem 1.5). Hence the assertion follows since $K \subset\subset \Omega$ and $T_0 \in (0,T)$ were arbitrary. ∎

In the next lemma we show another useful property of the maximal solution u of (1). If u_0 is symmetric with maximum in $x = 0$ then u is also symmetric with respect to x and $\max\limits_{x\in\Omega} u(x,t) = u(0,t)$ for all $t \in [0,T)$.

Lemma 2.5
Let assumption (2) *be fulfilled,* $a > 0$*,* $\Omega := (-a,a)$*,* $u_0 \in C^3(\bar{\Omega})$ *with* $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$*,* $(u_0)_x(x) \leq 0$ $\forall x \in (0,a)$ *and* $u_0(a) = u_0(-a) = 0$ *and let* u *be the maximal solution of* (1) *with maximal time of existence* T *that we get in Lemma* 2.2.
Then $u(x,t) = u(-x,t)$ *for all* $(x,t) \in \bar{\Omega} \times [0,T)$ *and* $u_x \leq 0$ *in* $(0,a) \times (0,T)$.

Proof. We define $v(x,t) := u(-x,t)$ for $(x,t) \in \bar{\Omega} \times [0,T)$.
Let $T_0 \in (0,T)$, $\varepsilon \in (0,\varepsilon_0)$ and u_ε be like in Lemma 2.2.
Then we have $v(x,0) = u(x,0) < u_\varepsilon(x,0)$ $\forall x \in \bar{\Omega}$ because u_0 is symmetric. We also have $0 = v(x,t) = u(x,t) < u_\varepsilon(x,t)$ for $t \in [0,T_0]$ and $x \in \{a,-a\}$. Hence we get $v < u_\varepsilon$ on Σ_{T_0}. Since $v_t = v^p v_{xx} + v^q + \kappa v^r v_x^2$ in $(-a,a) \times (0,T_0)$ we get $v \leq u_\varepsilon$ in $[-a,a] \times [0,T_0)$ by Corollary 1.2.
As $\varepsilon \in (0,\varepsilon_0)$ was arbitrary, we get $v \leq u$ in $[-a,a] \times [0,T_0)$ with $\varepsilon \searrow 0$. Hence we have $v \leq u$ in $[-a,a] \times [0,T)$ because $T_0 \in (0,T)$ was arbitrary.
So we get for $(x,t) \in [-a,a] \times [0,T)$:
$u(-x,t) = v(x,t) \leq u(x,t) = v(-x,t) \leq u(-x,t)$ and we have proved the first part of the assertion.
Now we have $u_x \in C^0([0,a) \times [0,T)) \cap C^{2,1}((0,a) \times (0,T))$ by Lemma 2.4 and $w := u_x$ fulfills in $(0,a) \times (0,T)$:
$w_t = u^p w_{xx} + pu^{p-1} w w_x + qu^{q-1} w + \kappa r u^{r-1} w^3 + 2\kappa u^r w w_x$
where we consider u as fixed function which is given. Hence we get $u_x \leq 0$ in $(0,a) \times (0,T)$ by Theorem 1.1 since $u_x \leq 0$ on the parabolic boundary of $(0,a)\times(0,T)$ by our assumptions on u_0, because u is symmetric and because $u > 0$ in $\Omega\times(0,T)$ by Corollary 2.3 and $u|_{\partial\Omega} = 0$. So the assertion is proved. ∎

We now prove that the maximal solution u of (1) fulfills $u_t \geq 0$ in $\Omega \times (0,T)$, whenever the conditions of the last lemma are fulfilled with $p + 1 \leq q$ and the initial data u_0 have a special property.

Lemma 2.6
Suppose $p+1 \leq q$ *so that assumption* (2) *is fulfilled,* $a > 0$, $\Omega = (-a,a)$, $u_0 \in C^3(\bar{\Omega})$ *with* $u_0(x) = u_0(-x) > 0 \ \forall x \in \Omega$, $(u_0)_x(x) \leq 0 \ \forall x \in (0,a)$, $u_0(a) = u_0(-a) = 0$ *and* $(u_0)_{xx} + (u_0)^{q-p} \geq 0$ *in* $\bar{\Omega}$.
Then the maximal solution u *of* (1) *that we get in Lemma* 2.2 *fulfills* $u_t \geq 0$ *in* $\Omega \times (0,T)$.

Proof. First we construct a different approximating sequence of u from the one in Lemma 2.2, because we need better properties than u_ε in Lemma 2.2 have.
Let $T_0 \in (0,T)$, $\varepsilon_0 > 0$ and u_ε for $\varepsilon \in (0,\varepsilon_0]$ be defined as in Lemma 2.2.
Let $M := 2\|u_{\varepsilon_0}\|_{L^\infty(\Omega\times[0,T_0])} + 1$, $\varepsilon \in (0,\varepsilon_0)$, $f_\varepsilon(t) := \varepsilon + t[\varepsilon^p(u_0)_{xx}(a) + \varepsilon^q + \kappa\varepsilon^r((u_0)_x(a))^2]$ for $t \geq 0$, and let $\rho_\varepsilon(t)$ and $\varphi_{s,\varepsilon}(t)$ be defined as in the proof of Theorem 2.1.
As in the proof of Theorem 2.1, there is a maximal time of existence $T^\varepsilon > 0$ and $u^\varepsilon \in C^0(\bar{\Omega} \times [0,T^\varepsilon)) \cap C^{2,1}(\Omega \times (0,T^\varepsilon))$, so that u^ε is a solution of

$$(u^\varepsilon)_t \;=\; \rho_\varepsilon(u^\varepsilon)(u^\varepsilon)_{xx} + \varphi_{q,M,\varepsilon}(u^\varepsilon) + \kappa\varphi_{r,M,\varepsilon}(u^\varepsilon)(u^\varepsilon)_x^2 \quad \text{in} \quad \Omega \times (0,T^\varepsilon)$$

with $u^\varepsilon(x,t) = u_0(x) + f_\varepsilon(t)$ on the parabolic boundary Σ_{T^ε}.
Since $f_\varepsilon(t) \geq \varepsilon$ for $t \geq 0$ (because $(u_0)_{xx}(a) \geq -(u_0)^{q-p}(a) = 0$), we have $u^\varepsilon \geq \varepsilon$ on the parabolic boundary Σ_{T^ε}. Hence we get $u^\varepsilon \geq \varepsilon$ in $\Omega \times (0,T^\varepsilon)$ by Theorem 1.1.
Like in the proof of Lemma 2.2 we also get $\hat{T} > T_0$ so that $u^\varepsilon \leq M$ in $\bar{\Omega} \times [0,\hat{T}]$ for all $\varepsilon \in (0,\varepsilon_1)$, where $\varepsilon_1 \in (0,\frac{\varepsilon_0}{2})$ is chosen so that $f_{\varepsilon_1}(t) < \varepsilon_0$ for $t \in [0,T^{\varepsilon_0/2}]$.
So in fact u^ε fulfills in $\Omega \times (0,\hat{T})$

$$(u^\varepsilon)_t \;=\; (u^\varepsilon)^p(u^\varepsilon)_{xx} + (u^\varepsilon)^q + \kappa(u^\varepsilon)^r((u^\varepsilon)_x)^2$$

Similar to the proof of Theorem 2.1 we can show that there is $v \in C^0(\bar{\Omega} \times [0,\hat{T}]) \cap C^{2,1}(\Omega \times (0,\hat{T}])$ so that v is a solution of (1) in $\Omega \times (0,\hat{T})$ with $v|_{t=0} = u_0$ so that $u^\varepsilon \to v$ in $C^{2,1}(\bar{K} \times [t_0,\hat{T}])$ for $\varepsilon \searrow 0$ and for all $K \subset\subset \Omega$ and all $t_0 \in (0,\hat{T})$.
Let u be the maximal solution of (1) with $u|_{t=0} = u_0$ that we get in Lemma 2.2.
Since $T_0 < \hat{T}$ and $f_\varepsilon(t) \geq \varepsilon$ for $t \geq 0$ we have $u^\varepsilon > u$ on Σ_{T_0} for $\varepsilon \in (0,\varepsilon_1)$. So by Corollary 1.2 we get $u^\varepsilon \geq u$ in $\Omega \times (0,T_0)$ for all $\varepsilon \in (0,\varepsilon_1)$. Hence we get with $\varepsilon \searrow 0$:
$v \geq u$ in $\Omega \times (0,T_0)$.
Since u is a maximal solution of (1) by Lemma 2.2, we have in fact $v = u$ in $\Omega \times (0,T_0)$.
So we have constructed another approximating sequence of u.
Since u is symmetric by Lemma 2.5 and also u^ε is symmetric with a similar proof, we get for $\varepsilon \in (0,\varepsilon_1)$:

$$\begin{aligned} ((u^\varepsilon)^p(u^\varepsilon)_{xx} + (u^\varepsilon)^q + \kappa(u^\varepsilon)^r((u^\varepsilon)_x)^2)(\pm a,0) &= \varepsilon^p(u_0)_{xx}(a) + \varepsilon^q + \kappa\varepsilon^r((u_0)_x(a))^2 \\ &= f'_\varepsilon(0) \end{aligned}$$

Hence u^ε fulfills the first compatibility condition and so $u^\varepsilon \in C^{2,1}(\bar{\Omega} \times [0,T_0))$ by Theorem 1.3 for $\varepsilon \in (0,\varepsilon_1)$.

We now show $(u^\varepsilon)_t \geq 0$ in $\Omega \times [0, T_0)$. Let $\varepsilon \in (0, \varepsilon_1)$ and $w(x,t) := (u^\varepsilon)_t(x,t)$ for $t \in [-a, a] \times [0, T_0)$. Because of the regularity of u^ε we have for $x \in [-a, a]$

$$\begin{aligned} w(x,0) &= (u_0(x)+\varepsilon)^p((u_0)_{xx}(x) + (u_0(x)+\varepsilon)^{q-p} + \kappa(u_0(x)+\varepsilon)^{r-p}((u_0)_x(x))^2) \\ &\geq (u_0(x)+\varepsilon)^p((u_0)_{xx}(x) + (u_0(x))^{q-p}) \geq 0 \end{aligned}$$

because of the assumptions on u_0 and since $q - p \geq 1 > 0$.
We also have $w(\pm a, t) \geq 0$ for $t \in [0, T)$ because $w(\pm a, t) = f'_\varepsilon(t) \geq \varepsilon^q \geq 0$, since $(u_0)_{xx}(a) \geq -(u_0)^{q-p}(a) = 0$ by our assumption on u_0.
In $(-a, a) \times (0, T_0)$ we get

$$\begin{aligned} w_t &= (u^\varepsilon)_{tt} = ((u^\varepsilon)^p(u^\varepsilon)_{xx} + (u^\varepsilon)^q + \kappa(u^\varepsilon)^r((u^\varepsilon)_x)^2)_t \\ &= p(u^\varepsilon)^{p-1}(u^\varepsilon)_{xx}w + (u^\varepsilon)^p w_{xx} + q(u^\varepsilon)^{q-1}w + \kappa r(u^\varepsilon)^{r-1}((u^\varepsilon)_x)^2 w + 2\kappa(u^\varepsilon)^r(u^\varepsilon)_x w_x \end{aligned}$$

since $u^\varepsilon \in C^\infty(\Omega \times (0, T_0))$ by Theorem 1.5 (with a similar proof as in Lemma 2.4) where the coefficients of this parabolic equation are in $C^0(\bar{\Omega} \times [0, T))$ because of the regularity of u^ε.
So by Theorem 1.1 we have $(u^\varepsilon)_t \geq 0$ in $\Omega \times (0, T_0)$ for $\varepsilon \in (0, \varepsilon_1)$.
Hence the assertion follows with $\varepsilon \searrow 0$, since $u^\varepsilon \to u$ in $C^{2,1}(\bar{K} \times [t_0, T_0])$ for each $K \subset\subset \Omega$ and each $t_0 \in (0, T_0)$ and since $T_0 \in (0, T)$ was arbitrary. ∎

The following remark shows that there exist initial data that fulfill the conditions of the preceding lemma if $p + 1 = q$ and $a > \frac{\pi}{2}$ or if $p + 1 < q$ and $a > 0$, and these cases are analysed in Section 5.

Remark 2.7
Let $a > 0$ and $\Omega = (-a, a)$. In case of $p + 1 = q$ with $a \geq \frac{\pi}{2}$ and in case of $p + 1 < q$ there exists $u_0 \in C^3(\bar{\Omega})$ that fulfills $u_0(x) = u_0(-x) > 0\ \forall x \in \Omega$, $(u_0)_x(x) \leq 0\ \forall x \in (0, a)$, $u_0(a) = u_0(-a) = 0$ and $(u_0)_{xx} + (u_0)^{q-p} \geq 0$ in $\bar{\Omega}$.

Proof. In case of $p + 1 = q$ and $a \geq \frac{\pi}{2}$ we define $u_0(x) := \cos(\frac{\pi}{2a}x)$ for $x \in \bar{\Omega}$. It can easily be verified that u_0 fulfills the conditions given above because

$$(u_0)_{xx}(x) = -\frac{\pi^2}{4a^2}u_0(x) \geq -u_0(x)$$

for all $x \in \bar{\Omega}$ due to $a \geq \frac{\pi}{2}$.

In case of $p+1 < q$ and arbitrary $a > 0$ let $\beta \geq (\frac{64}{a^2})^{\frac{1}{q-p-1}} e^2$. Then we define $u_0(x) := \beta e^{\frac{2x^2}{x^2-a^2}}$ for $x \in (-a, a)$ and $u_0(\pm a) = 0$. Hence we have $u \in C^0(\bar{\Omega}) \cap C^3(\Omega)$ with $u_0(x) = u_0(-x) > 0\ \forall x \in \Omega$ and $u_0(a) = u_0(-a) = 0$. Furthermore we get for $x \in (-a, a)$

$$\begin{aligned}(u_0)_x(x) &= -\beta\frac{4a^2x}{(x^2-a^2)^2}e^{\frac{2x^2}{x^2-a^2}}\\ (u_0)_{xx}(x) &= 4a^2\beta\frac{3x^4+2a^2x^2-a^4}{(x^2-a^2)^4}e^{\frac{2x^2}{x^2-a^2}}\\ (u_0)_{xxx}(x) &= 16a^2\beta x^3\frac{-3x^4-6a^2x^2+5a^4}{(x^2-a^2)^6}e^{\frac{2x^2}{x^2-a^2}}\end{aligned}$$

So we have $u_0 \in C^3(\bar{\Omega})$ with $(u_0)_x(\pm a) = (u_0)_{xx}(\pm a) = (u_0)_{xxx}(\pm a) = 0$ and $(u_0)_x(x) \leq 0$ for $x \in (0, a)$.
Finally we have to show that $(u_0)_{xx}+(u_0)^{q-p} \geq 0$ is fulfilled in $\bar{\Omega}$. This inequality is fulfilled for $x = \pm a$ because $(u_0)_{xx}(\pm a) = u_0(\pm a) = 0$. Moreover we get for $|x| \in [\frac{a}{\sqrt{2}}, a)$

$$\begin{aligned}(u_0)_{xx}(x)+(u_0)^{q-p}(x) &\geq (u_0)_{xx}(x)\\ &\geq 4a^2\beta\frac{2a^2x^2-a^4}{(x^2-a^2)^4}e^{\frac{2x^2}{x^2-a^2}} \geq 0\end{aligned}$$

due to $x^2 \geq \frac{a^2}{2}$. Furthermore we have for $|x| \leq \frac{a}{\sqrt{2}}$

$$\begin{aligned}(u_0)_{xx}(x)+(u_0)^{q-p}(x) &= \beta e^{\frac{2x^2}{x^2-a^2}}(4a^2\frac{3x^4+2a^2x^2-a^4}{(x^2-a^2)^4}+(\beta e^{\frac{2x^2}{x^2-a^2}})^{q-p-1})\\ &\geq \beta e^{\frac{2x^2}{x^2-a^2}}(4a^2\frac{-a^4}{(x^2-a^2)^4}+\beta^{q-p-1}e^{(q-p-1)\frac{2\frac{a^2}{2}}{\frac{a^2}{2}-a^2}})\\ &\geq \beta e^{\frac{2x^2}{x^2-a^2}}(\frac{-4a^6}{(\frac{a^2}{2}-a^2)^4}+\beta^{q-p-1}e^{-2(q-p-1)})\\ &= \beta e^{\frac{2x^2}{x^2-a^2}}(\frac{-64}{a^2}+\beta^{q-p-1}e^{-2(q-p-1)}) \geq 0\end{aligned}$$

due to the choice of β and because $q-p > 1$. Hence u_0 fulfills the conditions given above and the assertion is proved. ■

The following lemma is a consequence of the Sturmian theorem. We remark that the assertion of the following lemma is not true for $x_0 = 0$ which we can see with the help of Lemma 2.5 if u_0 is symmetric.

Lemma 2.8
Let $a > 0$, $\Omega := (-a, a)$, assumptions (2) and (4) be fulfilled and u be the maximal solution of problem (1) evolving from u_0 that we get in Lemma 2.2. Then for all $t_0 \in (0, T)$ and all $x_0 \in \Omega$, $x_0 \neq 0$, the equality $u_x(x_0, t) = 0$ holds for at most a finite number of times $t \in (t_0, T)$.

Proof. We fix $t_0 \in (0,T)$ and assume that there were $x_0 \neq 0$ and infinitely many $t_k \in (t_0,T)$ so that $u_x(x_0,t_k) = 0$ for all $k \in \mathbb{N}$. We may assume without loss of generality $x_0 \in (0,a)$.
Let $G := (2x_0 - a, a)$, $v(x,t) := u(2x_0 - x, t)$ and $z(x,t) := u(x,t) - v(x,t)$ for $(x,t) \in G \times [t_0,T)$.
Then we have $z(2x_0 - a, t) = u(2x_0 - a, t) - u(a,t) = u(2x_0 - a, t) > 0$ and $z(a,t) = u(a,t) - u(2x_0 - a, t) = -u(2x_0 - a, t) < 0$ for all $t \in [t_0,T)$, since $u(a,t) = 0$ and $u(x,t) > 0$ for all $t \in (0,T)$ by Corollary 2.3.
We claim that the number of zeroes of $z(t)$ in G is finite and nonincreasing on (t_0,T) and drops whenever $z(t)$ has a multiple zero.
Because, for all $s_0 \in (t_0,T)$, z is continuous in $\bar{G} \times [t_0,s_0]$, we get a subdomain $G' \subset\subset G$ with $z \neq 0$ in $(\bar{G} \setminus G') \times [t_0,s_0]$ since $z \neq 0$ on $\partial G \times [t_0,T)$.
We also have $z \in C^\infty(\bar{G}' \times [t_0,s_0])$ by Lemma 2.4 and

$$\begin{aligned}
z_t &= u^p u_{xx} + u^q + \kappa u^r u_x^2 - v^p v_{xx} - v^q - \kappa v^r v_x^2 \\
&= u^p z_{xx} + u^{q-1} z + \kappa(u^r u_x + v^r v_x) z_x + (u^p - v^p) v_{xx} + (u^{q-1} - v^{q-1}) v \\
&\quad + \kappa u_x v_x (u^r - v^r) \\
&= u^p z_{xx} + \kappa(u^r u_x + v^r v_x) z_x + u^{q-1} z + v_{xx} \Big(\int_0^1 p(v + \tau(u-v))^{p-1} d\tau\Big) z \\
&\quad + v\Big(\int_0^1 (q-1)(v + \tau(u-v))^{q-2} d\tau\Big) z + \kappa u_x v_x \Big(\int_0^1 r(v+\tau(u-v))^{r-1} d\tau\Big) z \\
&=: A(x,t) z_{xx} + B(x,t) z_x + C(x,t) z
\end{aligned}$$

in $G' \times (t_0,s_0)$ with coefficients $A, B, C \in C^\infty(\bar{G}' \times [t_0,s_0])$ (since $u, v \in C^\infty(\bar{G}' \times [t_0,s_0])$ by Lemma 2.4 and $u, v > 0$ in $\bar{G}' \times [t_0,s_0]$ by Corollary 2.3). So Theorem 1.9 asserts that the number $N(t;G')$ of zeroes in G' has the properties just stated. By construction of G' we have $N(t;G) = N(t;G')$ for all $t \in [t_0,s_0]$, so we have proved the claim because $s_0 \in (t_0,T)$ was arbitrary.
By definition of z we have $z(x_0,t) = 0$ for all $t \in [t_0,T)$ and by assumption $z_x(x_0,t_k) = 2u_x(x_0,t_k) = 0$ for all $k \in \mathbb{N}$, so that $z(t_k)$ has a multiple zero for all $k \in \mathbb{N}$. But the claim given above implies that the $\mathbb{N}$-valued function $N(\cdot,G)$ drops by at least one at each $t = t_k$ which is a contradiction. ■

3 Some previous results concerning degenerate diffusion equations without gradient terms

In this section we present some previous results concerning the following problem:

$$\begin{aligned} v_t &= v^p \Delta v + v^q \quad \text{in} \quad \Omega \times (0,T) \\ v|_{\partial\Omega} &= 0 \\ v|_{t=0} &= v_0 \end{aligned} \tag{10}$$

where

$$p > 0 \quad \text{and} \quad q > 0 \tag{11}$$

are fixed parameters,

$$\Omega \subset \mathbb{R}^n \qquad \text{is a bounded domain with smooth boundary} \tag{12}$$

and

$$v_0 \in C^0(\bar{\Omega}) \qquad \text{is positive in } \Omega \quad \text{and} \quad v_0|_{\partial\Omega} = 0 \tag{13}$$

The solution v shall fulfill $v \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$.

This problem was studied intensively during the last ten years. It was shown that the properties of the solution v depend on the ratio of $p+1$ and q. We present some results concerning the question if the solution v of (10) exists globally for all $t > 0$ or if blow-up in finite time occurs. In the first case we also give results concerning the asymptotic behaviour of v for $t \to \infty$. In the second case we present some results concerning the question if single point blow-up or regional blow-up occurs.
We first consider the case $p+1 > q$. In this case the solution v of (10) always exists globally. A proof of the next theorem can be found in [Wie2].

Theorem 3.1
If assumptions (11), (12) *and* (13) *are fulfilled with* $q < p+1$ *and the initial data* $v_0 \in C^1(\bar{\Omega})$ *fulfill* $0 < c_0 \le v_0(x) dist(x, \partial\Omega)^{-1} \le c_1$ *in* Ω, *the problem* (10) *has a global solution* $v \in C^0(\bar{\Omega} \times [0,\infty))$, *which is positive and smooth in* $\Omega \times (0,\infty)$. *This solution is unique in the class of nonnegative functions* $w \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T))$, $T > 0$.
Furthermore, independent of v_0, $\lim\limits_{t\to\infty} \max\limits_{x\in\bar{\Omega}} |v(x,t) - W(x)| = 0$, *where* W *denotes the unique positive solution of* $\Delta W = -W^{q-p}$ *in* Ω *with* $W = 0$ *on* $\partial\Omega$.

Next we consider the case $p+1 = q$, where the size of the principal eigenvalue λ_1 of the Laplacian in Ω plays an important role. If $\lambda_1 > 1$, the solution v of (10) always exists globally and tends to zero for $t \to \infty$. A proof of the following theorem can be found in [Wie2].

Theorem 3.2
If assumptions (11)*,* (12) *and* (13) *are fulfilled with* $p+1=q$*,* $\lambda_1 > 1$ *and the initial data* $v_0 \in C^1(\bar{\Omega})$ *fulfill* $0 < c_0 \le v_0(x) dist(x, \partial\Omega)^{-1} \le c_1$ *in* Ω*, there exists a unique solution* v *of problem* (10)*, which is positive in* $\Omega \times (0,\infty)$ *and tends uniformly to zero. More precisely, independent of* v_0*,* $\lim_{t\to\infty} \max_{x\in\bar{\Omega}} |v(x,t)(1+pt)^{\frac{1}{p}} - W(x)| = 0$*, where* $W > 0$ *in* Ω*,* $\Delta W + W = -W^{1-p}$ *in* Ω*,* $W|_{\partial\Omega} = 0$*.*

If $p+1=q$ and $\lambda_1 = 1$ the solution v of (10) also exists globally, but tends to a stationary solution of (10) for $t \to \infty$. This stationary solutions are eigenfunctions of $-\Delta$ in Ω with eigenvalue λ_1.
We have to give a few definitions before we present the next theorem. Let Θ be the principal eigenfunction with $\Delta\Theta + \lambda_1\Theta = 0$ in Ω, $\Theta|_{\partial\Omega} = 0$, $0 \le \Theta \le 1$ in Ω and $\max_{x\in\bar{\Omega}} \Theta(x) = 1$. There are some constants d_0 and d_1 with $0 < d_0 \le \Theta(x) dist(x,\partial\Omega)^{-1} \le d_1$ in Ω and $\Theta \in C^\infty(\bar{\Omega})$ because Ω is a smoothly bounded domain. Let
$\omega(v_0) := \{w \in L^2(\Omega) |\ \exists (t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $t_k \to \infty$ and $v(t_k) \to w$ in $L^2(\Omega)$ for $k \to \infty\}$ denote the ω - limit set.
A proof of the following theorem can be found in [Wie2].

Theorem 3.3
If assumptions (11)*,* (12) *and* (13) *are fulfilled with* $p+1=q$*,* $\lambda_1 = 1$ *and the initial data* $v_0 \in C^1(\bar{\Omega})$ *fulfill* $0 < c_0 \le v_0(x) dist(x,\partial\Omega)^{-1} \le c_1$ *in* Ω*, there is a unique global solution* v *of problem* (10)*, and we have the estimate* $\frac{c_0}{d_1}\Theta(x) \le v(x,t) \le \frac{c_1}{d_0}\Theta(x)$ *in* $\bar{\Omega} \times (0,\infty)$*.*
Furthermore we have $\omega(v_0) \subset \{\gamma\Theta \mid \gamma \in \mathbb{R},\ \frac{c_0}{d_1} \le \gamma \le \frac{c_1}{d_0}\}$*.*

Now we consider the case $p+1=q$ with $\lambda_1 < 1$, where we have to distinguish between $p < 2$ and $p \ge 2$. We have a finite blow-up time instead of a global solution in the former cases. We define the monotone blow-up set $S_\star := \{x \in \Omega \mid \limsup_{t\nearrow T} v(x,t) = \infty\}$ and the blow-up set $S := \{x \in \bar{\Omega} \mid \exists ((x_k,t_k))_{k\in\mathbb{N}} \subset \Omega \times (0,T)$ such that $x_k \to x$ and $v(x_k,t_k) \to \infty$ as $k \to \infty\}$
where $T \in (0,\infty)$ is a finite blow-up time with $\limsup_{t\nearrow T} \|v(t)\|_{L^\infty(\Omega)} = \infty$.
A proof of the next theorem can be found in [Win2].

Theorem 3.4
If assumptions (11)*,* (12) *and* (13) *are fulfilled with* $p+1=q$*,* $p < 2$ *and* $\lambda_1 < 1$*, there is a uniquely determined maximal existence time* $T < \infty$ *and a unique function*
$v \in C^0(\bar{\Omega} \times [0,T)) \cap C^\infty(\Omega \times (0,T))$ *fulfilling* $v > 0$ *in* $\Omega \times [0,T)$ *and solving* (10) *in the classical sense.*
Furthermore we have $|S_\star| > 0$ *and there is* $C_0 > 0$ *with* $\frac{p^{-\frac{1}{p}}}{(T-t)^{\frac{1}{p}}} \le \|v(t)\|_{L^\infty(\Omega)} \le \frac{C_0}{(T-t)^{\frac{1}{p}}}$ *for all* $t \in (0,T)$*.*

We now define the fast blow-up set

$$S_f := \{x \in \bar{\Omega} \quad | \quad \exists ((x_k,t_k))_{k\in\mathbb{N}} \subset \Omega \times (0,T) \quad \text{such that} \quad x_k \to x \quad \text{and} \\ (T-t_k)^{\frac{1}{p}} v(x_k,t_k) \to \infty \quad \text{as} \quad k \to \infty\}$$

If $p+1 = q$, $\lambda_1 > 1$ and $p \geq 2$ we also have blow-up in finite time and we do not only have $|S| > 0$. We even have that the fast blow-up set fulfills $|S_f| > 0$ (by definition of S_f we easily see that $S_f \subset S$). A proof of the next theorem for the case $p \geq 2$ can be found in [Win3].

Theorem 3.5
If assumptions (12) *and* (13) *are fulfilled with* $p+1 = q$, $\lambda_1 < 1$ *and* $p \geq 2$, *there is a uniquely determined maximal existence time* $T < \infty$ *and a unique function* $v \in C^0(\bar{\Omega} \times [0,T)) \cap C^\infty(\Omega \times (0,T))$ *fulfilling* $v > 0$ *in* $\Omega \times [0,T)$ *and solving* (10) *in the classical sense.*
Furthermore we have $\limsup_{t \nearrow T} (T-t)^{\frac{1}{p}} \|v(t)\|_{L^\infty(\Omega)} = \infty$ *and* $|S_f| > 0$.

Last we consider the case $p+1 < q$ where the behaviour of the solution v of (10) is again different from the former cases. If the initial data v_0 are small then v exists globally and if v_0 is large enough then v blows up in finite time. We give a theorem which is similar to Theorem 5.1 in [Win1].

Theorem 3.6
Let assumptions (12) *and* (13) *be fulfilled with* $p+1 < q$ *and* $p \geq 1$. *Then there is a classical solution* v *of* (10) *and we have:*

(i) *There exists a one-parameter family* $(w_a)_{a>0}$ *of radially symmetric positive functions* $w_a(x) \in C^0(\mathbb{R}^n)$ *vanishing at infinity with* $w_a(0) = a$ *such that whenever* $v_0 \leq w_a$ *in* Ω *then the corresponding solution* v *of* (10) *exists globally and obeys the decay estimate* $\|v(t)\|_{L^\infty(\Omega)} \leq c(1+t)^{-1/(q-1)}$.

(ii) *For each* $w \in C^0(\bar{\Omega})$ *with* $w > 0$ *in* Ω *and* $w = 0$ *on* $\partial\Omega$ *there is* $b_0 > 0$ *such that if* $v_0 = bw$ *with* $b \geq b_0$, *then any classical solution* v *of* (10) *evolving from* v_0 *blows up in finite time.*

Proof. We can show similarly to the proof of Lemma 2.1 in [Win1] that for every $v_0 \in C^0(\bar{\Omega})$ with $v_0 > 0$ in Ω and $v_0 = 0$ on $\partial\Omega$ a solution v of (10) exists.
To prove (i) let $a > 0$ and let $w_a \in C^0(\mathbb{R}^n)$ vanishing at infinity with $w(0) = a$ be the radially symmetric function of Theorem 5.1 (i) in [Win1]. We also assume $v_0 \leq w_a$ in Ω. By Theorem 5.1 (i) in [Win1] we know that there exists $w \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ with $w_t = w^p \Delta w + w^q$ in $\mathbb{R}^n \times (0,\infty)$, $w(x,0) = w_a(x)$ for $x \in \mathbb{R}^n$, $w > 0$ in $\mathbb{R}^n \times [0,\infty)$ and $\|w(t)\|_{L^\infty(\Omega)} \leq c(1+t)^{\frac{-1}{q-1}}$ for $t \geq 0$.

In particular w fulfills $w_t = w^p \Delta w + w^q$ in $\Omega \times (0,\infty)$ with $w \geq v$ on the parabolic boundary of $\Omega \times (0,T)$, where v is the solution of (10) evolving from v_0 and T is the maximal existence time of v. So we have $0 \leq v \leq w$ in $\Omega \times (0,T)$ by Theorem 1.1 (where we use $p,q \geq 1$). Hence we must have $T = \infty$, since T is the maximal existence time of v and $w \in L^\infty(\Omega \times (0,t))$ for every $t \in (0,\infty)$ because $\|w(s)\|_{L^\infty(\Omega)} \leq c(1+s)^{\frac{-1}{q-1}}$ for $s \geq 0$. Hence part (i) is proved.
To prove (ii), let $w \in C^0(\bar{\Omega})$ with $w > 0$ in Ω and $w = 0$ on $\partial\Omega$.
Since $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, we can prove in the same way as in the proof of Theorem 5.1 (ii) in [Win1] that there exist $c_0, c_2 > 0$, so that the corresponding solution of (10) with $v_0 = bw$ blows up in finite time, whenever b fulfills

$$\left(\frac{1}{p-1}\int_\Omega w^{1-p}\Theta\right)b^{1-p} \;\leq\; c_0$$

in the case $p > 1$ or

$$-\int_\Omega \ln(bw)\Theta \;\leq\; c_2$$

in the case $p = 1$,
where in both cases $\Theta \geq 0$ is an eigenfunction of $-\Delta$ in Ω corresponding to the principal eigenvalue λ_1 of Ω with $\int_\Omega \Theta = 1$.
So in both cases there exists $b_0 > 0$ so that the solution v of (10) evolving from $v_0 = bw$ blows up in finite time for every $b \geq b_0$. ■

We now give a result concerning the blow-up profile of the solution v of (10), if $p+1 = q$, $\Omega = (-a,a)$ with $\lambda_1 < 1$ and v_0 is nonincreasing on $[0,a]$. The blow-up profile L of v is defined as $L(x) := \lim\limits_{t \nearrow T} \frac{v(x,t)}{\max_{y\in\bar{\Omega}} v(y,t)}$ for $x \in \bar{\Omega}$ where T is the blow-up time of v, if this limit exists. In our case L exists and is a well-known function. A proof of the following theorem can be found in [Win3]. An analogous theorem was already proved before in [Wie3] with more assumptions on the initial data v_0.

Theorem 3.7
Let $p+1 = q$, $p \geq 2$, $a > \frac{\pi}{2}$ and $\Omega := (-a,a)$ and $v_0 \in C^0(\bar{\Omega})$ so that v_0 is nonincreasing in $[0,a]$ with $v_0 > 0$ in $(-a,a)$, $v_0(\pm a) = 0$ and $v_0(x) = v_0(-x)$ for $x \in [-a,a]$.
Then the corresponding solution v of (10) blows up in finite time T, $L(x) := \lim\limits_{t\nearrow T} \frac{v(x,t)}{v(0,t)}$ exists for $x \in [-a,a]$ and we get independent of v_0

(i) $\lim\limits_{t\nearrow T} \frac{1}{pv(0,t)^p(T-t)} = 0$

(ii) $L(x) := \begin{cases} \cos(x) & , \quad \text{for} \quad |x| \leq \frac{\pi}{2} \\ 0 & , \quad \text{for} \quad \frac{\pi}{2} < |x| \leq a \end{cases}$

(iii) *The blow up set S is given by $\bar{S} = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and we have $S = S_f$.*

4 The case $r = p - 1$ in $u_t = u^p u_{xx} + u^q + \kappa u^r u_x^2$

In this section we study solutions of (1) with $r = p - 1$. With the help of the following lemma, we can transform a solution of (1) to a solution of (10), so we can get some properties of the solutions of (1) with the help of the last section.
In this section u shall denote a solution of (1), but u does not have to be the maximal solution of (1) that we get in Lemma 2.2.

Lemma 4.1
Let assumptions (2) and (4) be fulfilled with $r = p - 1$, $\Omega = (a, b)$ and let $u \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))$ be a solution of (1) evolving from u_0. Then $v(y, s) := u^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s)$ fulfills

$$v_s = v^{\frac{p}{\kappa+1}} v_{yy} + v^{\frac{q+\kappa}{\kappa+1}} \quad \text{in} \quad G \times (0, (\kappa+1)T) \quad \text{with} \quad v|_{\partial G} = 0 \quad \text{and} \quad v|_{s=0} = v_0$$

where $G := (\sqrt{\kappa+1}\, a, \sqrt{\kappa+1}\, b)$ and $v_0(y) := (u_0)^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}} y)$ for $y \in \bar{G}$.

Proof. Let $v(y, s) := u^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s)$ for $(y, s) \in \bar{G} \times [0, (\kappa+1)T)$ with G defined as above. Then we get for $(y, s) \in G \times (0, (\kappa+1)T)$

$$\begin{aligned} v_s(y, s) &= u^\kappa(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) u_t(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) \\ v_y(y, s) &= \sqrt{\kappa+1} u^\kappa(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) u_x(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) \\ v_{yy}(y, s) &= (u^\kappa u_{xx} + \kappa u^{\kappa-1} (u_x)^2)(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) \end{aligned}$$

and thus we have

$$\begin{aligned} (v_s - v^{\frac{p}{\kappa+1}} v_{yy} - v^{\frac{q+\kappa}{\kappa+1}})(y, s) &= (u^\kappa u_t - u^p (u^\kappa u_{xx} + \kappa u^{\kappa-1} (u_x)^2) - u^{q+\kappa})(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) \\ &= (u^\kappa (u_t - u^p u_{xx} - u^q - \kappa u^{p-1} (u_x)^2))(\frac{1}{\sqrt{\kappa+1}} y, \frac{1}{\kappa+1} s) \\ &= 0 \end{aligned}$$

in $G \times (0, (\kappa+1)T)$ since u is a solution of (1) in $\Omega \times (0, T)$ with $r = p - 1$.
In addition we get for $y \in \bar{G}$ and $s \in (0, (\kappa+1)T)$

$$\begin{aligned} v(\sqrt{\kappa+1}\, a, s) &= u^{\kappa+1}(a, \frac{1}{\kappa+1} s) = 0 \\ v(\sqrt{\kappa+1}\, b, s) &= u^{\kappa+1}(b, \frac{1}{\kappa+1} s) = 0 \\ v(y, 0) &= u^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}} y, 0) = (u_0)^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}} y) \end{aligned}$$

Hence the assertion is proved. ■

Now we can give some properties of the solutions of (1) in case of $r = p - 1$. For the rest of this section let v, v_0 and G be as in the previous lemma.
First we analyse for different values of $p + 1$ and q, if a solution u of (1) exists globally or if we have blow-up in finite time. In the case of global existence we also give some results concerning the asymptotic behaviour of u for $t \to \infty$ and in the case of blow-up we analyse if single point blow-up or regional blow-up occurs. The following corollaries correspond to the theorems given in the last section.

Corollary 4.2
Let assumption (2) be fulfilled, $r = p - 1$, $p + 1 > q$, $\Omega = (a, b)$ and $u_0 \in C^0(\bar{\Omega})$ so that $(u_0)^{\kappa+1} \in C^1(\bar{\Omega})$ with $0 < c_0 \leq (u_0)^{\kappa+1}(x) dist(x, \partial\Omega)^{-1} \leq c_1 < \infty$ in Ω, $u_0 > 0$ in Ω and $u_0|_{\partial\Omega} = 0$. Then there is a unique global solution $u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))$ of problem (1).
Furthermore, independent of u_0, $\lim_{t\to\infty} \max_{x\in\bar{\Omega}} |u^{\kappa+1}(x, t) - W(\sqrt{\kappa + 1}\, x)| = 0$, where W denotes the unique positive solution of $W_{yy} = -W^{\frac{q+\kappa-p}{\kappa+1}}$ in $G := (\sqrt{\kappa + 1}\, a, \sqrt{\kappa + 1}\, b)$ with $W = 0$ on ∂G.

Proof. Since $p + 1 > q$ we also have $\frac{p}{\kappa+1} + 1 > \frac{q+\kappa}{\kappa+1}$.
From $c_0 \leq (u_0)^{\kappa+1}(x) dist(x, \partial\Omega)^{-1} \leq c_1$ in Ω we get $\frac{c_0}{\sqrt{\kappa+1}} \leq v_0(y) dist(y, \partial G)^{-1} \leq \frac{c_1}{\sqrt{\kappa+1}}$ in G, because $dist(y, \partial G) = \sqrt{\kappa + 1}\, dist(\frac{1}{\sqrt{\kappa+1}} y, \partial\Omega)$.
So a global solution v of $v_s = v^{\frac{p}{\kappa+1}} v_{yy} + v^{\frac{q+\kappa}{\kappa+1}}$ in $G \times (0, \infty)$ with $v|_{\partial G} = 0$ and $v|_{s=0} = v_0$ exists and this solution is unique and positive in $G \times (0, \infty)$ by Theorem 3.1. Hence we can show like in Lemma 4.1 that $u(x, t) := v^{\frac{1}{\kappa+1}}(\sqrt{\kappa + 1}\, x, (\kappa + 1)t)$ for $(x, t) \in \Omega \times [0, \infty)$ is a solution of (1) with $u|_{t=0} = u_0$.
From the definition of v, we get for W defined as in the assertion and for $t \in (0, \infty)$

$$\begin{aligned}\max_{x\in\bar{\Omega}} |u^{\kappa+1}(x, t) - W(\sqrt{\kappa + 1}\, x)| &= \max_{x\in\bar{\Omega}} |v(\sqrt{\kappa + 1}\, x, (\kappa + 1)t) - W(\sqrt{\kappa + 1}\, x)| \\ &= \max_{y\in\bar{G}} |v(y, (\kappa + 1)t) - W(y)|\end{aligned}$$

So the assertion follows by Lemma 4.1 and Theorem 3.1. ■

We now consider the case $p+1 = q$, where $\Omega = (a, b)$ is an interval. We remark that in this case the solution u of (1) blows up for $b - a > \frac{\pi}{\sqrt{\kappa+1}}$, whereas the solution v of (10) only blows up for $b - a > \pi$ (which is equivalent to the condition that the principal eigenvalue λ_1 of Ω fulfills $\lambda_1 > 1$) by the theorems given in the last section. Furthermore we always get regional blow-up for solutions u of (1) in case of $b - a > \frac{\pi}{\sqrt{\kappa+1}}$.

Corollary 4.3
Let assumption (2) *be fulfilled,* $r = p-1$, $p+1 = q$, $\Omega = (a,b)$ *with* $b - a < \frac{\pi}{\sqrt{\kappa+1}}$ *and* $u_0 \in C^0(\bar\Omega)$ *so that* $(u_0)^{\kappa+1} \in C^1(\bar\Omega)$ *with* $0 < c_0 \le (u_0)^{\kappa+1}(x) dist(x, \partial\Omega)^{-1} \le c_1 < \infty$ *in* Ω, $u_0 > 0$ *in* Ω *and* $u_0|_{\partial\Omega} = 0$. *Then there is a unique global solution* u *of* (1) *which tends uniformly to zero. More precisely, independent of* u_0,
$\lim_{t\to\infty} \max_{x\in\bar\Omega} |u^{\kappa+1}(x,t)(1+pt)^{\frac{\kappa+1}{p}} - W(\sqrt{\kappa+1}\, x)| = 0$, *where* $W > 0$ *in* $G := (\sqrt{\kappa+1}\, a, \sqrt{\kappa+1}\, b)$, $W_{yy} + W = -W^{1-\frac{p}{\kappa+1}}$ *in* G *and* $W|_{\partial G} = 0$.

Proof. Since $p+1 = q$ we also have $\frac{p}{\kappa+1} + 1 = \frac{q+\kappa}{\kappa+1}$ and the principal eigenvalue $\lambda_1(G)$ of the Laplacian in G fulfills $\lambda_1(G) > 1$ due to $b - a < \frac{\pi}{\sqrt{\kappa+1}}$. Like in the proof of Corollary 4.2 we get $\frac{c_0}{\sqrt{\kappa+1}} \le v_0(y) dist(y, \partial G)^{-1} \le \frac{c_1}{\sqrt{\kappa+1}}$ in G.
So a global solution v of $v_s = v^{\frac{p}{\kappa+1}} v_{yy} + v^{\frac{q+\kappa}{\kappa+1}}$ in $G \times (0,\infty)$ with $v|_{\partial G} = 0$ and $v|_{s=0} = v_0$ exists and this solution is unique and positive in $G \times (0,\infty)$ by Theorem 3.2. Hence we can show like in Lemma 4.1 that $u(x,t) := v^{\frac{1}{\kappa+1}}(\sqrt{\kappa+1}\, x, (\kappa+1)t)$ for $(x,t) \in \Omega \times [0,\infty)$ is a solution of (1) with $u|_{t=0} = u_0$.
For $t \in (0,\infty)$ we get by the definition of v for W defined as in the assertion

$$\begin{aligned} & \max_{x\in\bar\Omega} |u^{\kappa+1}(x,t)(1+pt)^{\frac{\kappa+1}{p}} - W(\sqrt{\kappa+1}\, x)| \\ = & \max_{x\in\bar\Omega} |v(\sqrt{\kappa+1}\, x, (\kappa+1)t)(1 + \frac{p}{\kappa+1}(\kappa+1)t)^{\frac{\kappa+1}{p}} - W(\sqrt{\kappa+1}\, x)| \\ = & \max_{y\in\bar G} |v(y, (\kappa+1)t)(1 + \frac{p}{\kappa+1}(\kappa+1)t)^{\frac{\kappa+1}{p}} - W(y)| \end{aligned}$$

Hence the assertion follows by Lemma 4.1 and Theorem 3.2. ■

Corollary 4.4
Suppose assumption (2) *is fulfilled,* $r = p-1$, $p+1 = q$, $\Omega = (a,b)$ *with* $b - a = \frac{\pi}{\sqrt{\kappa+1}}$ *and* $u_0 \in C^0(\bar\Omega)$ *so that* $(u_0)^{\kappa+1} \in C^1(\bar\Omega)$ *with* $0 < c_0 \le (u_0)^{\kappa+1}(x) dist(x, \partial\Omega)^{-1} \le c_1 < \infty$ *in* Ω, $u_0 > 0$ *in* Ω *and* $u_0|_{\partial\Omega} = 0$. *Then there is a unique global solution* u *of* (1), *and we have the estimate*

$$\frac{c_0}{d_1\sqrt{\kappa+1}}\Theta(\sqrt{\kappa+1}\, x) \;\le\; u^{\kappa+1}(x,t) \le \frac{c_1}{d_0\sqrt{\kappa+1}}\Theta(\sqrt{\kappa+1}\, x) \quad \text{in} \quad \bar\Omega \times (0,\infty)$$

where $\Theta(y) = \sin(y - \sqrt{\kappa+1}\ a)$ *for* $y \in G := (\sqrt{\kappa+1}\ a, \sqrt{\kappa+1}\ b)$ *is the principal eigenfunction of the Laplacian in* G *with* $0 \le \Theta \le 1$ *in* G *and* $d_0, d_1 \in (0,\infty)$ *are some constants with* $0 < d_0 \le \Theta(y) dist(y, \partial G)^{-1} \le d_1$ *in* G.

Proof. Since $p+1 = q$ we also have $\frac{p}{\kappa+1} + 1 = \frac{q+\kappa}{\kappa+1}$ and the principal eigenvalue $\lambda_1(G)$ of the Laplacian in G fulfills $\lambda_1(G) = 1$ due to $b - a = \frac{\pi}{\sqrt{\kappa+1}}$. Like in the proof of Corollary 4.2 we get $\frac{c_0}{\sqrt{\kappa+1}} \le v_0(y) dist(y, \partial G)^{-1} \le \frac{c_1}{\sqrt{\kappa+1}}$ in G.

So a global solution v of $v_s = v^{\frac{p}{\kappa+1}} v_{yy} + v^{\frac{q+\kappa}{\kappa+1}}$ in $G \times (0, \infty)$ with $v|_{\partial G} = 0$ and $v|_{s=0} = v_0$ exists and this solution is unique and positive in $G \times (0, \infty)$ by Theorem 3.3. Hence we can show like in Lemma 4.1 that $u(x,t) := v^{\frac{1}{\kappa+1}}(\sqrt{\kappa+1}\, x, (\kappa+1)t)$ for $(x,t) \in \Omega \times [0, \infty)$ is a solution of (1) with $u|_{t=0} = u_0$. Furthermore we have

$$\frac{c_0}{d_1\sqrt{\kappa+1}}\Theta(\sqrt{\kappa+1}\, x) \quad \le \quad v(\sqrt{\kappa+1}\, x, (\kappa+1)t) \le \frac{c_1}{d_0\sqrt{\kappa+1}}\Theta(\sqrt{\kappa+1}\, x)$$

for $(x,t) \in \bar{\Omega} \times (0, \infty)$ by Theorem 3.3. Hence the assertion follows by Lemma 4.1 since $v(\sqrt{\kappa+1}\, x, (\kappa+1)t) = u^{\kappa+1}(x,t)$. ■

Corollary 4.5
Suppose $r = p - 1$, $p + 1 = q$, $1 \le p < 2(\kappa+1)$, $\kappa > 0$, $\Omega = (a,b)$, $b - a > \frac{\pi}{\sqrt{\kappa+1}}$ and assumption (4) is fulfilled. Then there is a uniquely determined maximal existence time $T < \infty$ and a unique function $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^\infty(\Omega \times (0,T))$ fulfilling $u > 0$ in $\Omega \times [0,T)$ and solving (1) in the classical sense.
Furthermore we have $|S_\star| > 0$ and there is $C > 0$ with $\frac{p^{-\frac{1}{p}}}{(T-t)^{\frac{1}{p}}} \le \|u(t)\|_{L^\infty(\Omega)} \le \frac{C}{(T-t)^{\frac{1}{p}}}$ for all $t \in (0,T)$.

Proof. Since $p + 1 = q$ we also have $\frac{p}{\kappa+1} + 1 = \frac{q+\kappa}{\kappa+1}$, $\frac{p}{\kappa+1} < 2$ due to $p < 2(\kappa+1)$ and the principal eigenvalue $\lambda_1(G)$ of the Laplacian in G fulfills $\lambda_1(G) < 1$ due to $b - a > \frac{\pi}{\sqrt{\kappa+1}}$.
So by Lemma 4.1 and Theorem 3.4 there is a uniquely determined maximal existence time $(\kappa+1)T < \infty$ and a unique solution v of (10) with v_0 as in Lemma 4.1, which is positive and smooth in $G \times (0, (\kappa+1)T)$. Hence the first part of the assertion follows by Lemma 4.1.
By Theorem 3.4 the monotone blow-up set $\tilde{S}_\star$ of v fulfills $|\tilde{S}_\star| > 0$. So we get for the monotone blow-up set $S_\star$ of u that $|S_\star| = \frac{1}{\sqrt{\kappa+1}}|\tilde{S}_\star| > 0$ by Lemma 4.1.
By Lemma 4.1 and Theorem 3.4 we also have $C_0 > 0$ with

$$\frac{(\frac{p}{\kappa+1})^{-\frac{\kappa+1}{p}}}{((\kappa+1)(T-t))^{\frac{\kappa+1}{p}}} \quad \le \quad \|v((\kappa+1)t)\|_{L^\infty(G)} = \|u(t)\|^{\kappa+1}_{L^\infty(\Omega)} \le \frac{C_0}{((\kappa+1)(T-t))^{\frac{\kappa+1}{p}}}$$

for $t \in (0,T)$. So the assertion follows with $C := C_0^{\frac{1}{\kappa+1}}\, (\kappa+1)^{\frac{-1}{p}}$. ■

Corollary 4.6
Assume $r = p-1$, $p+1 = q$, $p \ge 2(\kappa+1)$, $\kappa > 0$, $\Omega = (a,b)$, $b - a > \frac{\pi}{\sqrt{\kappa+1}}$ and assumption (4) is fulfilled. Then there is a uniquely determined maximal existence time $T < \infty$ and a unique function $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^\infty(\Omega \times (0,T))$ fulfilling $u > 0$ in $\Omega \times [0,T)$ and solving (1) in the classical sense.
Furthermore we have $\limsup_{t \nearrow T} (T-t)^{\frac{1}{p}} \|u(t)\|_{L^\infty(\Omega)} = \infty$ and $|S_f| > 0$.

Proof. Since $\frac{p}{\kappa+1} + 1 = \frac{q+\kappa}{\kappa+1}$, $\frac{p}{\kappa+1} \geq 2$ and the principal eigenvalue $\lambda_1(G)$ of the Laplacian in G fulfills $\lambda_1(G) < 1$ due to $b - a > \frac{\pi}{\sqrt{\kappa+1}}$, there is by Lemma 4.1 and Theorem 3.5 a uniquely determined maximal existence time $(\kappa+1)T < \infty$ and a unique solution v of (10) with v_0 as in Lemma 4.1, which is positive and smooth in $G \times (0, (\kappa+1)T)$. So the first part of the assertion follows by Lemma 4.1.
By Lemma 4.1, we get for $(x,t) \in \Omega \times (0,T)$

$$(T-t)^{\frac{1}{p}} u(x,t) \quad = \quad (\kappa+1)^{\frac{-1}{p}} [((\kappa+1)T - (\kappa+1)t)^{\frac{\kappa+1}{p}} v(\sqrt{\kappa+1}\, x, (\kappa+1)t)]^{\frac{1}{\kappa+1}}$$

Hence the assertion follows by Theorem 3.5 and the preceding equation, because the fast blow-up set S_f of u fulfills $|S_f| = \frac{1}{\sqrt{\kappa+1}} |\tilde{S}_f|$ by Lemma 4.1, where $\tilde{S}_f$ is the fast blow-up set of v. ■

Corollary 4.7
Suppose $r = p - 1$, $p + 1 < q$, $p \geq \kappa + 1$, $\kappa > 0$, $\Omega = (a,b)$ and assumption (4) is fulfilled. Then there is a classical solution u of (1) and we have:

(i) *There exists a one-parameter family $(\tilde{w}_a)_{a>0}$ of radially symmetric positive functions $\tilde{w}_a(x) \in C^0(\mathbb{R})$ vanishing at infinity with $\tilde{w}_a(0) = a$ such that whenever $u_0 \leq \tilde{w}_a$ in Ω then the corresponding maximal solution u of (1) exists globally and obeys the decay estimate*

$$\|u(t)\|_{L^\infty(\Omega)} \leq c(1 + (\kappa+1)t)^{-1/(q-1)}$$

(ii) *For each $w \in C^3(\bar{\Omega})$ with $w > 0$ in Ω and $w = 0$ on $\partial\Omega$ there is $b_0 > 0$ so that if $u_0 = bw$ with $b \geq b_0$ then any positive classical solution u of (1) evolving from u_0 blows up in finite time.*

Proof. We have $\frac{p}{\kappa+1} + 1 < \frac{q+\kappa}{\kappa+1}$ and $\frac{p}{\kappa+1} \geq 1$ due to our assumptions on p and q. Then (ii) and the first part of (i) follow by Lemma 4.1 and Theorem 3.6. We also have the following estimate for $t > 0$ by Lemma 4.1 and Theorem 3.6, with $\tilde{w}_a(x) := (w_{a^{\kappa+1}}(\sqrt{\kappa+1}\, x))^{\frac{1}{\kappa+1}}$ for $x \in \mathbb{R}$ (where $w_{a^{\kappa+1}}$ is defined as in Theorem 3.6), because $v_0 \leq w_{a^{\kappa+1}}$ in G if $u_0 \leq \tilde{w}_a$ in Ω. By the definition of v we get

$$\|u(t)\|_{L^\infty(\Omega)} \quad = \quad \|v((\kappa+1)t)\|_{L^\infty(G)}^{\frac{1}{\kappa+1}} \leq c(1 + (\kappa+1)t)^{\frac{-1}{(\kappa+1)(\frac{q+\kappa}{\kappa+1} - 1)}} = c(1 + (\kappa+1)t)^{\frac{-1}{q-1}}$$

where c is chosen suitably. So the assertion follows. ■

We now study the blow-up profile of a solution u of (1) in a special case similar to Theorem 3.7. We remark that if we vary the parameter κ, the size of the blow-up set and the arguments of the cos - function in the blow-up profile vary, and also the exponent of the cos - function in the blow-up profile varies so that the blow-up profile changes very much for variable κ.

Corollary 4.8
Let $r = p-1$, $p+1 = q$, $p \geq 2(\kappa+1)$, $\kappa > 0$, $a > \frac{\pi}{2\sqrt{\kappa+1}}$, $\Omega := (-a,a)$ and $u_0 \in C^3(\bar{\Omega})$ so that u_0 is nonincreasing in $[0,a]$ with $u_0 > 0$ in $(-a,a)$, $u_0(\pm a) = 0$ and $u_0(x) = u_0(-x)$ for $x \in [-a,a]$.
Then the corresponding solution u of (1) blows up in finite time T, $L(x) := \lim\limits_{t \nearrow T} \frac{u(x,t)}{u(0,t)}$ exists for $x \in [-a,a]$ and we get independent of u_0

(i) $\lim\limits_{t \nearrow T} \frac{1}{pu(0,t)^p(T-t)} = 0$

(ii) $L(x) := \begin{cases} (\cos(\sqrt{\kappa+1}\,x))^{\frac{1}{\kappa+1}} & , \ \textit{for} \ |x| \leq \frac{\pi}{2\sqrt{\kappa+1}} \\ 0 & , \ \textit{for} \ \frac{\pi}{2\sqrt{\kappa+1}} < |x| \leq a \end{cases}$

(iii) *The blow-up set S is given by $\bar{S} = [-\frac{\pi}{2\sqrt{\kappa+1}}, \frac{\pi}{2\sqrt{\kappa+1}}]$.*

Proof. Let v and v_0 be as in Lemma 4.1. Then v_0 fulfills $v_0(y) = (u_0)^{\kappa+1}(\frac{1}{\sqrt{\kappa+1}}y)$ for $y \in \bar{G}$.
So due to the assumptions on u_0, v_0 fulfills the assumptions of Theorem 3.7. We also have $\frac{p}{\kappa+1} + 1 = \frac{q+\kappa}{\kappa+1}$ and $G = (-\sqrt{\kappa+1}\,a, \sqrt{\kappa+1}\,a)$ with $\sqrt{\kappa+1}\,a > \frac{\pi}{2}$. Hence a solution u of (1) exists and the assertion follows by Lemma 3.1 and Theorem 3.7, where v has blow-up time $(\kappa+1)T$ and we use for (i) the equation

$$\frac{p}{\kappa+1} v(0,(\kappa+1)t)^{\frac{p}{\kappa+1}}((\kappa+1)T - (\kappa+1)t) \quad = \quad pu^p(0,t)(T-t)$$

■

5 The blow-up set of solutions to $u_t = u^p u_{xx} + u^q + \kappa u^r u_x^2$

In this section we consider the maximal solution u of problem (1) evolving from $u_0 \in C^3(\bar{\Omega})$ that we get in Lemma 2.2, where $T \in (0,\infty]$ is suitably chosen, so that T is the maximal time of existence of u.
We first formulate a short lemma that shows once again why the results in Section 3 are very interesting for our studies of the solutions of (1). In particular it follows from the lemma that if v is a solution of (10) evolving from u_0 and v blows up in finite time, then u blows up in finite time, too, where u is the maximal solution of (1) evolving from u_0 as described above.

Lemma 5.1
Let assumptions (2)*,* (3) *and* (4) *be fulfilled, u be the maximal solution of* (1) *evolving from u_0 and v be a solution of* (10) *evolving from $v_0 := u_0 \in C^3(\bar{\Omega})$. Then $v \le u$ in $\bar{\Omega} \times [0,T)$, where T is the minimum of the maximal existence times of u and v.*

Proof. Let $T_0 \in (0,T)$, $\varepsilon \in (0,\varepsilon_0)$ and let u_ε be the solution of (9) that we get in Lemma 2.2. Then $u_\varepsilon \ge \varepsilon$ in $\bar{\Omega} \times [0,T_0)$. So we have $(u_\varepsilon)_t \ge (u_\varepsilon)^p (u_\varepsilon)_{xx} + (u_\varepsilon)^q$ in $\Omega \times (0,T_0)$. Hence, by Corollary 1.2, we have $u_\varepsilon \ge v$ in $\bar{\Omega} \times [0,T_1)$ where T_1 is the minimum of T_0 and the maximal existence time of v, since $u_\varepsilon \ge v$ on Σ_{T_1}.
Since $\varepsilon \in (0,\varepsilon_0)$ was arbitrary, $u_\varepsilon(x,t) \searrow u(x,t)$ for $\varepsilon \searrow 0$ and all $(x,t) \in \Omega \times (0,T_1)$ and $v = u$ on Σ_{T_1} the assertion follows because $T_0 \in (0,T)$ was arbitrary. ■

We restrict to the case that $\Omega = (-a,a)$ for $a > 0$ and u_0 is symmetric, monotonely decreasing for $x > 0$ and has its maximum in $x = 0$. We consider different cases for p, q and r so that we know that the maximal solution u of (1) blows up in finite time and we examine if u only blows up in a single point or if the blow-up set S fulfills $|S| > 0$, which is called regional blow-up. In the case of regional blow-up we can also give an estimate from below for the size of the blow-up set for some special cases.
We now consider the case $p + 1 < q$ with u_0 large enough so that the maximal solution u of (1) blows up in finite time. We will see that the size of the blow-up set depends on the size of r.
First we give a lemma to show that if u_0 fulfills the conditions of the next two theorems except for the condition that the corresponding maximal solution u of (1) blows up in finite time, there is $b \ge 1$ so that bu_0 fulfills all conditions of the next two theorems, especially the corresponding solution u of (1) with $u|_{t=0} = bu_0$ blows up in finite time.

Lemma 5.2
Suppose assumption (2) *is fulfilled with $p + 1 < q$, $a > 0$, $\Omega := (-a,a)$ and $u_0 \in C^3(\bar{\Omega})$ with $u_0(x) = u_0(-x)$ $\forall x \in \Omega$, $(u_0)_x(x) \le 0$ $\forall x \in (0,a)$, $u_0(a) = u_0(-a) = 0$ and $(u_0)_{xx} + (u_0)^{q-p} \ge 0$ in $\bar{\Omega}$.*
Then there is $b_1 \ge 1$, so that bu_0 also fulfills these conditions and the corresponding maximal solution u of (1) *with $u|_{t=0} = bu_0$ blows up in finite time for all $b \ge b_1$.*

PROOF. Let $b \geq 1$. Then $bu_0 \in C^3(\bar{\Omega})$ has the following properties due the assumptions on u_0:
$bu_0(x) = bu_0(-x)\ \forall x \in \Omega$, $(bu_0)_x(x) \leq 0\ \forall x \in (0,a)$, $bu_0(a) = bu_0(-a) = 0$ and

$$(bu_0)_{xx} + (bu_0)^{q-p} \geq -b(u_0)^{q-p} + b^{q-p}(u_0)^{q-p} \geq -b(u_0)^{q-p} + b(u_0)^{q-p} \geq 0$$

in $\bar{\Omega}$ due to $b \geq 1$, $q-p > 1$ and the assumptions on u_0.
So bu_0 also fulfills the conditions that we assume for u_0.
From Theorem 3.6 we know that there exists $b_0 > 0$, so that the corresponding solution v of (10) with $v_0 = bu_0$ blows up in finite time for every $b \geq b_0$. Then also the corresponding solution u of (1) with $u|_{t=0} = bu_0$ blows up in finite time for every $b \geq b_0$ by Lemma 5.1. Hence the assertion follows with $b_1 := \max\{b_0, 1\}$. ■

In the next theorem we consider the case $p+1 < q$ with $r < q-2$. In this case we can show that the only blow-up point of u is $x = 0$, if u_0 fulfills the conditions of the last lemma so that u blows up in finite time. The proof of the next theorem is done according to an idea demonstrated in [FMcL1]. This idea is to show that the function $J(x,t) := u_x(x,t) + c(x)u^\alpha(x,t)$ fulfills $J \leq 0$ in $I \times [t_0, T)$ if we assume that there is a blow-up point $x \neq 0$, where $I \subset \Omega$ is a suitably chosen interval, T is the blow-up time of u and $c(x)$, α and t_0 are suitably chosen. This leads to a contradiction.

Theorem 5.3
Let assumption (2) *be fulfilled with* $p+1 < q$, $r < q-2$, $a > 0$, $\Omega := (-a,a)$, $u_0 \in C^3(\bar{\Omega})$ *with* $u_0(x) = u_0(-x) > 0\ \forall x \in \Omega$, $(u_0)_x(x) \leq 0\ \forall x \in (0,a)$, $u_0(a) = u_0(-a) = 0$, $(u_0)_{xx} + (u_0)^{q-p} \geq 0$ *in* $\bar{\Omega}$, *let* u *be the maximal solution of* (1) *evolving from* u_0 *and suppose* u_0 *is large enough so that there exists* $T \in (0,\infty)$ *with* $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$.
Then $x = 0$ *is the only blow-up point of* u.

PROOF. By Lemma 5.2 we know that if u_0 fulfills the conditions of our assertion except for the condition that the corresponding solution u of (1) blows up in finite time, there is $b_1 \geq 1$ so that bu_0 fulfills all conditions of our assertion for $b \geq b_1$, especially the corresponding solution u of (1) with $u|_{t=0} = bu_0$ blows up in finite time for $b \geq b_1$.
Hence we can choose u_0 large enough, so that the corresponding solution of (1) blows up with blow-up time $T \in (0,\infty)$, so that $u \in C^0(\bar{\Omega} \times [0,T))$ and $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$.
From Lemma 2.5 we know that $u(0,t) = \max_{x \in \bar{\Omega}} u(x,t)$ for all $t \in [0,T)$. So $x = 0$ is in fact a blow-up point since $u \geq 0$ in $\bar{\Omega} \times [0,T)$ by Corollary 2.3.
Now we assume that there exists $x \in (0,a)$ which is a blow-up point. Hence there exist $(x_n)_{n\in\mathbb{N}} \subset \Omega$ and $(t_n)_{n\in\mathbb{N}} \subset (0,T)$ with $u(x_n,t_n) \to \infty$, $x_n \to x$ and $t_n \to T$ for $n \to \infty$. So there are $y \in (0,a)$ and $n_0 \in \mathbb{N}$ with $y \leq x_n$ for $n \geq n_0$. Hence we have $u(y,t_n) \to \infty$ for $n \to \infty$, since $u(y,t_n) \geq u(x_n,t_n)$ for $n \geq n_0$ by Lemma 2.5.
We choose $t_0 \in (0,T)$ with $u(y,t) \geq M\ \forall t \in [t_0,T)$ and $u_x(y,t) < 0$ for all $t \in [t_0,T)$, which is possible because of our assumption and by Lemma 2.5, Lemma 2.6 and Lemma 2.8.
Then we have $u(x,t) \geq M\ \forall (x,t) \in [0,y] \times [t_0,T)$ by Lemma 2.5.

We also choose $\delta \in (0, y)$ with $u_x(x, t_0) < 0$ for all $x \in [\delta, y]$ (which is possible since $u_x \in C^0(\Omega \times (0, T))$), $\alpha > 1$ with $\alpha < \min\{q - p, q - 1 - r\}$ and $M > 0$ with $(\alpha - q)u^{q-1} + \frac{\pi^2}{(y-\delta)^2}u^p + 2\alpha\frac{\pi}{y-\delta}u^{p+\alpha-1} + 2\kappa\frac{\pi}{y-\delta}u^{\alpha+r} + p\frac{\pi}{y-\delta}u^{p+\alpha-1} \le 0$ for $u \ge M$, which is possible because α was suitably chosen.
Then we define $\Theta(x) := \sin(\frac{x-\delta}{y-\delta}\pi)$ for $x \in [\delta, y]$ and $J(x, t) := u_x(x, t) + \varepsilon\Theta(x)u^\alpha(x, t)$, where $\varepsilon \in (0, 1)$ is chosen so that $J(x, t_0) \le 0$ for all $x \in [\delta, y]$ which is possible because of our choice of t_0 and δ and because u is continuous.
So we get in $[\delta, y] \times [t_0, T)$: (we use $u \ge 0$, $\alpha > 1$ and $\varepsilon \in (0, 1)$)

$$\begin{aligned}
J_t &= u_{xt} + \varepsilon\Theta(x)\alpha u^{\alpha-1}u_t \\
&= u^p u_{xxx} + pu^{p-1}u_x u_{xx} + qu^{q-1}u_x + \kappa r u^{r-1}(u_x)^3 + 2\kappa u^r u_x u_{xx} \\
&\quad +\varepsilon\alpha\Theta(x)u^{\alpha-1}(u^p u_{xx} + u^q + \kappa u^r(u_x)^2) \\
&= u^p J_{xx} + pu^{p-1}u_x u_{xx} + qu^{q-1}u_x + \kappa r u^{r-1}(u_x)^3 + 2\kappa u^r u_x u_{xx} \\
&\quad +\varepsilon\alpha\Theta(x)u^{\alpha-1}(u^q + \kappa u^r(u_x)^2) - u^p(-\frac{\pi^2}{(y-\delta)^2}\varepsilon\Theta(x)u^\alpha + 2\varepsilon\Theta'(x)\alpha u^{\alpha-1}u_x \\
&\quad +\varepsilon\Theta(x)\alpha(\alpha-1)u^{\alpha-2}(u_x)^2) \\
&= u^p J_{xx} + (pu^{p-1} + 2\kappa u^r)u_x J_x + qu^{q-1}u_x + \kappa r u^{r-1}(u_x)^3 \\
&\quad +\varepsilon\alpha\Theta(x)u^{\alpha-1}(u^q + \kappa u^r(u_x)^2) - u^p(-\frac{\pi^2}{(y-\delta)^2}\varepsilon\Theta(x)u^\alpha + 2\varepsilon\alpha\Theta'(x)u^{\alpha-1}u_x \\
&\quad +\varepsilon\alpha(\alpha-1)\Theta(x)u^{\alpha-2}(u_x)^2) - (pu^{p-1} + 2\kappa u^r)u_x(\varepsilon\Theta'(x)u^\alpha + \varepsilon\Theta(x)\alpha u^{\alpha-1}u_x) \\
&= u^p J_{xx} + (pu^{p-1} + 2\kappa u^r)u_x J_x + [qu^{q-1} - 2\varepsilon\alpha\Theta'(x)u^{p+\alpha-1} - 2\kappa\varepsilon\Theta'(x)u^{\alpha+r} \\
&\quad -p\varepsilon\Theta'(x)u^{p+\alpha-1}]J + \kappa r u^{r-1}(u_x)^3 + \varepsilon\alpha\Theta(x)u^{\alpha-1}(u^q + \kappa u^r(u_x)^2) \\
&\quad -u^p(-\frac{\pi^2}{(y-\delta)^2}\varepsilon\Theta(x)u^\alpha + \varepsilon\alpha(\alpha-1)\Theta(x)u^{\alpha-2}(u_x)^2) \\
&\quad -(pu^{p-1} + 2\kappa u^r)\varepsilon\alpha\Theta(x)u^{\alpha-1}(u_x)^2 - [qu^{q-1} - 2\varepsilon\alpha\Theta'(x)u^{p+\alpha-1} \\
&\quad -2\kappa\varepsilon\Theta'(x)u^{\alpha+r} - p\varepsilon\Theta'(x)u^{p+\alpha-1}]\varepsilon\Theta(x)u^\alpha \\
&= u^p J_{xx} + B(x,t)J_x + C(x,t)J + \kappa r u^{r-1}(u_x)^3 + \varepsilon\alpha\Theta(x)(\kappa u^{\alpha+r-1} \\
&\quad -(\alpha-1)u^{\alpha+p-2} - 2\kappa u^{\alpha+r-1} - pu^{\alpha+p-2})(u_x)^2 + \varepsilon\Theta(x)u^\alpha((\alpha-q)u^{q-1} \\
&\quad +\frac{\pi^2}{(y-\delta)^2}u^p + 2\varepsilon\alpha\Theta'(x)u^{p+\alpha-1} + 2\kappa\varepsilon\Theta'(x)u^{\alpha+r} + p\varepsilon\Theta'(x)u^{p+\alpha-1}) \\
&\overset{u_x \le 0}{\le} u^p J_{xx} + B(x,t)J_x + C(x,t)J + \varepsilon\Theta(x)u^\alpha((\alpha-q)u^{q-1} + \frac{\pi^2}{(y-\delta)^2}u^p \\
&\quad +2\alpha\frac{\pi}{y-\delta}u^{p+\alpha-1} + 2\kappa\frac{\pi}{y-\delta}u^{\alpha+r} + p\frac{\pi}{y-\delta}u^{p+\alpha-1}) \\
&\le u^p J_{xx} + B(x,t)J_x + C(x,t)J
\end{aligned} \tag{14}$$

due to the choice of M, where u^p, B and C are continuous functions in $[\delta, y] \times [t_0, T)$ since $u \in C^\infty(\Omega \times (0, T))$ by Lemma 2.4.

Since $\Theta(\delta) = \Theta(y) = 0$, we get $J \leq 0$ on the parabolic boundary of $(\delta, y) \times [t_0, T)$ due to the choice of ε. So by (14) and Theorem 1.1 we have $J \leq 0$ in $[\delta, y] \times [t_0, T)$, which is equivalent to $-u_x \geq \varepsilon\Theta(x)u^\alpha$ in $[\delta, y] \times [t_0, T)$.
By integration we get for $t \in [t_0, T)$ with $G(s) := \frac{-1}{1-\alpha} s^{1-\alpha}$ for $s \geq 0$

$$
\begin{aligned}
G(u(y,t)) - G(u(\delta,t)) &= \frac{-1}{1-\alpha}((u(y,t))^{1-\alpha} - (u(\delta,t))^{1-\alpha}) = \int_\delta^y -\frac{u_x(s,t)}{u^\alpha(s,t)} ds \\
&\geq \int_\delta^y \varepsilon\Theta(s) ds = -\varepsilon\frac{y-\delta}{\pi} \cos(\frac{s-\delta}{y-\delta}\pi)|_{s=\delta}^{s=y} \\
&= \varepsilon\frac{y-\delta}{\pi}(1 - \cos(\frac{y-\delta}{y-\delta}\pi)) = 2\varepsilon\frac{y-\delta}{\pi} \\
&> 0
\end{aligned}
$$

which is a contradiction to our assumption since $G(u(y,t)) \to 0$ for $t \nearrow T$ because $u(y,t) \to \infty$ for $t \nearrow T$ and $\alpha > 1$ and $G(u(\delta,t)) \geq 0$ for $t \in (t_0, T)$.
The assertion now follows because $x \in (0, a)$ was arbitrary and u is symmetric by Lemma 2.5.

■

Now we can also prove for solutions v of problem (10) with $p + 1 < q$ that $x = 0$ is the only blow-up point of v if the initial data v_0 fulfill the same conditions like u_0 in the last theorem.

Corollary 5.4
Assume $p+1 < q$, $p \geq 1$, $a > 0$, $\Omega := (-a, a)$, $v_0 \in C^3(\bar{\Omega})$ with $v_0(x) = v_0(-x) > 0\ \forall x \in \Omega$, $(v_0)_x(x) \leq 0\ \forall x \in (0, a)$, $v_0(a) = v_0(-a) = 0$, $(v_0)_{xx} + (v_0)^{q-p} \geq 0$ in $\bar{\Omega}$ and v_0 large enough so that the solution v of (10) evolving from v_0 blows up in finite time. Then $x = 0$ is the only blow-up point of v.

Proof. The proof is similar as in Theorem 5.3, if we formally define $\kappa := 0$, because we can also prove similar assertions like in Corollary 2.3, Lemma 2.4, Lemma 2.5, Lemma 2.6 and Lemma 2.8 in this case. By Theorem 3.6 the assumptions on v_0 can be fulfilled if v_0 is chosen large enough. ■

In contrast to the last theorem we get regional blow-up for $p + 1 < q$ and $r > q - 2$, if u_0 fulfills the same conditions as in the last theorem. It remains open, if we have single point blow-up or regional blow-up in the case $p + 1 < q$ and $r = q - 2$. The proof of the next theorem is done according to an idea demonstrated in [Win2]. This idea is to show that the function $J(x,t) := u_x(x,t) + c(x)u^\alpha(x,t)$ fulfills $J \geq 0$ in $[0, \delta) \times [t_0, T)$, where T is the blow-up time of u and $\delta > 0$, $c(x)$, α and t_0 are suitably chosen. Then we can show that $[0, \delta)$ is contained in the blow-up set.

Theorem 5.5
Let assumption (2) be fulfilled with $p+1<q$, $r>q-2$, $a>0$, $\Omega := (-a,a)$, $u_0 \in C^3(\bar\Omega)$ with $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$, $(u_0)_x(x) \le 0$ $\forall x \in (0,a)$, $u_0(a) = u_0(-a) = 0$, $(u_0)_{xx} + (u_0)^{q-p} \ge 0$ in $\bar\Omega$, let u be the maximal solution of (1) evolving from u_0 and suppose u_0 is large enough so that there exists $T \in (0,\infty)$ with $\limsup_{t\nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$. Then the blow-up set S of u fulfills $|S|>0$.

Proof. Similar to the proof of Theorem 5.3 we can show by Lemma 5.2 that u blows up in finite time whenever u_0 is chosen large enough, because there exists $b_1 \ge 1$ so that the solution u of (1) with $u|_{t=0} = bu_0$ blows up in finite time for all $b \ge b_1$. By Lemma 2.5 we know that u is symmetric and that $u(0,t) = \max_{x\in\bar\Omega} u(x,t)$ for $t \in (0,T)$ where T shall denote the finite blow-up time of u. So $x=0$ is a blow-up point since $u \ge 0$ in $\bar\Omega \times [0,T)$ by Corollary 2.3.
By Lemma 2.6 we have $u_t \ge 0$ in $\Omega \times (0,T)$ due to our assumptions on u_0.
Since $r > q-2$, we can choose $\alpha \in (0,1)$ with $r > q-1-\alpha$. As $p+1<q$ we also have $\alpha + r - p > q-1-p > 0$.
Hence we can choose $M \ge 1$ large enough, so that the following conditions are fulfilled:
$-\frac{p^2}{2} + \frac{p}{2}(2\alpha+p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p+\alpha-1)u^{2\alpha-2} - \kappa(\alpha+r)u^{2\alpha+r-1-p} \ge 0$ if $u \ge M$,
$-p + (2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p} \ge 0$ if $u \ge M$ and
$(\alpha-q)u^{q-1-p} + \beta(-p + (2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) \ge 0$ if $u \ge M$ and $\beta \ge 1$.
Since $x=0$ is a blow-up point we can choose $t_0 \in (0,T)$ so that $u(0,t_0) \ge 2M$. From Lemma 2.5 we know that there is $\delta > 0$ so that $u_{xx}(x,t_0) \le 0$ for $x \in (-\delta,\delta)$ and $u(x,t_0) \ge M$ for $x \in (-\delta,\delta)$ since u is continuous.
So we can choose $\beta \ge 1$ large enough, so that $-\beta \le u_{xx}(x,t_0) \le 0$ and $u(x,t_0) \ge M$ for $x \in [0,x_\infty]$ with $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}}$.
We now define $J(x,t) := u_x(x,t) + c(x)u^\alpha(x,t)$ for $(x,t) \in (0,x_\infty) \times [t_0,T)$ with
$c(x) := \sqrt{\frac{2\beta}{p}}\tan(\sqrt{\frac{p\beta}{2}}x)$.
We will show $J \ge 0$ in $(0,x_\infty) \times (t_0,T)$.
First we have $u \ge M$ in $(0,x_\infty) \times (t_0,T)$ due to the choice of t_0 and x_∞ and because $u_t \ge 0$ in $\Omega \times (0,T)$. We now use $c'(x) = \frac{p}{2}c(x)^2 + \beta$ and $c''(x) = p\,c(x)c'(x)$ for $x \in (0,x_\infty)$.
Hence we have in $(0,x_\infty) \times (t_0,T)$

$$\begin{aligned}
J_t &= u_{xt} + c\alpha u^{\alpha-1}u_t \\
&= u^p u_{xxx} + pu^{p-1}u_x u_{xx} + qu^{q-1}u_x + \kappa r u^{r-1}(u_x)^3 + 2\kappa u^r u_x u_{xx} \\
&\quad + c\alpha u^{\alpha-1}(u^p u_{xx} + u^q + \kappa u^r (u_x)^2) \\
&= u^p J_{xx} + pu^{p-1}u_x u_{xx} + qu^{q-1}u_x + \kappa r u^{r-1}(u_x)^3 + 2\kappa u^r u_x u_{xx} \\
&\quad + c\alpha u^{\alpha-1}(u^q + \kappa u^r (u_x)^2) - u^p(c'' u^\alpha + 2c'\alpha u^{\alpha-1}u_x + c\alpha(\alpha-1)u^{\alpha-2}(u_x)^2)
\end{aligned}$$

$$
\begin{aligned}
&= u^pJ_{xx} + (pu^{p-1} + 2\kappa u^r)u_xJ_x + qu^{q-1}u_x + \kappa ru^{r-1}(u_x)^3 \\
&\quad +c\alpha u^{\alpha-1}(u^q + \kappa u^r(u_x)^2) - u^p(c''u^\alpha + 2c'\alpha u^{\alpha-1}u_x + c\alpha(\alpha-1)u^{\alpha-2}(u_x)^2) \\
&\quad -(pu^{p-1} + 2\kappa u^r)u_x(c'u^\alpha + c\alpha u^{\alpha-1}u_x) \\
&= u^pJ_{xx} + (pu^{p-1} + 2\kappa u^r)u_xJ_x + [qu^{q-1} + \kappa ru^{r-1}(u_x)^2 + \kappa c\alpha u^{\alpha-1}u^r u_x - 2c'\alpha u^{\alpha-1}u^p \\
&\quad -c\alpha(\alpha-1)u^{\alpha-2}u^pu_x - pc'u^\alpha u^{p-1} - 2\kappa c'u^\alpha u^r - pc\alpha u^{\alpha-1}u^{p-1}u_x - 2\kappa c\alpha u^{\alpha-1}u^ru_x]J \\
&\quad +c\alpha u^{\alpha-1}u^q - c''u^{p+\alpha} - cu^\alpha[qu^{q-1} + \kappa ru^{r-1}(u_x)^2 + \kappa c\alpha u^{\alpha-1}u^ru_x - 2c'\alpha u^{\alpha-1}u^p \\
&\quad -c\alpha(\alpha-1)u^{\alpha-2}u^pu_x - pc'u^\alpha u^{p-1} - 2\kappa c'u^\alpha u^r - pc\alpha u^{\alpha-1}u^{p-1}u_x - 2\kappa c\alpha u^{\alpha-1}u^ru_x] \\
&= u^pJ_{xx} + (pu^{p-1} + 2\kappa u^r)u_xJ_x + [qu^{q-1} + \kappa ru^{r-1}(u_x)^2 + \kappa c\alpha u^{\alpha-1+r}u_x - 2c'\alpha u^{\alpha-1+p} \\
&\quad -c\alpha(\alpha-1)u^{\alpha-2+p}u_x - pc'u^{\alpha+p-1} - 2\kappa c'u^{\alpha+r} - pc\alpha u^{\alpha-2+p}u_x - 2\kappa c\alpha u^{\alpha-1+r}u_x \\
&\quad -cu^\alpha(\kappa ru^{r-1}u_x + \kappa c\alpha u^{\alpha-1+r} - c\alpha(\alpha-1)u^{\alpha-2+p} - c\alpha pu^{\alpha-2+p} - 2\kappa c\alpha u^{\alpha-1+r} \\
&\quad -\kappa cru^{\alpha-1+r})]J + c\alpha u^{\alpha-1+q} - c''u^{p+\alpha} - cu^\alpha[qu^{q-1} - 2c'\alpha u^{\alpha-1+p} - pc'u^{\alpha-1+p} \\
&\quad -2\kappa c'u^{r+\alpha}] + c^2u^{2\alpha}[\kappa c\alpha u^{\alpha-1+r} - c\alpha(\alpha-1)u^{\alpha-2+p} - pc\alpha u^{\alpha-2+p} - 2\kappa c\alpha u^{\alpha-1+r}] \\
&\quad -c^3u^{3\alpha}\kappa ru^{r-1} \\
&= u^pJ_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-q)u^{q-1-p} - \frac{c''}{c} \\
&\quad +c'((2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) - c^2(\alpha(p+\alpha-1)u^{2\alpha-2} + \kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&= u^pJ_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-q)u^{q-1-p} - pc' \\
&\quad +c'((2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) - c^2(\alpha(p+\alpha-1)u^{2\alpha-2} + \kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&= u^pJ_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-q)u^{q-1-p} + \beta(-p + (2\alpha+p)u^{\alpha-1} \\
&\quad +2\kappa u^{\alpha+r-p}) + c^2(-\frac{p^2}{2} + \frac{p}{2}(2\alpha+p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p+\alpha-1)u^{2\alpha-2} \\
&\quad -\kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&\geq u^pJ_{xx} + B(x,t)J_x + C(x,t)J \qquad (15)
\end{aligned}
$$

due to our choice of M, where u^p, B and C are continuous functions in $[0, x_\infty) \times [t_0, T)$ because $u \in C^\infty(\Omega \times (0,T))$ by Lemma 2.4.
Since $\tan(y) \geq y$ for $y \in (0, \frac{\pi}{2})$, $M \geq 1$ and $\alpha > 0$, we have for $x \in (0, x_\infty)$

$$J(x,t_0) = u_x(x,t_0) + c(x)u^\alpha(x,t_0) \geq -\beta x + \beta x M^\alpha \geq 0$$

As $u_x(0,t) = 0$ for $t \in (0,T)$ by Lemma 2.5, we get $J(0,t) \geq 0$ for $t \in [t_0, T)$.
Next let $T_1 \in (t_0, T)$. Since $u \in C^{2,1}([0, x_\infty] \times [t_0, T_1])$ there exists $d > 0$ with $|u_x| \leq d$ in $[0, x_\infty] \times [t_0, T_1]$. Now we choose $x_0 \in (0, x_\infty)$ so that $M^\alpha c(x_0) \geq d$. Then we have $J(x,t) \geq -d + c(x_0)M^\alpha \geq 0$ for $(x,t) \in [x_0, x_\infty) \times [t_0, T_1]$. Hence we get $J \geq 0$ on the parabolic boundary of $[0,x] \times [t_0, T_1]$ for all $x \in [x_0, x_\infty)$. So we have $J \geq 0$ in $[0,x] \times [t_0, T_1]$ for all $x \in [x_0, x_\infty)$ by (15) and Theorem 1.1 and hence $J \geq 0$ in $(0, x_\infty) \times [t_0, T_1]$.
Hence it follows that $J \geq 0$ in $(0, x_\infty) \times (t_0, T)$ because $T_1 \in (t_0, T)$ was arbitrary.
Let $x_0 \in (0, x_\infty)$. Then we have $u_x \geq -c(x_0)u^\alpha$ in $(0, x_0) \times (t_0, T)$. Since $u_t \geq 0$ in $\Omega \times (0,T)$, $\alpha \in (0,1)$ and $x = 0$ is a blow-up point, we can choose $t_1 \in (t_0, T)$ so that $u(0,t)^{1-\alpha} - (1-\alpha)c(x_0)x_0 > 0$ for $t \in (t_1, T)$.

Hence we have

$$u(x,t) \;\geq\; (-(1-\alpha)c(x_0)x + u(0,t)^{1-\alpha})^{\frac{1}{1-\alpha}} \geq (-(1-\alpha)c(x_0)x_0 + u(0,t)^{1-\alpha})^{\frac{1}{1-\alpha}}$$

for $(x,t) \in (0,x_0) \times (t_1,T)$. So the interval $(0,x_0)$ is contained in the blow-up set, since $x=0$ is a blow-up point and $\alpha \in (0,1)$. Hence by the symmetry of u (by Lemma 2.5) the interval $(-x_\infty, x_\infty)$ is contained in the blow-up set and the assertion follows. ■

For some special cases we can give an estimate from below for the size of the blow-up set, because it is sometimes better to have such an estimate than only to know that $|S| > 0$ is fulfilled.

Corollary 5.6
Let assumption (2) be fulfilled with $p+1 < q$, $r > q-2$, $a > 0$, $\Omega := (-a,a)$, $u_0 \in C^3(\bar{\Omega})$ with $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$, $(u_0)_x(x) \leq 0$ $\forall x \in (0,a)$, $u_0(a) = u_0(-a) = 0$, $(u_0)_{xx} + (u_0)^{q-p} \geq 0$ in $\bar{\Omega}$, let u be the maximal solution of (1) evolving from u_0 and suppose u_0 is large enough so that there exists $T \in (0,\infty)$ with $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$. Then there exist $\alpha \in (0,1)$ with $r > q-1-\alpha$ depending only on q and r and a constant $M_0 := \max\{1, (\frac{p+2}{\kappa})^{\frac{1}{\alpha+r-p}}, (\frac{2(\alpha+r)}{p})^{\frac{1}{1-\alpha}}\}$ depending only on p, q, r and κ, so that whenever there exist $\delta \in (0,a)$ and $M \geq M_0$ so that $M^{r+\alpha+1-q} \geq \frac{q2p\delta^2}{\kappa\pi^2}$, $u_0(x) \geq M$ and $-\frac{\pi^2}{2p\delta^2} \leq (u_0)_{xx}(x) \leq 0$ for $x \in [-\delta,\delta]$, then the blow-up set S of u contains the interval $(-\delta,\delta)$ and in particular $|S| \geq 2\delta$.

Proof. We can choose $\alpha \in (0,1)$ so that $r > q-1-\alpha$ (like in the proof of Theorem 5.5) and $M_0 \geq 1$ as in the assertion. Then the following conditions are fulfilled for $u \geq M_0$ due to $\alpha \in (0,1)$, $M_0 \geq 1$ and $\alpha + r - p > 0$:

$$\begin{aligned}
& -\frac{p^2}{2} + \frac{p}{2}(2\alpha+p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p+\alpha-1)u^{2\alpha-2} - \kappa(\alpha+r)u^{2\alpha+r-1-p} \\
\geq\; & -\frac{p^2}{2} + \frac{p\kappa}{2}u^{\alpha+r-p} - p + \frac{p\kappa}{2}u^{\alpha+r-p} - \kappa(\alpha+r)u^{2\alpha+r-1-p} \\
\geq\; & -\frac{p^2}{2} + \frac{p\kappa}{2}\frac{p+2}{\kappa} - p + \kappa u^{2\alpha+r-1-p}(\frac{p}{2}u^{1-\alpha} - (\alpha+r)) \\
\geq\; & \kappa u^{2\alpha+r-1-p}(\frac{p}{2}\frac{2(\alpha+r)}{p} - (\alpha+r)) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
-p + (2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p} \;&\geq\; -p + \kappa u^{\alpha+r-p} + \kappa u^{\alpha+r-p} \\
&\geq\; -p + \kappa\frac{p+2}{\kappa} + \kappa u^{\alpha+r-p} \geq \kappa u^{\alpha+r-p}
\end{aligned}$$

We can choose $\beta := \frac{\pi^2}{2p\delta^2}$, $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}} = \delta$, $t_0 = 0$ and M as in the assertion, too.
Then we also have

$$\begin{aligned}(\alpha - q)u^{q-1-p} + \beta(-p + (2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) &\geq -qu^{q-1-p} + \beta\kappa u^{\alpha+r-p} \\ &= u^{q-1-p}(-q + \beta\kappa u^{r+\alpha+1-q}) \\ &\geq u^{q-1-p}(-q + \beta\kappa M^{r+\alpha+1-q}) \geq 0\end{aligned}$$

if $u \geq M$ because $r + \alpha + 1 - q > 0$.
Hence we can show like in the proof of Theorem 5.5 (because $u_x \in C^0(\Omega \times [0, T))$ by Lemma 2.4) that the interval $(-\delta, \delta)$ is contained in the blow-up set S, so the assertion is proved. ■

We now study the case $p + 1 = q$, where we assume that the principal eigenvalue of the Laplacian in Ω is smaller than 1. So we can guarantee that the maximal solution u of (1) evolving from u_0 blows up in finite time. In the case $r \leq p - 1$ we can show with a proof similar to the one of Theorem 5.5 that the blow-up set S fulfills $|S| > 0$. In contrast to Theorem 5.5 we do not need $u_t \geq 0$ in $\Omega \times (0, T)$, so that the assumptions on u_0 are fulfilled by more functions than in Theorem 5.5.

Theorem 5.7
Let assumption (2) *be fulfilled with* $p+1 = q$, $r \leq p-1$, $a > \frac{\pi}{2}$, $\Omega := (-a, a)$ *and* $u_0 \in C^3(\bar{\Omega})$ *with* $u_0(x) = u_0(-x) > 0\ \forall x \in \Omega$, $(u_0)_x(x) \leq 0\ \forall x \in (0, a)$ *and* $u_0(a) = u_0(-a) = 0$ *and let* u *be the maximal solution of* (1) *evolving from* u_0*. Then there exists* $T \in (0, \infty)$ *with* $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$ *and the blow-up set* S *of* u *fulfills* $|S| > 0$*.*

Proof. Because of the assumptions on u_0 there is $\delta > 0$ with $(u_0)_{xx} \leq 0$ in $(-\delta, \delta)$. So we can choose $\beta \geq \frac{p}{2}$, so that $x_\infty < a$ and

$$-\varepsilon \leq (u_0)_{xx} \leq 0 \quad \text{in} \quad [0, x_\infty] \tag{16}$$

where $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}}$ and $\varepsilon := \frac{\beta\, u_0(0)}{1+\frac{\pi^2}{4p}}$.
Due to our assumption on a, we have $\lambda_1 < 1$ for the principal eigenvalue λ_1 of the Laplacian in Ω. So by Theorem 3.4, Theorem 3.5 and Lemma 5.1 u blows up in finite time with blow-up time T.
We now define $J(x,t) := u_x(x,t) + c(x)u(x,t)$ for $(x,t) \in (0, x_\infty) \times [0, T)$ with
$c(x) := \sqrt{\frac{2\beta}{p}}\tan(\sqrt{\frac{p\beta}{2}}x)$.
We will show $J \geq 0$ in $(0, x_\infty) \times (0, T)$.
Due to our choice of ε and x_∞ we have $\varepsilon = \frac{\beta\, u_0(0)}{1+\frac{x_\infty^2}{2}\beta}$.
So we get (with $\tan(y) \geq y$ in $(0, \frac{\pi}{2})$) for $x \in (0, x_\infty)$

$$\begin{aligned} J(x,0) &= (u_0)_x(x) + c(x)u_0(x) \overset{(16)}{\geq} -\varepsilon x + \beta x(u_0(0) - \frac{\varepsilon}{2}x_\infty^2) \\ &= x(\beta\, u_0(0) - \varepsilon(1 + \frac{x_\infty^2}{2}\beta)) = 0 \end{aligned}$$

By Lemma 2.5 we also have $J(0,t) = 0$ for $t \in [0,T)$.
We now use $c'(x) = \frac{p}{2}c(x)^2 + \beta$ and $c''(x) = p\, c(x)c'(x)$ for $x \in (0, x_\infty)$.
Hence we have in $(0, x_\infty) \times (0,T)$ (like in the proof of Theorem 5.5 with $\alpha = 1$ and $q = p+1$)

$$\begin{aligned} J_t &= u^p J_{xx} + (pu^{p-1} + 2\kappa u^r)u_x J_x + [(p+1)u^p + \kappa r u^{r-1}(u_x)^2 + \kappa c u^r u_x - 2c'u^p \\ &\quad -pc'u^p - 2\kappa c' u^{r+1} - pcu^{p-1}u_x - 2\kappa c u^r u_x - cu(\kappa r u^{r-1}u_x + \kappa c u^r - cpu^{p-1} \\ &\quad -2\kappa c u^r - \kappa c r u^r)]J + cu^{p+1} - c''u^{p+1} - cu[(p+1)u^p - 2c'u^p - pc'u^p - 2\kappa c' u^{r+1}] \\ &\quad +c^2u^2[\kappa c u^r - pcu^{p-1} - 2\kappa c u^r] - \kappa c^3 u^3 r u^{r-1} \\ &= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+1}(-p - \frac{c''}{c} + c'(2 + p + 2\kappa u^{1+r-p}) \\ &\quad -c^2(p + \kappa(r+1)u^{1+r-p})) \\ &= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+1}[-p - pc' + pc' + 2c' - pc^2 \\ &\quad +\kappa u^{1+r-p}(2c' - (r+1)c^2)] \\ &= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+1}[-p + 2\beta + \kappa u^{r+1-p}((p-r-1)c^2 + 2\beta)] \\ &\geq u^p J_{xx} + B(x,t)J_x + C(x,t)J \end{aligned}$$

because $\beta \geq \frac{p}{2}$, $r \leq p-1$ and $u \geq 0$, $c \geq 0$ in $(0, x_\infty) \times (0,T)$. u^p, B and C are continuous functions in $[0, x_\infty) \times [0,T)$ since $u \in C^\infty(\Omega \times (0,T))$ and $u_x \in C^0(\Omega \times [0,T))$ by Lemma 2.4 and hence we can show $J \geq 0$ in $(0, x_\infty) \times (0,T)$ similar as in the proof of Theorem 5.5 because there is $M > 0$ with $u \geq M > 0$ in $[0, x_\infty] \times [0,T)$ by Corollary 2.3.
So for $x_0 \in (0, x_\infty)$ we have $u_x \geq -c(x_0)u$ in $(0, x_0) \times (0,T)$ and hence
$u(x,t) \geq e^{-c(x_0)x_0}u(0,t)$ in $(0, x_0) \times (0,T)$.
By Lemma 2.5, $u(0,t) = \max_{x \in \bar{\Omega}} u(x,t)$ for $t \in (0,t)$ and so $x = 0$ is a blow-up point because $u \geq 0$ in $\bar{\Omega} \times [0,T)$ by Corollary 2.3. Hence by the symmetry of u (see Lemma 2.5) we get that every $x_0 \in (-x_\infty, x_\infty)$ is a blow-up point. So the assertion follows. ■

For some special cases we can give an estimate from below for the size of the blow-up set which is sometimes useful. But in contrast to Corollary 5.6 here we can only show that $(-\delta, \delta)$ is contained in the blow-up set if $\delta \leq \frac{\pi}{p}$. In Corollary 5.6 δ can be arbitrarily large, if u_0 is suitably chosen.

Corollary 5.8
Assume that assumption (2) is fulfilled with $p+1 = q$, $r \leq p-1$, $a > \frac{\pi}{2}$, $\Omega := (-a,a)$ and $u_0 \in C^3(\bar{\Omega})$ with $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$, $(u_0)_x(x) \leq 0$ $\forall x \in (0,a)$ and $u_0(a) = u_0(-a) = 0$.

If there exists $\delta \in (0, a)$ with $\delta \leq \frac{\pi}{p}$ so that $-\frac{\pi^2}{2p\delta^2} \frac{u_0(0)}{1+\frac{\pi^2}{4p}} \leq (u_0)_{xx}(x) \leq 0$ for $x \in [-\delta, \delta]$, then the maximal solution u of (1) evolving from u_0 blows up in finite time and the blow-up set S of u contains the interval $(-\delta, \delta)$ and in particular $|S| \geq 2\delta$.

Proof. We can choose $\beta := \frac{\pi^2}{2p\delta^2}$ and $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}} = \delta$.
Then $\beta \geq \frac{p}{2}$ and $-\varepsilon \leq (u_0)_{xx} \leq 0$ in $[0, \delta]$ with $\varepsilon := \frac{\beta u_0(0)}{1+\frac{\pi^2}{4p}}$ due to the assumptions on δ and u_0.
So we can show just like in the proof of Theorem 5.7 that the interval $(-\delta, \delta)$ is contained in the blow-up set S and hence the assertion is proved. ■

Last we consider the case $p + 1 = q$ and $r > p - 1$ with the same conditions on Ω and u_0 as in Theorem 5.7, but like in Theorem 5.5 we additionally need $u_t \geq 0$ in $\Omega \times (0, T)$. Like in Theorem 5.7 we can show here with a similar proof that the blow-up set S of u fulfills $|S| > 0$.

Theorem 5.9
Let assumption (2) be fulfilled with $p+1 = q$, $r > p-1$, $a > \frac{\pi}{2}$, $\Omega := (-a, a)$ and $u_0 \in C^3(\bar{\Omega})$ with $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$, $(u_0)_x(x) \leq 0$ $\forall x \in (0, a)$, $u_0(a) = u_0(-a) = 0$ and $(u_0)_{xx} + u_0 \geq 0$ in $\bar{\Omega}$ and let u be the maximal solution of (1) evolving from u_0. Then there exists $T \in (0, \infty)$ with $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\Omega)} = \infty$ and the blow-up set S of u fulfills $|S| > 0$.

Proof. Due to our assumption on a, we have $\lambda_1 < 1$ for the principal eigenvalue λ_1 of the Laplacian in Ω. So by Theorem 3.4, Theorem 3.5 and Lemma 5.1 u blows up in finite time with blow-up time T. We also know by Lemma 2.5 that $x = 0$ is a blow-up point, because $u(0, t) = \max_{x \in \bar{\Omega}} u(x, t)$ for $t \in (0, T)$ and $u \geq 0$ in $\bar{\Omega} \times [0, T)$ by Corollary 2.3.
By Lemma 2.6 we have $u_t \geq 0$ in $\Omega \times (0, T)$ due to our assumptions on u_0.
Since $r > p - 1$, we can choose $\alpha \in (0, 1)$ so that $r + \alpha > p$.
Hence we can choose $M \geq 1$ large enough, so that the following conditions are fulfilled:
$-\frac{p^2}{2} + \frac{p}{2}(2\alpha + p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p + \alpha - 1)u^{2\alpha-2} - \kappa(\alpha + r)u^{2\alpha+r-1-p} \geq 0$ if $u \geq M$,
$-p + (2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p} \geq 0$ if $u \geq M$ and
$(\alpha - 1 - p) + \beta(-p + (2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) \geq 0$ if $u \geq M$ and $\beta \geq 1$.
Since $x = 0$ is a blow-up point we can choose $t_0 \in (0, T)$ so that $u(0, t_0) \geq 2M$. From Lemma 2.5 we know that there is $\delta > 0$ so that $u_{xx}(x, t_0) \leq 0$ for $x \in (-\delta, \delta)$ and $u(x, t_0) \geq M$ for $x \in (-\delta, \delta)$ since u is continuous.
Hence we can choose $\beta \geq 1$ large enough, so that $-\beta \leq u_{xx}(x, t_0) \leq 0$ and $u(x, t_0) \geq M$ for $x \in [0, x_\infty]$ with $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}}$.
We now define $J(x, t) := u_x(x, t) + c(x)u^\alpha(x, t)$ for $(x, t) \in (0, x_\infty) \times [t_0, T)$ with
$c(x) := \sqrt{\frac{2\beta}{p}} \tan(\sqrt{\frac{p\beta}{2}} x)$.
We will show $J \geq 0$ in $(0, x_\infty) \times (t_0, T)$.
As $\tan(y) \geq y$ for $y \in (0, \frac{\pi}{2})$, $M \geq 1$ and $\alpha > 0$, we have for $x \in (0, x_\infty)$

$$J(x,t_0) = u_x(x,t_0) + c(x)u^\alpha(x,t_0) \geq -\beta x + \beta x M^\alpha \geq 0$$

We get $J(0,t) \geq 0$ for $t \in (t_0,T)$ since $u_x(0,t) = 0$ for $t \in (0,T)$ by Lemma 2.5.
As $u_t \geq 0$ in $\Omega \times (0,T)$, we have $u \geq M$ in $(0,x_\infty) \times (t_0,T)$ due to the choice of t_0 and x_∞.
Hence we have in $(0,x_\infty) \times (t_0,T)$ (like in the proof of Theorem 5.5 with $q = p+1$)

$$\begin{aligned}
J_t &= u^p J_{xx} + (pu^{p-1} + 2\kappa u^r)u_x J_x + [(p+1)u^p + \kappa r u^{r-1}(u_x)^2 + \kappa c\alpha u^{\alpha-1+r}u_x \\
&\quad -2c'\alpha u^{\alpha-1+p} - c\alpha(\alpha-1)u^{\alpha-2+p}u_x - pc'u^{\alpha+p-1} - 2\kappa c'u^{\alpha+r} - pc\alpha u^{\alpha-2+p}u_x \\
&\quad -2\kappa c\alpha u^{\alpha-1+r}u_x - cu^\alpha(\kappa r u^{r-1}u_x + \kappa c\alpha u^{\alpha-1+r} - c\alpha(\alpha-1)u^{\alpha-2+p} - c\alpha p u^{\alpha-2+p} \\
&\quad -2\kappa c\alpha u^{\alpha-1+r} - \kappa c r u^{\alpha-1+r})]J + c\alpha u^{\alpha+p} - c''u^{p+\alpha} - cu^\alpha[(p+1)u^p - 2c'\alpha u^{\alpha-1+p} \\
&\quad -pc'u^{\alpha-1+p} - 2\kappa c'u^{r+\alpha}] + c^2u^{2\alpha}[\kappa c\alpha u^{\alpha-1+r} - c\alpha(\alpha-1)u^{\alpha-2+p} - pc\alpha u^{\alpha-2+p} \\
&\quad -2\kappa c\alpha u^{\alpha-1+r}] - \kappa c^3 u^{3\alpha} r u^{r-1} \\
&= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-1-p) - \frac{c''}{c} \\
&\quad +c'((2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) - c^2(\alpha(p+\alpha-1)u^{2\alpha-2} + \kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-1-p) - pc' \\
&\quad +c'((2\alpha+p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) - c^2(\alpha(p+\alpha-1)u^{2\alpha-2} + \kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&= u^p J_{xx} + B(x,t)J_x + C(x,t)J + cu^{p+\alpha}[(\alpha-1-p) + \beta(-p + (2\alpha+p)u^{\alpha-1} \\
&\quad +2\kappa u^{\alpha+r-p}) + c^2(-\frac{p^2}{2} + \frac{p}{2}(2\alpha+p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p+\alpha-1)u^{2\alpha-2} \\
&\quad -\kappa(r+\alpha)u^{2\alpha+r-1-p})] \\
&\geq u^p J_{xx} + B(x,t)J_x + C(x,t)J
\end{aligned}$$

due to our choice of M, where u^p, B and C are continuous functions in $[0,x_\infty) \times [t_0,T)$ because $u \in C^\infty(\Omega \times (0,T))$ by Lemma 2.4. So it follows like in the proof of Theorem 5.5 that $J \geq 0$ in $(0,x_\infty) \times (t_0,T)$.
Let $x_0 \in (0,x_\infty)$. Then we have $u_x \geq -c(x_0)u^\alpha$ in $(0,x_0) \times (t_0,T)$.
Since $u_t \geq 0$ in $\Omega \times (0,T)$, $\alpha \in (0,1)$ and $x = 0$ is a blow-up point, we can choose $t_1 \in (t_0,T)$ so that $u(0,t)^{1-\alpha} - (1-\alpha)c(x_0)x_0 > 0$ for $t \in (t_1,T)$.
Hence we have

$$u(x,t) \geq (-(1-\alpha)c(x_0)x + u(0,t)^{1-\alpha})^{\frac{1}{1-\alpha}} \geq (-(1-\alpha)c(x_0)x_0 + u(0,t)^{1-\alpha})^{\frac{1}{1-\alpha}}$$

for $(x,t) \in (0,x_0) \times (t_1,T)$. So the interval $(0,x_0)$ is contained in the blow-up set, since $x = 0$ is a blow-up point and $\alpha \in (0,1)$. Hence by the symmetry of u (by Lemma 2.5) the interval $(-x_\infty, x_\infty)$ is contained in the blow-up set and the assertion follows. ■

For some special cases we can give an estimate from below for the size of the blow-up set and like in Corollary 5.6 we can show for arbitrarily large $\delta > 0$ that $(-\delta,\delta)$ is contained in the blow-up set if u_0 is suitably chosen.

Corollary 5.10
Let assumption (2) *be fulfilled with* $p+1 = q$, $r > p-1$, $a > \frac{\pi}{2}$, $\Omega := (-a, a)$ *and* $u_0 \in C^3(\bar{\Omega})$ *with* $u_0(x) = u_0(-x) > 0$ $\forall x \in \Omega$, $(u_0)_x(x) \le 0$ $\forall x \in (0, a)$, $u_0(a) = u_0(-a) = 0$ *and* $(u_0)_{xx} + u_0 \ge 0$ *in* $\bar{\Omega}$ *and let* u *be the maximal solution of* (1) *evolving from* u_0.
Then there exist $\alpha \in (0,1)$ *with* $r > p - \alpha$ *depending only on* r *and* p *and a constant* $M_0 := \max\{1, (\frac{p+2}{\kappa})^{\frac{1}{\alpha+r-p}}, (\frac{2(\alpha+r)}{p})^{\frac{1}{1-\alpha}}\}$ *depending only on* p, r *and* κ, *so that whenever there exist* $\delta \in (0, a)$ *and* $M \ge M_0$ *so that* $M^{r+\alpha-p} \ge \frac{(p+1)2p\delta^2}{\kappa\pi^2}$, $u_0(x) \ge M$ *and* $-\frac{\pi^2}{2p\delta^2} \le (u_0)_{xx}(x) \le 0$ *for* $x \in [-\delta, \delta]$, *then the blow-up set* S *of* u *contains the interval* $(-\delta, \delta)$ *and in particular* $|S| \ge 2\delta$.

Proof. We can choose $\alpha \in (0,1)$ so that $r > p - \alpha$ (like in the proof of Theorem 5.9) and $M_0 \ge 1$ as in the assertion. Then the following conditions are fulfilled for $u \ge M_0$ due to $\alpha \in (0,1)$, $M_0 \ge 1$ and $\alpha + r - p > 0$:

$$\begin{aligned}
& -\frac{p^2}{2} + \frac{p}{2}(2\alpha + p)u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \alpha(p + \alpha - 1)u^{2\alpha-2} - \kappa(\alpha + r)u^{2\alpha+r-1-p} \\
\ge \; & -\frac{p^2}{2} + \frac{p\kappa}{2}u^{\alpha+r-p} - p + \frac{p\kappa}{2}u^{\alpha+r-p} - \kappa(\alpha + r)u^{2\alpha+r-1-p} \\
\ge \; & -\frac{p^2}{2} + \frac{p\kappa}{2}\frac{p+2}{\kappa} - p + \kappa u^{2\alpha+r-1-p}(\frac{p}{2}u^{1-\alpha} - (\alpha + r)) \\
\ge \; & \kappa u^{2\alpha+r-1-p}(\frac{p}{2}\frac{2(\alpha + r)}{p} - (\alpha + r)) \ge 0
\end{aligned}$$

and

$$\begin{aligned}
-p + (2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p} \; & \ge \; -p + \kappa u^{\alpha+r-p} + \kappa u^{\alpha+r-p} \\
& \ge \; -p + \kappa\frac{p+2}{\kappa} + \kappa u^{\alpha+r-p} \ge \kappa u^{\alpha+r-p}
\end{aligned}$$

We can also choose $\beta := \frac{\pi^2}{2p\delta^2}$, $x_\infty := \frac{\pi}{2}\sqrt{\frac{2}{p\beta}} = \delta$, $t_0 = 0$ and M as in the assertion.
Then we also have

$$\begin{aligned}
(\alpha - 1 - p) + \beta(-p + (2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p}) \; & \ge \; -1 - p + \beta\kappa u^{\alpha+r-p} \\
& \ge \; -1 - p + \beta\kappa M^{r+\alpha-p} \ge 0
\end{aligned}$$

if $u \ge M$ because $r + \alpha - p > 0$.
Hence we can show like in the proof of Theorem 5.9 (because $u_x \in C^0(\Omega \times [0, T))$ by Lemma 2.4) that the interval $(-\delta, \delta)$ is contained in the blow-up set S, so the assertion is proved. ■

We remark that Theorem 5.7, Corollary 5.8, Theorem 5.9 and Corollary 5.10 remain true for $a \le \frac{\pi}{2}$, if we can ensure that the maximal solution u of (1) evolving from u_0 blows up in finite time. For example Theorem 5.7 and Corollary 5.8 remain true in the case $r = p - 1$ for $a > \frac{\pi}{2\sqrt{\kappa+1}}$ by Corollary 4.5 and Corollary 4.6.

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