Qualitative behavior of solutions to parabolic equations with different types of diffusion

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Contents

Part I

Summary

1. Introduction

Diffusion is an important aspect of various processes in the natural sciences. Correspondingly, there are a lot of mathematical models involving partial differential equations (PDEs) with a diffusion term. This thesis provides a collection of articles concerning qualitative properties of diffusion equations. Thereby, different types of diffusion, including nonlinear and degenerate ones, are studied. In addition, some of these equations are coupled via nonlinear terms modeling different kinds of taxis. The first step in the qualitative analysis of such equations is usually the proof of local well-posedness, i.e. the local existence (in time) and uniqueness of a solution. Then a natural question is whether the solution exists globally or ceases to exist after a finite time, e.g., due to a blowup phenomenon. If possible we also determine the large time behavior of global solutions, for instance the convergence to stationary solutions.

The simplest diffusion equation is the linear heat equation

$$u_t = \Delta u,$$

where here and in the sequel $u(x,t)$ depends on the spatial variable $x \in \Omega \subset \mathbb{R}^n$ and the time $t \geq 0$, and $\Delta = \Delta_x$ denotes the Laplace operator with respect to x. Although this equation was used to describe the temperature evolution in particular settings, it turned out that many biological, chemical or physical phenomena involving diffusion cannot be described adequately by purely linear equations. In addition, nonlinear diffusion equations often offer a richer behavior than linear ones, also from the mathematical point of view. Therefore, our studies will focus on nonlinear equations. The first part of this thesis is concerned with scalar equations and motivated from the theoretical point of view, while in the second part systems of equations which model biological phenomena are studied.

We start our considerations with a semilinear parabolic equation, where in presence of linear diffusion the nonlinearity is provided by a source term of order zero, a model case being the power type function u^p. Namely, in Article 1 we study the Cauchy problem for

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

with $p > 1$ and nonnegative initial data. After the pioneering work of Fujita ([45]) a rich variety of large time behaviors for solutions to (1.1) was shown, among them finite and infinite time blowup, as well as convergence to nonzero steady states and to zero. For a broad overview we refer to the monograph [89] and the references therein. In particular, it was shown in [44] that if p is larger than a critical (so-called Joseph-Lundgren) exponent, there are solutions to (1.1) converging to zero as $t \to \infty$ with arbitrary slow polynomial rates of convergence. In Article 1, we prove that for the same range of p even very slow rates of convergence to zero exist, which are slower than any polynomial rate and are rarely observed in parabolic equations. In particular, arbitrary negative powers of iterated logarithms occur as convergence rates for (1.1) for suitably chosen initial data. More details about these results are provided in Chapter 2.

Next, we consider a quasilinear equation, where apart from a nonlinear source term also nonlinear diffusion is included. Two common nonlinear generalizations of the linear diffusion Δu are the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ and the porous medium type diffusion $\nabla \cdot (u^{m-1}\nabla u) = \frac{1}{m}\Delta(u^m)$. Here the cases $p > 2$ and $m > 1$ are called slow diffusion, as diffusion is slowed down for small values of ∇u or u, respectively. Correspondigly, the cases $p \in (1,2)$ and $m \in (0,1)$ are called fast diffusion, while $p = 2$ and $m = 1$ correspond to linear diffusion.
Now we start from the semilinear diffusion equation

$$u_t = \Delta u + |\nabla u|^q \tag{1.2}$$

with $q > 0$, where the source term of order zero in (1.1) is replaced by a corresponding first order term. (1.2) is on the one hand known as viscous Hamilton-Jacobi equation, which appears as viscosity approximation of Hamilton-Jacobi equations in control theory, and on the other hand it is called the generalized deterministic Kardar-Parisi-Zhang equation. The latter was proposed to describe the evolution of the profile of a growing interface for instance in the context of ballistic deposition (see e.g. [64, 69]). A short overview of results concerning the behavior of solutions to (1.2) is given e.g. in [89, Section 40] and the references provided therein.
We intend to study the interplay of diffusion and the nonlinear source term in a quasilinear generalization of (1.2). As the source $|\nabla u|^q$ depends solely on ∇u but not on u itself, it seems reasonable to generalize the linear diffusion by the p-Laplacian operator which does not depend on u either. Therefore, in Articles 2–4 we study nonnegative solutions to the quasilinear equation

$$u_t = \Delta_p u + |\nabla u|^q, \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.3}$$

with $p > 2$ and $q > 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$, endowed with homogeneous Dirichlet boundary conditions. Since the diffusion degenerates when $\nabla u = 0$, we cannot expect the existence of classical solutions and hence consider either weak or viscosity solutions. Concerning the large time behavior of nonnegative global solutions to (1.3), $q = p-1$ turns out to be a critical exponent. One important reason is that zero is the only stationary solution in case of $q \geq p-1$, whereas the comparison principle for the stationary equation is not valid in the case $q < p-1$ and non-zero steady states may exist. In Article 2, we study the case $q < p-1$ in the one-dimensional setting $n = 1$. We show the existence of

a family of nonnegative steady states for (1.3) and prove that for any sufficiently regular initial data there exists a global weak solution to (1.3) which converges to one of the steady states as $t \to \infty$. These results are generalized in Article 3 to arbitrary dimensions $n \geq 2$ within the concept of radially symmetric viscosity solutions, when Ω is the unit ball in $\mathbb{R}^n$. Both results strongly rely on the available classification of the stationary solutions and such a classification as well as the large time behavior of solutions to (1.3) remain open for general domains $\Omega \subset \mathbb{R}^n$.
Finally, we show in Article 4 that in case of $q \geq p-1$ for any nonnegative continuous initial data u_0, which is assumed to be sufficiently small for $q > p$, there exists a unique global in time viscosity solution u to (1.3) which converges to zero as $t \to \infty$. In contrast to the very slow convergence rates obtained for (1.1), here u converges to zero with a fixed polynomial convergence rate. More precisely, after a suitable rescaling of time, we prove convergence to a unique spatial profile in the large time limit. In fact, there are only two different profiles, one for $q > p-1$ and one for the borderline case $q = p-1$, which both do not depend on the initial data. Altogether, our results show that, like for (1.2) in the case $p = 2$, the large time behavior of global solutions to (1.3) is the same as for the pure diffusion equation $u_t = \Delta_p u$ if $q > p-1$, while it strongly depends on the source term $|\nabla u|^q$ in the case $q < p-1$. Moreover, we provide an indication that for $q > p$ a smallness assumption on u_0 is indeed necessary, since we show that for large u_0 there is no Lipschitz continuous weak solution to (1.3), which exists globally in time. This has been further strengthened in [18], where the occurrence of finite time gradient blowup for weak solutions to (1.3) is proved for large u_0 and $q > p$, meaning that u remains bounded and ∇u becomes unbounded after a finite time. More details about the results from Articles 2–4 are presented in Chapter 3.

After the foregoing works, where we studied the influence of the interplay between diffusion and further nonlinearities of zeroth or first order on the large time behavior of solutions to scalar diffusion equations, we next study in Articles 5–8 systems of parabolic equations including a strong coupling via cross diffusion terms which rely on different types of taxis. The migration of various cell populations relies at least partially on taxis, which means directed movement in response to the gradient of some stimulus. Here we consider only cell motions up such gradients. Furthermore, we account for either chemotaxis, where cells are attracted by a diffusible chemical (called chemoattractant), or a combination of the latter mechanism with haptotaxis, where the (insoluble) attractant substance is not moving (e.g. a component of the extracellular matrix). Denoting by u the cell density and by v and w the concentrations of the chemo- and haptotactic attractant, respectively, the system

$$\begin{cases} u_t = \nabla \cdot (\phi(u,w)\nabla u) - \nabla \cdot (\psi(u,v)\nabla v) - \nabla \cdot (\xi(u,w)\nabla w) + g_1(u,v,w), \\ v_t = \Delta v + g_2(u,v,w), \\ w_t = g_3(u,v,w), \end{cases} \tag{1.4}$$

provides a general framework for the models considered in Articles 5–8. Therein (1.4) is always imposed in $\Omega \times (0,T)$ together with homogeneous Neumann boundary conditions and appropriate initial conditions, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Thereby, the first equation in (1.4) describes the evolution of the cell density in presence of diffusion, chemo- and haptotaxis. Here, the diffusivity ϕ is of a generalized porous medium type, which reflects e.g. when ϕ is decreasing with respect to u the reduced cell movement at places where cells are densely packed, and ψ and ξ are the respective chemo- and haptotactic sensitivities. Moreover, the chemoattractant v satisfies a parabolic equation, while the substance w involved in haptotaxis solves an ordinary differential equation (ODE). Apart from their dependence on u and ∇u, the taxis terms also depend on spatial derivatives of the respective attractant. Therefore, they are considerably more general than the nonlinearities present in (1.1) and (1.3) and moreover provide strong couplings between the equations in (1.4).

In Articles 5 and 6 we study variants of the well-known Keller-Segel chemotaxis model, which was introduced in [65] to model aggregation of the slime mold Dictyostelium discoideum. Several variants thereof have been proposed to describe the behavior of various types of cells and have been mathematically analyzed by numerous authors. The surveys [56, 58] and the references therein present different variants of the Keller-Segel model and mathematical results concerning the behavior of their solutions. In particular, it has been shown that aggregation phenomena do occur in these models. In Article 5 we study positive and radially symmetric classical solutions to the parabolic-parabolic Keller-Segel system formed by the subsystem of (1.4) involving only u and v with $g_1 \equiv 0$ in a ball $\Omega \subset \mathbb{R}^n$. Moreover, we assume that the diffusivity ϕ and the chemotactic sensitivity ψ depend only on u and that the cells produce the chemoattractant (see (4.1) for the precise model). In this setting an important issue is the investigation of finite-time blowup, in the sense that the cell density u becomes unbounded in finite time, which can be interpreted as an indication for the formation of aggregation. Quite a few results concerning finite-time blowup have been proved for parabolic-elliptic versions of the Keller-Segel model (see e.g. the introduction of Article 5 for a short overview), however, when the equation for v is genuinely parabolic there were only the three works [53, 30, 101] addressing this issue (for details we refer to Chapter 4). The results therein are valid for a linear chemotactic sensitivity ψ and only [30] includes nonlinear diffusion, i.e. non-constant ϕ, in the one-dimensional setting $n = 1$. Article 5 provides the first result of finite-time blowup for a parabolic-parabolic Keller-Segel model with both nonlinear diffusion and nonlinear chemotactic sensitivity. More precisely, for dimensions $n \geq 3$ we assume a certain relation between ϕ and ψ, which in the sequel is called *supercritical*, along with an at least linear growth of ψ. By generalizing the method from [101], we then prove that for any prescribed initial mass $m > 0$ of the cells there are initial data such that the corresponding solution to the Keller-Segel model blows up in finite time. The supercritical relation of ϕ and ψ is necessary for finite-time blowup (if the critical relation is suppressed), since in presence of the corresponding subcritical relation all solutions are global and bounded (see e.g. [95]). However, in view of the question whether the superlinear growth of the chemotactic sensitivity ψ is necessary, we only provide a partial answer. We show that if $\psi(u)$ decreases fast enough for large u then there exists a diffusivity ϕ such that the supercritical relation is satisfied, but for any positive initial mass of u there are solutions which exist globally in time and blow up in infinite time. The latter result is also unusual, since most results about infinite-time blowup in variants of the Keller-Segel model are only valid for specific, nonarbitrary initial cell masses like critical masses. The existence of a critical growth of ψ separating finite-time and infinite-time blowup remains open; partial answers are given by the refined results in our subsequent papers [3, 4]. More details about Article 5 are provided in Section 4.1.

In Article 6 we consider a chemotaxis system of Keller-Segel type within a slightly different setting. Instead of a single species we now study the competition between two cell populations in the presence of a common chemoattractant. More precisely, we assume that the movement of both species is governed by diffusion and chemotaxis and that both species produce the chemoattractant. Moreover, suppose that they proliferate and compete for resources like nutrients or space, such that their mutual competition takes place according to the classical Lotka-Volterra dynamics. For examples of such species we refer to the introduction of Article 6 and the references therein. We further assume for simplicity that the chemoattractant diffuses much faster than each of the two species, so that its dynamics can be approximated by an elliptic instead of a parabolic equation, and that both species move according to linear diffusion and a linear chemotactic sensitivity. Altogether, we study a parabolic-elliptic variant of the Keller-Segel model supplementary involving competition terms (see (4.11) for the precise model). One basic question is whether in the large time behavior for this system both species coexist, meaning that both population densities converge to a positive steady state as $t \to \infty$, or if competitive exclusion occurs in the sense that one population outcompetes the other such that the latter converges to zero while the former converges to a positive steady state as $t \to \infty$. The coexistence case was studied in [97] and we study competitive exclusion in Article 6. Namely, we show that for the same competition parameters, which imply competitive exclusion for the classical Lotka-Volterra ODE system, competitive exclusion occurs as well for all positive solutions of the Keller-Segel system. This result, which is valid independent of the diffusivity constants, requires the smallness of the chemotactic sensitivities when compared to the proliferation rates from the competition terms. However, it remains open whether a similar behavior can also be observed for larger chemotactic sensitivities. More details about these results can be found in Section 4.2.

Finally, in Articles 7 and 8 we study multiscale models for cancer cell migration. In particular, they contain a Keller-Segel chemotaxis model as a subsystem. Cancer cells migrate through the surrounding tissue in order to reach blood vessels and distal organs, where they initiate further tumors, called metastases. Thereby, the cancer cell migration is influenced by various processes (including diffusion, chemotaxis, and haptotaxis) taking place at different spatial and temporal scales. These scales range from the subcellular level (*microscopic* scale) up to the level of cell and tissue populations (*macroscopic* scale). In addition, the microscopic processes happen at much shorter time scales than the macroscopic ones. In order to model the migration of a cancer cell population we couple a system of PDEs for the macroscopic quantities with a system of ODEs for the subcellular dynamics. In that way we obtain a continuous micro-macro model which is a

rather new approach in the context of cancer cell migration. Such models allow a more detailed modeling than purely macroscopic population models and provide a simplified multiscale approach as compared to models including also the intermediate mesoscale of cell-cell and/or cell-tissue interactions. More details on related multiscale approaches for cancer cell migration are presented in Chapter 5 and in the references therein.
In Article 7, we propose a micro-macro model which focuses on the influence of cell contractivity on cancer cell migration. Thereby, cell contractivity describes the ability of the cancer cell to modify its shape according to its environment. On the macroscopic scale, our model accounts for the densities of cancer cells and tissue fibers in the extracellular matrix (ECM) as well as for the proteolytic rests, which are resulting from the ECM fiber degradation by the cancer cells. Here, the tissue fibers and proteolytic rests are the respective haptotactic and chemotactic attractants for the cancer cells, so that the macroscopic part of our model has the structure (1.4). This system is further coupled with an ODE system which models the binding of cell surface receptors (called integrins) to tissue fibers and proteolytic residuals on the microscopic scale. The coupling between these two scales is provided by the cell contractivity function, which on the one hand is influenced by the integrin dynamics and on the other hand affects the macroscopic cancer cell density. The latter is reflected by an explicit dependence of the diffusivity ϕ and of the haptotactic sensitivity ξ (from (1.4)) on the contractivity function, which is a new feature for continuous micro-macro models. The precise model (5.1) and details about the modeling are provided in Chapter 5. In particular, when considering only cancer cells and proteolytic rests on the macroscale, our model reduces to a Keller-Segel chemotaxis system. Moreover, the purely macroscopic subsystem of type (1.4) is related to the competition model studied in Article 6. The main differences are the haptotaxis term, which provides an additional coupling between the competing cancer cells and tissue fibers apart from the Lotka-Volterra competition terms, and the lack of diffusion for the tissue fibers.
In Article 7, we further assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $n \leq 3$. In view of biologically motivated requirements, we aim at assuming rather modest regularity assumptions. Hence, we prove the local existence and uniqueness within the framework of weak solutions, which is a nontrivial issue, since the system consists of different types of equations which are coupled in a highly nonlinear way. Furthermore, we illustrate the solution behavior by numerical simulations. Afterwards, the global existence of a weak solution has been shown in [14] in a slightly more specific setting. In addition, our model provides a paradigm for further multiscale models in which subcellular processes and their effects on cancer cell migration can be described in a more detailed manner. In particular, the proofs of local and global existence provide a framework for such models. More details about Article 7 are given in Chapter 5.

In Article 8 we focus on a different aspect of cancer cell migration, namely acid-mediated tumor invasion. Cancer cells are able to upregulate some biological mechanisms which cause the acidification of their neighborhood. This in turn leads to apoptosis of normal cells, which cannot survive in an acidic surrounding, and hence provides space for cancer invasion. An acidic tumor environment is generated when cancer cells regulate their intracellular acidity for instance by increasing extrusion of intracellular protons through membrane transporters. Hence, we propose a multiscale model for acid-mediated cancer invasion by accounting for the densities of cancer cells and normal cells and for the concentration of extracellular protons on the macroscopic scale, as well as for the intracellular proton concentration on the microscopic scale. Two features included in our model are pH taxis and a time-varying carrying capacity for the cancer cells due to the effects of acidity. Here the chemotaxis mechanism is called pH taxis, as cancer cells move up the gradient of proton concentration. Altogether, in Article 8 we obtain a micro-macro model, where a macroscopic system of type (1.4) is coupled with an ODE for the intracellular protons (see Chapter 5 for the model (5.5) and the modeling). The micro-macro models proposed in Articles 7 and 8 have a few structural differences. On the macrolevel, in Article 8 on the one hand we do not account for haptotaxis, since now the interactions of cancer cells with normal cells instead of tissue fibers are studied. On the other hand, we include time varying carrying capacities. The subcellular dynamics in Article 8 now consist of only one ODE, whereas the coupling between integrins and cell contractivity on the microscale is included in Article 7. However, it turns out that only the consideration of haptotaxis has a strong impact on the mathematical analysis as it implies weaker regularity properties than chemotaxis.

In Article 8 we prove the global existence of a weak solution in a general framework. Again, the setting of weak solutions is motivated by biological considerations. Moreover, we provide conditions on data and parameters implying the uniqueness and the uniform boundedness of the solution. Finally, we illustrate the solution behavior for different choices of the carrying capacity of cancer cells by numerical simulations. Details about Article 8 are presented in Chapter 5.

To summarize, the results contained in this thesis contribute to the knowledge on the qualitative behavior of solutions to different types of diffusion equations. Thereby, models ranging from nonlinear scalar equations up to multiscale models for cancer cell migration are studied. In view of the increasing complexity of the models, the experience acquired during the development and application of methods to prove the results of former articles often contributed substantially to the understanding of solution behaviors for more complex models and to the development of appropriate methods for the proofs of upcoming outcomes. In the subsequent Chapters 2–5 the results obtained in Articles 1–8 and the different methods used in the proofs are presented in a more detailed way. Part II of this thesis contains the collection of Articles 1–8.

2. Very slow convergence rates for a semilinear heat equation

In this chapter we summarize the results and methods from [8] (Article 1) and start by presenting our result in the context of related works.
We study the large time behavior of nonnegative classical solutions to the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

with $p > 1$ and initial data $u_0 \in C^0(\mathbb{R}^N)$. In spite of its simple structure, (2.1) offers a variety of behaviors (see [89] and the references therein for an overview). Here we study convergence rates of global solutions converging to zero and first notice that positive global-in-time solutions to (2.1) exist if and only if $p > p_F$, where $p_F := \frac{N+2}{N}$ is the Fujita exponent (see [45] and, for the case $p = p_F$, [52, 68]). Concerning rates of convergence to zero of nonnegative solutions to (2.1), in case of $p > p_F$ conditions on the initial data were found implying the same convergence rates as for the linear heat equation $u_t = \Delta u$ (see e.g. [40, 73]). The slowest of these rates is the self-similar rate $t^{-\frac{1}{p-1}}$ in the sense that for some initial data there are positive constants K_1, K_2 such that

$$K_1(t+1)^{-\frac{1}{p-1}} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq K_2(t+1)^{-\frac{1}{p-1}} \quad \text{for all } t \geq 0.$$

Slower convergence rates were found in [49], where the existence of global solutions satisfying

$$t^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to \infty \qquad \text{as } t \to \infty$$

was shown in case of $p > p_c$. Here,

$$p_c := \begin{cases} \infty & \text{for } N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N \geq 11, \end{cases}$$

is the Joseph-Lundgren exponent (see [63]), which satisfies $p_c > \frac{N}{N-2} > 1$ for $N \geq 11$. By studying the exact convergence rates of the above solutions from [49], arbitrarily slow polynomial (or algebraic) rates of convergence to zero for (2.1) were found in [44]. We extend this result in Article 1 and prove the existence of very slow convergence rates, for instance logarithmic rates, for $p > p_c$. We now present the latter two results within a more general framework which also includes rates of infinite-time blowup.

In case of $p > p_c$, on the one hand a singular radially symmetric steady state to (2.1) exists, namely

$$\varphi_\infty(|x|) := L|x|^{-m}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where

$$m := \frac{2}{p-1} \quad \text{and} \quad L := \{m(N-2-m)\}^{\frac{1}{p-1}}.$$

On the other hand, (2.1) has a family of radially symmetric regular positive steady states, which exist for $p \geq \frac{N+2}{N-2}$ and $N \geq 3$, but are only strictly ordered and stable, e.g. with respect to certain weighted L^∞-norms ([49]), in the case $p \geq p_c$. It turns out that the decay of u_0 as $|x| \to \infty$ is very important for the large time behavior of solutions to (2.1) and that the corresponding behavior of the regular steady states separates solutions converging to zero from those blowing up in infinite time. More precisely, we denote by λ_1 the smaller and by λ_2 the larger positive root of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0,$$

which has two distinct positive roots if and only if $p > p_c$.

Then assuming that $\eta \in C^2([0,\infty))$ is positive and belongs to certain function classes which will be specified soon, the following convergence rates were shown:
If $u_0 \in C^0(\mathbb{R}^N)$ satisfies

$$0 \leq u_0(x) < \varphi_\infty(|x|) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \tag{2.2}$$

(which guarantees the global-in-time existence of the solution u to (2.1), see [87]) as well as

$$\varphi_\infty(|x|) - c_1|x|^{-m-\lambda_1}\eta(|x|) \leq u_0(x) \leq \varphi_\infty(|x|) - c_2|x|^{-m-\lambda_1}\eta(|x|), \quad |x| > R, \tag{2.3}$$

with some positive constants c_1, c_2, and R, then for $p > p_c$ there exist positive constants C_1 and C_2 such that the solution to (2.1) satisfies

$$C_1\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_2\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \quad \text{for all } t \geq 0. \tag{2.4}$$

This behavior was first shown for algebraic functions $\eta(z) := z^\alpha$, $z \geq 0$. The case $\alpha \in (-(\lambda_2-\lambda_1+2), 0)$ along with its optimality was established [38, 42, 79], which implies that the corresponding solutions to (2.1) blow up in infinite time with arbitrarily slow algebraic blowup rates. Correspondingly, arbitrary slow algebraic rates of convergence to zero were detected in [44] for the case $\alpha \in (0, \lambda_1)$. In the borderline case $\alpha = 0$, the solutions are bounded and bounded away from zero and according to (2.3) they can be bounded from above and below by suitable regular positive steady states. This case requires a more detailed study of the initial behavior as $|x| \to \infty$ as compared to (2.3) and results concerning the stability of these positive steady states and arbitrarily slow algebraic rates of convergence toward them can be found e.g. in [43, 48, 61, 87, 88].
After these slow algebraic rates were established, it turned out that even slower rates like logarithmic ones occur. Such rates are rarely observed for parabolic equations. More

precisely, given any number of iterations of the logarithm and any $\alpha \neq 0$, for $z_0 > 0$ large enough the function $\eta(z) := (\ln(\ln(\ldots(\ln(z+z_0))\ldots)))^\alpha$, $z \geq 0$, is positive and has the property that (2.2) and (2.3) imply the solution behavior (2.4). First, the case of very slow rates of infinite-time blowup corresponding to $\alpha < 0$ were proved by Fila, King, Winkler, and Yanagida in [41]. Inspired by this result, we prove the corresponding very slow rates of convergence to zero for $\alpha > 0$ in Article 1. In fact, both results are valid for a more general class of functions η including the logarithmic functions given above. As the general conditions on η are quite technical, we confine ourselves to present only our result from Article 1 in full generality (see Section 2.1 below). The corresponding general result for very slow blowup rates is given in [41].
Later on, by slightly adapting the methods used to prove these results, we further established very slow rates of convergence to positive regular steady states for $p > p_c$ (see [10]) as well as slow algebraic rates of convergence to zero and to positive steady state in the critical case $p = p_c$ (see [11, 12] and, for corresponding rates of infinite-time blowup, [39]).

2.1 Results

In order to present the full result from Article 1, we first define the class of functions η for which the behavior (2.4) is shown. The aim is that η is slowly increasing as $z \to \infty$. More precisely, we assume that $\eta \in C^2([0,\infty))$ fulfills

$$\eta(z) > 0, \quad \eta'(z) > 0 \quad \text{and} \quad \eta''(z) \leq 0 \quad \text{for all } z \geq 0, \tag{2.5}$$

η increases slowly near infinity in the sense that

$$\frac{z\eta'(z)}{\eta(z)} \to 0 \quad \text{as } z \to \infty, \tag{2.6}$$

and satisfies

$$\left|\frac{z\eta''(z)}{\eta'(z)}\right| \leq C_\eta \quad \text{for all } z \geq 0 \tag{2.7}$$

with some constant $C_\eta > 0$. Furthermore, we require that for any $\alpha > 0$ and $\gamma > 0$

$$\eta(\gamma z^\alpha) \leq c_{\alpha,\gamma}\eta(z) \quad \text{for all } z \geq 1 \tag{2.8}$$

holds with some constant $c_{\alpha,\gamma} > 0$. Indeed, (2.8) is not a consequence of (2.5)–(2.7) (see the example given after (1.1.9) in Article 1).
In particular, due to (2.5) and (2.6), for any $\beta > 0$

$$\eta(z) \leq C_\beta z^\beta \quad \text{for all } z \geq 1$$

is satisfied with some $C_\beta > 0$. Moreover, $\eta(z) := (\ln(\ln(\ldots(\ln(z+z_0))\ldots)))^\alpha$ satisfies (2.5)–(2.8) for fixed $\alpha > 0$, if $z_0 > 0$ is chosen large enough. As (2.5)–(2.8) are also valid for some bounded functions η, we can prove (2.4) also for these functions η. However, the very slow rates of convergence to zero are obtained only for unbounded η.
In Article 1, we prove the following result (see Theorem 1.1.1).

Theorem 2.1 *Let $N \geq 11$, $p > p_c$ and assume that $u_0 \in C^0(\mathbb{R}^N)$ fulfills* (2.2) *and* (2.3)*, where η meets the conditions* (2.5)–(2.8)*. Then there are positive constants C_1 and C_2 such that the solution u of* (2.1) *satisfies* (2.4)*.*

2.2 Methods

The proof of Theorem 2.1 basically relies on the strategy developed by Fila, Winkler, and Yanagida in [44], where slow algebraic rates of convergence to zero were established. In a first step, in the radially symmetric setting, we use the self-similar change of variables to transform the solution u of (2.1) to a corresponding function v. If v is radially non-increasing, then (2.4) is equivalent to the behavior

$$v(0,s) \simeq e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0 \tag{2.9}$$

of v. In particular, instead of studying convergence to zero for u, we study rates of infinite-time blowup for v (see Section 1.2 in Article 1 and [44, Section 2] for more details concerning the transformation). In order to prove the blowup behavior (2.9) of v, we construct appropriate sub- and supersolutions to the transformed problem and use the comparison principle. Thereby, the supersolution is obtained by constructing two supersolutions, one in an inner region near $x = 0$ and another in a corresponding outer region bounded away from $x = 0$, and matching them. We build the sub- and supersolutions by inserting parts containing η into the corresponding functions from [44]. One advantage of the self-similar transformation is the structure of separated variables in the original functions. However, in order to include η in such a way that the initial behavior (2.3) is reflected, that the matching for the supersolution is possible, and that the sub- and supersolutions allow to prove (2.9), this structure gets lost in the new parts containing η and, in addition, we need to impose (2.8) on η. Both can be seen for instance in the definitions of v_{out} in Lemma 1.3.3 and of v_{sub} in Lemma 1.4.1, where in particular different values of β are used. In contrast to this, in the proof of the corresponding very slow rates of infinite-time blowup in [41] functions with separated variables could be used for the comparison argument and no condition corresponding to (2.8) had to be imposed, whereas conditions (2.5)–(2.7) are motivated by their analogs in [41].

3. Large time behavior for a quasilinear diffusive Hamilton-Jacobi equation

This chapter contains the summary of the results and methods from [9, 1, 6] (Articles 2–4). First we describe the connection of our results to other works.
Here we study nonnegative solutions to the diffusive Hamilton-Jacobi equation

$$\begin{cases} u_t = \Delta_p u + |\nabla u|^q, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $p \geq 2$, $q > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth enough boundary and u_0 is regular enough (at least continuous) and nonnegative with $u_0 \not\equiv 0$. One issue is to investigate in how far the competition between diffusion and the nonlinear gradient source term is reflected in the large time behavior of solutions to (3.1) and to identify optimal parameter regimes for each of the observed phenomena.

Equation (3.1) was first studied in the semilinear case $p = 2$, where classical solutions exist, and two critical exponents concerning the large time behavior were determined, namely $q = 1 (= p - 1)$ and $q = 2 (= p)$. In case of $q \in (0, 1)$, the existence of a continuum of stationary solutions to (3.1) was shown in [23]. In the one-dimensional case $n = 1$, the nonnegative steady states for (3.1) were explicitly calculated and shown to be an ordered one-parameter family. In addition, the convergence of each solution of (3.1) to one of these steady states was proved and the zero state was excluded as a limit function in case of nonnegative and nontrivial initial data, but also sign-changing solutions were studied (see [72]). However, the large time behavior in higher dimensions remained open.
In case of $q \geq 1$, the classical elliptic comparison principle implies that the zero state is the only stationary solution to (3.1). However, in this parameter regime, the nonlinear source term can cause finite-time gradient blowup for (3.1) in the sense that $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ is uniformly bounded, but $\|\nabla u(\cdot, t)\|_{L^\infty(\Omega)}$ becomes unbounded after a finite time. More precisely, while the solutions exist globally in time and are bounded in $C^1(\bar{\Omega})$ in case of $q \in (0, 2]$ for any u_0 (see e.g. [71]), for $q > 2$ this is only true if $\|u_0\|_{C^1(\bar{\Omega})}$ is below some positive threshold and otherwise finite-time gradient blowup occurs (see [17, 92]). Furthermore, it was shown in [26] that all global solutions converge to zero with an exponential

convergence rate for $q \geq 1$, where in the case $q > 1$ the decay rate and the limiting spatial profile are the same as for the linear heat equation, while for $q = 1$ the exponential decay rate is slower than in the latter case. For non-global solutions the gradient blowup takes place at the boundary $\partial\Omega$ and blowup rates as well as the size of the blowup set were studied (see e.g. [50, 74, 93]).

In the quasilinear case $p > 2$, the existence of classical solutions cannot be expected in view of the degeneracy of the diffusion. Instead, the concepts of weak and viscosity solutions turn out to be useful. In spite of some results for the Cauchy problem when $\Omega = \mathbb{R}^n$ (see e.g. the introduction of Article 2), the large time behavior for (3.1) in a bounded domain was fairly open in the quasilinear case $p > 2$ and we intended to provide further insights. Although our final results often parallel those from the case of linear diffusion, the proofs rely on different methods. Concerning the existence of multiple stationary solutions, we show that now $q = p - 1$ is critical, while $q = p$ turns out to be critical with respect to the existence of gradient blowup. We study the case $q < p - 1$ in Article 2 in the one-dimensional case $n = 1$ within the concept of weak solutions and in Article 3 in the context of radially symmetric viscosity solutions, when Ω is the unit ball in $\mathbb{R}^n$ and $n \geq 2$. In both cases we prove the existence of an ordered one-parameter family of steady states. For $n = 1$ this provides a classification of all weak steady states of arbitrary sign, while in the radial setting only all nonnegative, radially symmetric, and non-increasing stationary solutions in the viscosity sense are included. The available knowledge of the steady states then provides a starting point for proving the convergence of solutions of (3.1) to one of these stationary solutions and, like in the semilinear case, the zero state is excluded as limit for nonnegative and nontrivial solutions. Moreover, the steady states are flat in some subdomain of Ω (see Figure 2.1 in Article 2) and the results of Article 2 contain the behavior of solutions irrespective of their sign. Finally, while the proof of the large time behavior in Article 2 relies on the availability of a Liapunov functional, the theory of half-relaxed limits for viscosity solutions is used in Article 3.

The latter theory is again an important part of the method used in Article 4, where the large time behavior of global viscosity solutions satisfying the boundary condition in the classical sense is established in case of $q \geq p - 1$. Namely, we show that in this parameter regime all solutions exist globally in time and converge to zero with the polynomial convergence rate $t^{-\frac{1}{p-2}}$ as $t \to \infty$, where in the case $q > p$ we require in addition that u_0 is small enough. Moreover, for $q > p - 1$ the rescaled solution $t^{\frac{1}{p-2}} u(\cdot, t)$ converges to the same unique profile, which does not depend on u_0, as for the diffusion equation $u_t = \Delta_p u$, so that like in the case $p = 2$ the large time behavior of global solutions for large q is the diffusive one. However, in the critical case $q = p - 1$, the temporal decay rate is the same as for $q > p - 1$, but the limiting spatial profile changes, still being independent of u_0 (and implying the existence of a so called friendly giant). This shows a difference to the linear diffusion case, where the decay rate for $q = 1$ was slower than for $q > 1$, whereas now only the spatial profile changes.

Finally, we indicate in Article 4 that some smallness condition on u_0 is necessary for the global existence in case of $q > p$, as no global Lipschitz continuous weak solution to (3.1)

exists for large initial data. Later this was further studied in [18], where the occurrence of finite-time gradient blowup on the boundary $\partial\Omega$ was proved for $q > p$ and large initial data in the setting of weak solutions. This result confirms that blowup is the influence of the gradient source term in case of $q > p$, whereas global solutions show a purely diffusive behavior for these parameters. Moreover, the uniqueness of weak solutions, which remains open in Article 2, was shown for $q \geq \frac{p}{2}$ in [18]. Recent results on the exclusion of infinite-time gradient blowup and the size of the blowup set can be found e.g. in [19, 20].
Short, but certainly not complete overviews on further results on (3.1) in the whole space $\Omega = \mathbb{R}^n$ or for the case of a negative gradient term $-|\nabla u|^q$ can be found e.g. in [89] and the introduction of Article 2.

3.1 Results

Next, we summarize our results concerning the large time behavior for (3.1). The first theorem contains the results for $q < p - 1$ and $n = 1$ from Article 2 (see Lemma 2.2.1, Theorem 2.4.3, Theorem 2.4.4, and Corollary 2.4.6).

Theorem 3.1 *Assume that $p > 2$, $0 < q < p - 1$, and $\Omega := (-R, R) \subset \mathbb{R}$ with some $R > 0$.*

(a) Let $w \in C^1([-R, R])$ be a weak solution to the stationary problem corresponding to (3.1) in the sense that it satisfies $w(\pm R) = 0$ and

$$\int_{-R}^{R} \Big(-(|w_x|^{p-2}w_x)(x)\,\xi_x(x) + |w_x|^q(x)\,\xi(x)\Big)\mathrm{d}x = 0 \quad \text{for any } \xi \in C_0^\infty((-R,R)).$$

Then w is nonnegative and there is $\vartheta \in [0, R]$ such that $w = w_\vartheta$, where

$$w_\vartheta(x) := \frac{\tilde{c}_0}{\alpha}\left[(R-\vartheta)^\alpha - (|x| - \vartheta)_+^\alpha\right], \qquad x \in [-R, R],$$

for $\vartheta \in [0, R]$ with $\alpha := \frac{p-q}{p-1-q} > 1$ and $\tilde{c}_0 := \left(\frac{p-1-q}{p-1}\right)^{\frac{1}{p-1-q}} > 0$. In particular, $w_R \equiv 0$ in $\bar{\Omega}$.

(b) For any $u_0 \in C^1(\bar{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ there exists a global weak solution u to (3.1) in the sense of Definition 2.4.1. Furthermore, $\sup_{t\geq 0} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ is finite and there exists a unique $\vartheta_0 \in [0, R]$ such that $\|w_{\vartheta_0}\|_{C^0(\bar{\Omega})} = \lim_{t\to\infty} \|u(\cdot,t)\|_{C^0(\bar{\Omega})}$ and

$$\|u(\cdot,t) - w_{\vartheta_0}\|_{C^0(\bar{\Omega})} \to 0 \quad \text{as } t \to \infty$$

are fulfilled. In addition, in case of $u_0 \geq 0$ with $u_0 \not\equiv 0$ we have $\vartheta_0 < R$ (or equivalently $w_{\vartheta_0} \not\equiv 0$), whereas $\vartheta_0 = R$ (or equivalently $w_{\vartheta_0} \equiv 0$) is satisfied in case of $u_0 \leq 0$.

Let us add the following remark.

Remark 3.2 *Although the results in [9] (Article 2) are only stated for $p > 2$ and $1 < q < p-1$, they are indeed valid in the more general regime $p > 2$ and $0 < q < p-1$, as claimed above in Theorem 3.1. In fact, each proof contained in Article 2 remains true without any change, since the combination of the positivity of q with $q < p-1$ is actually sufficient in each step. In order to avoid changes in the introduction of the published paper [9], we confine ourselves with this remark and the footnote to* (2.1.3) *in Article 2.*

Next, we present the extension of the former one-dimensional results to the radial case. These results from Article 3 (see Theorem 3.1.1, Theorem 3.1.3, and Proposition 3.3.1) were even unknown in the semilinear case $p = 2$.

Theorem 3.3 *Suppose that $p \geq 2$, $0 < q < p-1$, and $\Omega := B_1(0) \subset \mathbb{R}^n$ is the unit ball with $n \geq 2$.*

(a) Let $w \in W^{1,\infty}(\Omega)$ be a radially symmetric and non-increasing viscosity solution to $-\Delta_p w - |\nabla w|^q = 0$ in Ω satisfying $w = 0$ on ∂B. Then there is $\vartheta \in [0,1]$ such that $w = w_\vartheta$, where

$$w_\vartheta(x) := c_0 \int\limits_{\max\{|x|,\vartheta\}}^{1} \left(\rho - \vartheta^\beta \rho^{-(\beta-1)}\right)^{\frac{1}{p-1-q}} \mathrm{d}\rho, \qquad x \in \bar{\Omega},$$

for $\vartheta \in [0,1]$ with $\beta := 1 + \frac{(N-1)(p-1-q)}{p-1} > 1$ and $c_0 := \left(\frac{p-1-q}{(p-1)\beta}\right)^{\frac{1}{p-1-q}} > 0$. In particular, we have $w_0(x) = \frac{c_0}{\alpha}\,(1-|x|^\alpha)$ for $x \in \bar{\Omega}$, where $\alpha := \frac{p-q}{p-1-q} > 1$, and $w_1 \equiv 0$ in $\bar{\Omega}$.

(b) For any radially symmetric and nonnegative $u_0 \in W_0^{1,\infty}(\Omega)$ with $u_0 \not\equiv 0$, there exists a unique global (radially symmetric) viscosity solution $u \in C^0(\bar{\Omega} \times [0,\infty))$ to (3.1)*. Moreover, $\sup_{t\geq 0} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ is finite and there is a unique $\vartheta_0 \in [0,1)$ such that*

$$\lim_{t\to\infty} \|u(\cdot,t) - w_{\vartheta_0}\|_{C^0(\bar{\Omega})} = 0.$$

In particular, $\vartheta_0 \in [0,1)$ implies $w_{\vartheta_0} \not\equiv 0$.

Finally, the results in case of $q \geq p-1$ from Article 4 (see Theorem 4.1.2, Theorem 4.1.4, Corollary 4.4.5, and Proposition 4.5.3) are collected in the following theorem.

Theorem 3.4 *Assume that $p > 2$, $q \geq p-1$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary (at least C^2), $n \in \mathbb{N}$, and $u_0 \in C^0(\bar{\Omega})$ is nonnegative such that $u_0 = 0$ on $\partial\Omega$ and $u_0 \not\equiv 0$.*

(a) In case of $q = p-1$, there is a unique global viscosity solution $u \in C^0(\bar{\Omega} \times [0,\infty))$ to (3.1) *in the sense of Definition 4.1.1 and*

$$\lim_{t\to\infty} \left\| t^{\frac{1}{p-2}}\, u(\cdot,t) - f \right\|_{C^0(\bar{\Omega})} = 0\,,$$

where $f \in C^0(\bar{\Omega})$ *is the unique positive viscosity solution to*

$$-\Delta_p f - |\nabla f|^{p-1} - \frac{f}{p-2} = 0 \quad in \quad \Omega, \quad f > 0 \quad in \quad \Omega\,, \quad f = 0 \quad on \quad \partial\Omega\,. \tag{3.2}$$

Furthermore, if $u_0 \in W^{1,\infty}(\Omega)$*, then* $\nabla u(\cdot,t) \in L^\infty(\Omega)$ *for all* $t \geq 0$ *and*

$$\ell[u_0] := \sup_{t\geq 0} \{\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}\} < \infty.$$

(b) Let $q > p-1$*. If* $q > p$*, assume further that there is a nonnegative* $G_0 \in W^{1,\infty}(\Omega)$ *satisfying* $G_0 = 0$ *on* $\partial\Omega$ *such that*

$$u_0(x) \leq \frac{G_0(x)}{\ell[G_0]}\,, \qquad x \in \bar{\Omega}\,, \tag{3.3}$$

where $\ell[G_0]$ *is defined in part (a). Then there is a unique global viscosity solution* $u \in C^0(\bar{\Omega} \times [0,\infty))$ *to* (3.1) *in the sense of Definition 4.1.1 and*

$$\lim_{t\to\infty} \left\| t^{\frac{1}{p-2}}\; u(\cdot,t) - f_0 \right\|_{C^0(\bar{\Omega})} = 0\,,$$

where $f_0 \in C^0(\bar{\Omega})$ *is the unique positive viscosity solution to*

$$-\Delta_p f_0 - \frac{f_0}{p-2} = 0 \quad in \quad \Omega, \quad f_0 > 0 \quad in \quad \Omega\,, \quad f_0 = 0 \quad on \quad \partial\Omega\,. \tag{3.4}$$

In addition, if $u_0 \in W^{1,\infty}(\Omega)$ *then* $\sup_{t\geq 0} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ *is finite.*

(c) Assume in addition that $u_0 \in W^{1,\infty}(\Omega)$*, let* $q > p$*, and define* $r := q/(q-p)$*. There is a constant* $\kappa > 0$ *depending on* Ω*,* p*, and* q *such that, if* $\|u_0\|_{L^{r+1}(\Omega)} > \kappa$*, then* (3.1) *has no global Lipschitz continuous weak solution.*

We add the following explanation concerning the uniform Lipschitz estimate from part (b).

Remark 3.5 *In case of* $q > p-1$*, the result that* $\sup_{t\geq 0} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ *is finite for Lipschitz continuous initial data is stated in Article 4 only for* $q \leq p$ *(see Corollary 4.4.5). However, the proof of the latter Corollary also covers the case* $q > p$*, provided that the additional assumption* (3.3) *is fulfilled. Indeed, it relies on* (4.4.18) *which is also valid in that case and proved in Section 4.5.2 of Article 4.*

Finally, we remark that while the existence and uniqueness of the positive viscosity solution f_0 to (3.4) was proved in [77], the corresponding result for the solution f to (3.2) is one contribution of Article 4. In particular, Theorem 3.4 implies in case of $q = p-1$ the uniform convergence of the solution u of (3.1) to the separated variables solution $u_\infty(x,t) := t^{-\frac{1}{p-2}} f(x)$ of (3.1) with an initial condition being identically infinite in Ω, while the limit function for $q > p-1$ is the corresponding solution $U_\infty(x,t) := t^{-\frac{1}{p-2}} f_0(x)$ of the diffusion equation $u_t = \Delta_p u$. The solutions u_∞ and U_∞ are also called friendly giants and we refer to the introduction of Article 4 for related results concerning other equations.

3.2 Methods

We first describe the methods used in Article 2 to prove Theorem 3.1 for $q < p-1$. While the classification of the stationary solutions given in part (a) is shown by straightforward calculations, the proof of the convergence of a weak solution to (3.1) towards one of these steady states strongly relies on the availability of a Liapunov functional, which is constructed by applying an idea of Zelenyak (see [103]). Although the latter method was already used in [72] for classical solutions in the semilinear case $p = 2$, our proof in the quasilinear case $p > 2$ requires different arguments. In particular, we do not prove the existence of a Liapunov functional for (3.1), but only for the regularized and strictly parabolic problems (2.3.1), and further need additional compactness properties in order to obtain the large time behavior of weak solutions to (3.1). To this end, by constructing suitable sub- and supersolutions and using an idea from [47], we first prove that the classical solutions u_ε to the approximate problems (2.3.1) satisfy a uniform spatial Lipschitz estimate as well as a uniform Hölder estimate with respect to time. Then, uniform estimates of $(u_\varepsilon)_t$ in $L^2(\Omega \times (0, \infty))$ and of the second spatial derivative are direct consequences of the Liapunov functional (see Lemma 2.3.3). Based on these estimates, we prove appropriate compactness properties which imply the existence of a weak solution u to (3.1) as a limit of the approximations u_ε (as $\varepsilon \searrow 0$) as well as the convergence of $u(\cdot, t)$ to one of the stationary solutions w_ϑ as $t \to \infty$. Therein, in view of the nonlinear diffusion, the pointwise convergence of the spatial derivatives $(u_\varepsilon)_x$ to u_x as well as of $u_x(\cdot, t)$ to $(w_\vartheta)_x$ are important and are provided by the two estimates derived from the Liapunov functional in conjunction with the Aubin-Lions lemma. Moreover, we are able to identify ϑ since on the one hand $\|w_\vartheta\|_{C^0(\bar{\Omega})}$ is strictly decreasing for $\vartheta \in [0, R]$ and on the other hand $\|u(\cdot, t)\|_{C^0(\bar{\Omega})}$ is non-increasing for $t \geq 0$.

As the existence of the Liapunov functional relies on the one-dimensional setting, we use a different method to prove the corresponding results of Theorem 3.3 in the higher-dimensional radially symmetric case in Article 3. Namely, we use the theory of viscosity solutions satisfying initial and boundary data in the classical sense (see e.g. the user's guide [31]). In a first step, we classify the nonnegative, Lipschitz continuous, radially symmetric and non-increasing viscosity solutions to the stationary problem corresponding to (3.1). As compared to the straightforward formal proof given in the beginning of Section 3.2 in Article 3 for more regular solutions, a number of comparison arguments for the radial stationary problem are used in the rigorous proof for viscosity solutions in order to prove the validity of (3.2.11) as well as the property that every Lipschitz continuous steady state in fact belongs to $C^1(\bar{\Omega})$. The existence and uniqueness of a uniformly Lipschitz continuous, radially symmetric viscosity solution to the parabolic problem (3.1) is then shown with the help of an approximation by solutions to regularized problems in conjunction with the stability theorem and the comparison principle for viscosity solutions. The proof of the large time behavior of the solution u to (3.1) relies on the method of half-relaxed limits introduced in [24]. As u is uniformly Lipschitz continuous, the half-relaxed limits, defined by

$$u_*(x) := \liminf_{(s,\varepsilon)\to(t,0)} u(x, \varepsilon^{-1}s) \quad \text{and} \quad u^*(x) := \limsup_{(s,\varepsilon)\to(t,0)} u(x, \varepsilon^{-1}s), \quad x \in \bar{\Omega},$$

are well-defined, do not depend on $t > 0$, and u_* and u^* are Lipschitz continuous viscosity super- and subsolutions, respectively, of the stationary problem corresponding to (3.1). In view of $u_* \leq u^*$ by definition, we aim to prove $u_* \geq u^*$ in order to show the equality of the two half-relaxed limits. As there is no comparison principle for the stationary problem, we use the additional properties that u_* and u^* are radially symmetric and non-increasing and have the same maximal value (see Lemma 3.4.1) in order to be able to apply certain comparison arguments which finally allow us to conclude that $u_* \geq u^*$. Therefore, the two half-relaxed limits are equal and coincide with one steady state w_{ϑ_0}, so that the theory from [21, 22] implies the claimed convergence of u to w_{ϑ_0}. Thereby, the proofs of both the classification of the steady states and the equality of the half-relaxed limits strongly rely on the radial setting and it is an open question whether corresponding results can be obtained for general solutions.

The proof of the large time behavior for the case $q \geq p - 1$ from Theorem 3.4 in Article 4 again relies on the theory of viscosity solutions and the method of half-relaxed limits. One main difference as compared to Article 3 is that in order to identify the spatial profiles f and f_0, the convergence is now proved for the rescaled function $v(x, t) := t^{\frac{1}{p-2}} u(x, t)$ and not for u itself. In particular, this requires more precise estimates for u reflecting the decay according to the rate $t^{-\frac{1}{p-2}}$. As a first step, we prove a comparison principle for the stationary problems (3.2) and (3.4) which generalizes a result from [28] and is crucial for the identification of the half-relaxed limits. In case of $q \in [p-1, p]$, the global existence of a unique viscosity solution u to (3.1) directly follows from [32]. We then prove Lipschitz estimates on the boundary and upper bounds for u (which both behave like $t^{-\frac{1}{p-2}}$ for large t) by constructing appropriate barrier functions and supersolutions, respectively. These estimates imply that the rescaled function v is uniformly Lipschitz continuous on the boundary and uniformly bounded, so that the half-relaxed limits corresponding to v are well-defined and Lipschitz continuous on the boundary. In addition, the large time behavior for the diffusion equation $z_t = \Delta_p z$ proved in [77] along with the comparison principle for (3.1) imply that the solution f_0 to (3.4) is a positive lower bound for both half-relaxed limits. This enables us to apply the above mentioned comparison principle for (3.2) or (3.4) to show that both half-relaxed limits coincide either with the solution f to (3.2) or f_0. Finally, we can apply the theory of half-relaxed limits as described for Article 3 to conclude that v converges to f in case of $q = p - 1$ and to f_0 in case of $q \in (p-1, p]$.

In the case $q > p$, the main issue is the well-posedness, because the result from [32] cannot be applied any more. However, we are able to construct a Lipschitz continuous supersolution to (3.1) by using a corresponding solution for (3.1) with $q = p - 1$. If the initial data are below this supersolution, the classical Perron method along with the comparison principle implies the existence and uniqueness of a global viscosity solution to (3.1). Then, the large time behavior can be proved just like in the case $q \in (p-1, p]$. The method from Article 4 also can be used for other equations, see e.g. [5] for the infinite heat equation $u_t = \Delta_\infty u$.

Finally, in the proof of the non-existence of a global Lipschitz continuous weak solution to

(3.1) in case of $q > p$ for large initial data (see Theorem 3.4(c)) we adapt the method from [54] to show that $\|u(\cdot,t)\|_{L^{r+1}(\Omega)}$ becomes unbounded after a finite time provided that it is large enough for $t = 0$.

Article 3 was mainly developed jointly by Philippe Laurençot and myself during my visit to the Université Paul Sabatier de Toulouse in June 2009. Guy Barles introduced us to the method of half-relaxed limits and explained common approaches for proofs of comparison arguments for viscosity solutions, which enabled Philippe Laurençot and me to develop part of the proofs from Article 3.
Article 4 was developed jointly by Philippe Laurençot and myself during his visit to the Universität Duisburg-Essen in January 2010 and my visit to the Université Paul Sabatier de Toulouse in March 2010.

4. Contributions to Keller-Segel chemotaxis models

In this chapter we present our contributions to Keller-Segel chemotaxis models contained in [2, 15] (Articles 5 and 6). Since both articles are related to different aspects of chemotaxis systems, we describe their results and methods in two sections. The first one contains finite-time blowup results for fully parabolic Keller-Segel systems, while the second is concerned with competition of two species in presence of chemotaxis.
As there is a huge literature on Keller-Segel models, we only present results which are very closely related to Articles 5 and 6. For a more general overview we refer e.g. to the surveys [56, 58].

4.1 Finite-time blowup in a quasilinear parabolic-parabolic Keller-Segel system

One feature of Keller-Segel models is the ability to describe aggregation phenomena for populations of cells. As finite-time blowup of the population density is an indication that aggregation can take place before the blowup time, this phenomenon has been studied by many authors. Here we present finite-time blowup results for the parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v), & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, ν is the outward unit normal on $\partial\Omega$, and u_0, v_0 are positive and sufficiently regular. Therein, u denotes the cell density and v the concentration of the chemical signal, which attracts the cells. We assume that the cell motion is governed by diffusion and chemotaxis and that the cells produce the chemical signal. Motivated by the volume-filling model derived in [55], we further require that the diffusivity ϕ and the chemotactic sensitivity ψ depend solely on u. Here we only study classical solutions to some variants of (4.1) and in the sequel *blowup* at time $T \in (0, \infty]$ means that $\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. For parabolic-elliptic simplifications of (4.1) (when the second equation in (4.1) is replaced by an elliptic one) many results of finite-time blowup are known, which mostly rely on the reduction of this system to a scalar parabolic equation or on the use of second moments, the first one being [62]. We refer e.g. to the introduction of Article 5 for a short, but certainly not complete overview about these results. However, for parabolic-parabolic Keller-Segel models, where these methods apparently do not work, before Article 5 there were only three results of finite-time blowup, namely those of Herrero and Velázquez from 1997 ([53]), of Cieślak and Laurençot from 2010 ([30]), and of Winkler proved in 2011 ([101]). In order to describe these results, which all rely on different methods, we look at two variants of (4.1).

We first consider the case $\phi(u) \equiv 1$ and $\psi(u) = u$ in (4.1), where both diffusion and chemotactic sensitivity are linear. This system is often called the classical or minimal Keller-Segel model. Concerning the existence of blowup for classical nonnegative solutions, it was proved in [86] that in dimension $n = 1$ all solutions to (4.1) are global and bounded. In dimension $n = 2$, the existence of a critical mass, which is 4π in the general setting and 8π when restricted to radially symmetric solutions, was shown. Namely, it was established in [46, 83] with two different proofs both using a Liapunov functional that all solutions with initial mass $m := \int_\Omega u_0(x)\,dx$ smaller than the critical mass are global and bounded, while it was shown in [60] that for any initial mass (which is required to be no multiple of 4π in the general setting) larger than the critical mass there exist unbounded solutions. The latter result was again proved by using the Liapunov functional, but it remained open whether these solutions blow up in finite or infinite time. In addition, in the radial case, for some masses larger than 8π some solutions blowing up in finite time were constructed in [53] by using the method of matched asymptotic expansions. It was further shown that the latter solutions develop a singularity of Dirac-δ type and the asymptotic behavior near the blowup time was described in a very detailed way. However, it remained open whether finite-time blowup is a generic phenomenon for large masses and $n = 2$ or depends on the particular choice of the solutions from [53]. It was revealed very recently in [81] by using the method from [101] described below that indeed for any initial mass larger than 8π there exists a large set of radially symmetric initial data such that the corresponding solution to (4.1) exhibits finite-time blowup, which therefore can be seen as a generic phenomenon in this context. In dimensions $n \geq 3$ it was proved in [101] that for any positive initial mass there exist radially symmetric initial data such that the corresponding solutions to (4.1) blow up in finite time and that the set of these initial data is dense with respect to a certain topology. In [101] the Liapunov functional was used in a new way by showing that for a supposedly global solution the corresponding Liapunov functional itself becomes unbounded after a finite time. Indeed [101] marked a breakthrough, because the method used there provides a framework which could be used (e.g. in our works and [81]) and possibly can be further adapted to prove results on finite-time blowup for fully parabolic Keller-Segel models.

As a second case, we describe the results known for the quasilinear variant of (4.1) when the diffusivity $\phi(u) = (u+1)^{-p}$ and the chemotactic sensitivity $\psi(u) = u(u+1)^{q-1}$ are both power type nonlinearities with $p, q \in \mathbb{R}$. This variant of (4.1) is related to the model with volume-filling effect proposed in [55] and serves as a prototype of a fully parabolic Keller-Segel model with both nonlinear diffusion and nonlinear chemotactic sensitivity. Although all results which we will present for this variant of (4.1) are indeed valid for more general choices of ϕ and ψ, for the ease of presentation we will only give the results of Article 5 in full generality (see the next subsection). For (4.1) with the power type nonlinearities ϕ and ψ, it was shown in [95] that in the subcritical case $p+q<\frac{2}{n}$ with $n\in\mathbb{N}$ all solutions are global and bounded. In addition, it was proved in [100] that in the supercritical case $p+q>\frac{2}{n}$ with $n\geq 2$ for any initial mass there exist unbounded solutions, but it remained open whether finite-time or infinite-time blowup occurs. Only in the one-dimensional case $n=1$, it was shown in [30] by using an identity of virial type and the boundedness of the Liapunov functional that finite-time blowup occurs in the supercritical case when $q=1$ and the initial mass is large enough. By generalizing the method from [101] described above, we prove in Article 5 that in the supercritical case $p+q>\frac{2}{n}$ with $n\geq 3$ and $q\geq 1$ for any positive initial mass there exist radially symmetric solutions blowing up in finite time. This result induced the question whether the superlinear growth of ψ (i.e. $q\geq 1$) is really necessary for the existence of finite-time blowup. As a partial answer, we further show in Article 5 that for any $q<1-n$ and $p\in\mathbb{R}$ such that $\frac{2}{n}<p+q\leq 1$ with $n\geq 3$ for any positive initial mass there exist radially symmetric solutions blowing up in infinite time. This reveals that for chemotactic sensitivities $\psi(u)$ decreasing fast enough as $u\to\infty$, infinite-time blowup is a generic phenomenon in the supercritical case and happens for any positive initial mass, while this usually occurs only for particular masses like critical ones in the context of Keller-Segel systems. The results from Article 5 are generalized to the two-dimensional case and further refined in our subsequent papers [3, 4] by slightly adapting the proofs from Article 5 in order to reduce the gap between those values of q enabling finite-time blowup and those implying blowup in infinite time. The current result is summarized in Corollary 4.4 below, but it is still open whether there exists a critical exponent q separating these two types of behavior. Recently, the local non-degeneracy of blowup points in case of $p=0$ was established in [80].

4.1.1 Results

In order to present the general results of Article 5, we need to introduce some notation. The results in the particular case, when the diffusivity ϕ and the chemotactic sensitivity ψ are power type nonlinearities, are given in Corollaries 4.3 and 4.4 below.
In general, let

$$\phi,\psi,\beta\in C^2([0,\infty))\quad\text{with}\quad \phi(s)>0,\quad \psi(s)=s\beta(s),\quad \beta(s)>0\quad\text{for } s\in[0,\infty) \tag{4.2}$$

be fulfilled. We further assume that there exist positive constants s_0, a, b such that

$$G(s):=\int_{s_0}^{s}\int_{s_0}^{\sigma}\frac{\phi(\tau)}{\psi(\tau)}\,d\tau\,d\sigma,\quad s>0,\qquad\text{and}\qquad H(s):=\int_0^s\frac{\sigma\phi(\sigma)}{\psi(\sigma)}\,d\sigma,\quad s\geq 0, \tag{4.3}$$

satisfy

$$G(s)\leq a\,s^{2-\alpha},\ s\geq s_0,\qquad\text{with some }\alpha>\frac{2}{n}, \tag{4.4}$$

as well as

$$H(s) \leq \gamma \cdot G(s) + b(s+1), \ s > 0, \qquad \text{with some } \gamma \in \left(0, \frac{n-2}{n}\right). \tag{4.5}$$

Here, ϕ and ψ satisfy the supercritical relation if (4.4) and (4.5) are fulfilled (which in the case of power type nonlinearities as defined above is equivalent to $p+q > \frac{2}{n}$). Furthermore, it is well-known that

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega G(u) \tag{4.6}$$

is a Liapunov functional for (4.1) with dissipation rate

$$\mathcal{D}(u,v) := \int_\Omega v_t^2 + \int_\Omega \psi(u) \cdot \left|\frac{\phi(u)}{\psi(u)}\nabla u - \nabla v\right|^2. \tag{4.7}$$

More precisely, any classical solution to (4.1) satisfies

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t),v(\cdot,t)) = -\mathcal{D}(u(\cdot,t),v(\cdot,t)) \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)), \tag{4.8}$$

where $T_{max}(u_0,v_0) \in (0,\infty]$ denotes the maximal existence time of (u,v) (see e.g. [100, Lemma 2.1]). Finally, for our finite-time blowup result we further need to impose a superlinear growth condition for the chemotactic sensitivity ψ, namely

$$\psi(s) \geq c_0\, s, \quad s \geq 0, \tag{4.9}$$

with some $c_0 > 0$.
With these conditions and notation, our main results from Article 5 are summarized in the following theorem (see Theorems 5.1.1, 5.1.2, and 5.1.6).

Theorem 4.1 *Suppose that $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and that (4.2) holds.*

(a) Assume further that (4.4), (4.5), and (4.9) are satisfied, and let $m > 0$ and $A > 0$ be given. Then there exist positive constants $T(m,A)$ and $K(m)$ such that for any

$$(u_0,v_0) \in \mathcal{B}(m,A) \ := \ \Big\{ (u_0,v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega) \ \Big| \ u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \bar\Omega,\ \int_\Omega u_0 = m,\ \|v_0\|_{W^{1,2}(\Omega)} \leq A, \text{ and } \mathcal{F}(u_0,v_0) \leq -K(m)\cdot(1+A^2) \Big\},$$

the corresponding solution (u,v) of (4.1) blows up at the finite time $T_{max}(u_0,v_0) \in (0,\infty)$, i.e. $\limsup_{t\nearrow T_{max}(u_0,v_0)} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$, where $T_{max}(u_0,v_0) \leq T(m,A)$.

(b) If (4.4) is satisfied, then for any $m > 0$ there exists $A > 0$ such that $\mathcal{B}(m,A) \neq \emptyset$. If (4.4) is fulfilled with some $\alpha > \frac{4}{n+2}$ and, moreover, $p \in (1, \frac{2n}{n+2})$ is such that $p > 2 - \alpha$, then for any $m > 0$ and $A > 0$, the set $\mathcal{B}(m,A)$ is dense with respect to the topology in $L^p(\Omega) \times W^{1,2}(\Omega)$ in the space of all radially symmetric positive functions $(u_0,v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ satisfying $\int_\Omega u_0 = m$ and $\|v_0\|_{W^{1,2}(\Omega)} < A$.

(c) *Assume that* (4.4) *and* (4.5) *are fulfilled, that* $\lim_{s\to\infty}\phi(s) = 0$*, and that there exist constants* $D > 0$*,* $D_1 > 0$*, and* $\gamma_1 > n$ *such that*

$$\frac{\beta(s)}{\phi(s)} \leq D \quad \textit{and} \quad \beta(s) \leq D_1 s^{-\gamma_1} \qquad \textit{for any } s > 0.$$

Then all solutions to (4.1) *exist globally in time and for any* $m > 0$ *there are radially symmetric global solutions* (u, v) *to* (4.1) *satisfying* $\int_\Omega u_0 = m$ *which blow up in infinite time, i.e.* $\limsup_{t\to\infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.

Let us add the following short remark.

Remark 4.2 *Although not explicitly stated in Theorem 5.1.6 of Article 5, its proof already shows that there is not only one solution blowing up in infinite time, but in fact such solutions exist for any positive initial mass as claimed in part (c) of Theorem 4.1 above. In the proof of Theorem 5.1.6, the latter claim is a consequence of [100, Theorem 5.1] used there, which indeed provides appropriate initial data for any positive mass.*

In the particular case of power type nonlinearities ϕ and ψ, Theorem 4.1 immediately implies the following results (see also Corollary 5.1.5).

Corollary 4.3 *Assume that* $\Omega = B_R(0) \subset \mathbb{R}^n$ *with some* $n \geq 3$ *and* $R > 0$*, and that* $\phi(s) = (s+1)^{-p}$ *and* $\psi(s) = s(s+1)^{q-1}$*,* $s \geq 0$*, with* $p, q \in \mathbb{R}$ *such that* $p + q > \frac{2}{n}$.

(a) *In case of* $q \geq 1$*, for any* $m > 0$ *there exist radially symmetric solutions* (u, v) *to* (4.1) *satisfying* $\int_\Omega u_0 = m$ *which blow up in finite time.*

(b) *In case of* $q < 1 - n$ *and* $\frac{2}{n} < p + q \leq 1$*, all solutions to* (4.1) *exist globally in time and for any* $m > 0$ *there are radially symmetric solutions* (u, v) *to* (4.1) *fulfilling* $\int_\Omega u_0 = m$ *which blow up in infinite time.*

The results from Article 5 were extended to the case $n = 2$ in [3] and further refined in [4]. Combining these results with Corollary 4.3, we have the following results for power type nonlinearities ϕ and ψ.

Corollary 4.4 *Assume that* $\Omega = B_R(0) \subset \mathbb{R}^n$ *with some* $n \geq 2$ *and* $R > 0$*, and that* $\phi(s) = (s+1)^{-p}$ *and* $\psi(s) = s(s+1)^{q-1}$*,* $s \geq 0$*, with* $p, q \in \mathbb{R}$ *such that* $p + q > \frac{2}{n}$.

(a) *If either* $q > \frac{2}{n}$ *and* $p \leq 0$ *or* $q \geq 1$ *is satisfied, then for any* $m > 0$ *there exist radially symmetric solutions* (u, v) *to* (4.1) *satisfying* $\int_\Omega u_0 = m$ *which blow up in finite time.*

(b) *If* $q < 0$ *and* $\frac{2}{n} - q < p < \frac{2}{n} - 2q$*, then all solutions to* (4.1) *exist globally in time and for any* $m > 0$ *there are radially symmetric solutions* (u, v) *to* (4.1) *fulfilling* $\int_\Omega u_0 = m$ *which blow up in infinite time.*

These results imply that in terms of q, which describes the growth of the chemotactic sensitivity ψ, for any $q > \frac{2}{n}$ there are p such that finite-time blowup occurs, while for $q < 0$ there are some p such that infinite-time blowup occurs, but finite-time blowup is excluded. However, it remains open whether there is a critical value $q \in [0, \frac{2}{n}]$ separating these two blowup types. The general results obtained in [3, 4] can be found in [3, Theorems 1.1 and 1.4] and [4, Theorems 1.1 and 1.3]. The main refinements in the proofs as compared to Article 5 are described in the next subsection.

4.1.2 Methods

For proving the finite-time blowup result given in Theorem 4.1(a), we adapt the method from [101], where the case $\phi(u) \equiv 1$ and $\psi(u) = u$ was studied, to the quasilinear case. The main idea is to show that the Liapunov functional $\mathcal{F}(u(t), v(t))$ defined in (4.6) tends to $-\infty$ after a finite time provided that it is small enough at $t = 0$. The main step towards this behavior consists of proving that each solution (u, v) to (4.1) starting from initial data (u_0, v_0), which satisfy all conditions raised in $\mathcal{B}(m, A)$ (defined in Theorem 4.1(a)) except the last one involving $\mathcal{F}(u_0, v_0)$, fulfills

$$\begin{aligned}\int_\Omega uv \;\; &\leq \;\; c_1\left(\left\|\Delta v - v + u\right\|_{L^2(\Omega)}^{2\theta} + \left\|\frac{\phi(u)}{\sqrt{\psi(u)}}\nabla u - \sqrt{\psi(u)}\nabla v\right\|_{L^2(\Omega)}^{2\theta} + 1\right) \\ &\quad + \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega G(u). \end{aligned} \tag{4.10}$$

for all $t \in (0, T_{max}(u_0, v_0))$ with some constants $c_1 > 0$ (depending on m and A) and $\theta \in (\frac{1}{2}, 1)$ (see Lemma 5.3.1). In view of (4.1), (4.7), and Young's inequality, this implies

$$\int_\Omega uv \leq c_2\left(\mathcal{D}^\theta(u, v) + 1\right) + \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega G(u), \qquad t \in (0, T_{max}(u_0, v_0)),$$

with some $c_2 > 0$, which means that we have estimated the only negative ingredient of the Liapunov functional $\mathcal{F}(u, v)$ by a sublinear power of the dissipation rate $\mathcal{D}(u, v)$ and positive ingredients of $\mathcal{F}(u, v)$. Inserting the latter inequality into (4.6), we have

$$\mathcal{F}(u, v) \geq -c_2\left(\mathcal{D}^\theta(u, v) + 1\right), \qquad t \in (0, T_{max}(u_0, v_0)),$$

(see Theorem 5.3.6), which in turn, when combined with (4.8) and the condition $\mathcal{F}(u_0, v_0) \leq -2c_2$, implies

$$\frac{d}{dt}\left[-\mathcal{F}(u(\cdot, t), v(\cdot, t))\right] \geq \left(\frac{-\mathcal{F}(u(\cdot, t), v(\cdot, t))}{2c_2}\right)^{\frac{1}{\theta}}, \qquad t \in (0, T_{max}(u_0, v_0)),$$

(see Lemma 5.4.1). Due to $\theta \in (0, 1)$, we conclude that $\mathcal{F}(u, v)$ and hence also its only negative ingredient $-\int_\Omega uv$ tend to $-\infty$ after a finite time. As Ω is bounded, the latter means that u blows up in finite time, since otherwise parabolic regularity theory applied to the second equation of (4.1) would imply the boundedness of both u and v. This completes the proof of part (a), as we can choose $K(m)$ such that $2c_2 = K(m) \cdot (1 + A^2)$ is satisfied corresponding to the last condition in $\mathcal{B}(m, A)$.
As compared to the original method from [101], apart from some more involved estimates our proof contains two important structural differences. First, in estimate (4.10) we include

positive ingredients of the Liapunov functional (namely the terms in the second line of (4.10)), which were not present in the corresponding estimate [101, (4.7)]. This is a consequence of the more general choices of ϕ and ψ, and is particularly reflected in the proof of Lemma 5.3.4. In the proof of (4.10), which altogether consists of Lemmas 5.3.1–5.3.5, $\int_\Omega uv$ is first estimated in terms of $\int_\Omega |\nabla v|^2$, which in turn is estimated appropriately by splitting the domain Ω into a small ball $B_{r_0}(0)$ and the remaining annulus, where the smallness of r_0 is needed to ensure the sublinear power of the dissipation rate $D(u,v)$ in (4.10). Mainly in Lemma 5.3.4, which contains the estimate on $\int_{B_{r_0}(0)} |\nabla v|^2$, the additional terms stemming from the Liapunov functional as well as the assumptions (4.4), (4.5), and the superlinear condition (4.9) play an important role.

The second difference is the explicit dependence on A in the condition $\mathcal{F}(u_0,v_0) \le -K(m)\cdot(1+A^2)$ raised in the set $\mathcal{B}(m,A)$ for the initial data. This is used in Theorem 4.1(b) to prove that for any initial mass $m>0$ there are suitable initial data in the sense that $\mathcal{B}(m,A) \neq \emptyset$ for some $A>0$. If α in (4.4) is large enough, we can directly adapt the proof from [101] to show even the density of $\mathcal{B}(m,A)$ in the sense claimed in the second part of Theorem 4.1(b), where we do not need the precise dependence on A of the upper bound for $\mathcal{F}(u_0,v_0)$. However, when α in (4.4) is close to $\frac{2}{n}$, which is the border between super- and subcritical case, we could only find appropriate initial data which imply that for any $m>0$ there is some $A>0$ such that $\mathcal{B}(m,A)\neq\emptyset$ (see the first part of Theorem 4.1(b)), if the upper bound on $\mathcal{F}(u_0,v_0)$ depends at most quadratically on A (see the proof of Theorem 5.1.2 starting from (5.4.7)). Moreover, in view of the definition of $\mathcal{F}$ in (4.6) the quadratic dependence on A seems to be optimal. In order to show that this quadratic dependence on A can be achieved in the sense that the above choice $2c_2 = K(m)\cdot(1+A^2)$ is possible, in the proof of part (a) we need to determine the precise dependence of the involved constants on A. In the slightly more general setting in Section 5.3 of Article 5 this corresponds to the dependence on M and B which both depend linearly on A (see the proof of Lemma 5.4.1). Thereby, we further refine some estimates from [101].

The proof of the infinite-time blowup result from Theorem 4.1(c) mainly relies on showing that there is $p>n$ such that $\sup_{t\in(\frac{T}{2},T)} \|u(\cdot,t)\|_{L^p(\Omega)} \le C(T)$ holds for any finite $T \le T_{max}(u_0,v_0)$. Parabolic regularity theory then implies uniform bounds in $L^\infty(\Omega)$ on ∇v and on u in $(\frac{T}{2},T)$ for any such T (see e.g. [29]), which implies the global existence of all solutions to (4.1). As unbounded solutions for any positive mass were already found in [100], the global existence result shows that they indeed blow up in infinite time.

This result of infinite-time blowup has been generalized in [3, 4] by showing bounds on $\|u(\cdot,t)\|_{L^p(\Omega)}$ for any $p\in[1,\infty)$. The latter enables us to conclude the global existence claimed in Corollary 4.4 from a more general result given in [95], which relies on an iterative method of Moser-Alikakos type (see [3, Section 4] and [4, Section 3] for details). Each of the finite-time blowup results in [3, 4] summarized in Corollary 4.4 mainly relies on the refinement of one estimate from the proof of Lemma 5.3.4 (which was described above). More precisely, in the two-dimensional setting we use a logarithmic Young inequality in order to handle the term r^{1-n} in the inequality below (5.3.30), which is not integrable at zero for $n=2$ (see [3, Lemma 2.4] for details). In order to get a finite-time blowup result also for some ψ with sublinear growth in dimensions $n\ge 3$, we use $\frac{s^2}{\psi(s)} \le L(G(s)+s+1)$

for $s > 0$ with some constant $L > 0$ instead of (4.9) in estimate (5.3.27) (see [4, Lemma 2.1 and (2.15)] for details).

Article 5 was jointly developed by Tomasz Cieślak and myself in December 2011 at the Universität Zürich, where both of us had a postdoc position.

4.2 Competitive exclusion in a two-species chemotaxis model

In this section we study the influence of chemotaxis on the competition of two biological species (e.g. cells or bacteria). Thereby we assume that these species are attracted by the same chemical signal, which they produce themselves, and that their movement is governed by diffusion and chemotaxis. Furthermore, we assume that the species proliferate and compete for resources like space or nutrients such that their competition can be modeled by the classical Lotka-Volterra dynamics. Finally, we assume that the chemical signal diffuses much faster than the species so that we may describe its dynamics by an elliptic instead of a parabolic equation. Denoting by u and v the population densities of the two species and by w the concentration of the chemical signal, in Article 6 we therefore consider the two-species Keller-Segel system

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), & (x,t)\in\Omega\times(0,\infty), \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-v-a_2 u), & (x,t)\in\Omega\times(0,\infty), \\ -\Delta w + \lambda w = ku + v, & (x,t)\in\Omega\times(0,\infty), \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & (x,t)\in\partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{4.11}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, u_0 and v_0 are continuous and nonnegative, d_i, μ_i, λ are positive and χ_i, a_i, k are nonnegative parameters. As compared to (4.1), the second species and the competition terms are included, while (4.11) is not fully parabolic and contains only linear diffusion and chemotactic sensitivities. Some examples of species which compete in presence of chemotaxis as well as mathematical models combining competition and chemotaxis are provided in the introduction of Article 6 and the references given therein. In order to present the influence of chemotaxis on the competition of u and v, we first recall that in the absence of diffusion and chemotaxis the classical Lotka-Volterra ODE-system

$$\begin{cases} u_t = \mu_1 u(1-u-a_1 v), & t\in(0,\infty), \\ v_t = \mu_2 v(1-v-a_2 u), & t\in(0,\infty), \end{cases} \tag{4.12}$$

with positive initial data has the following large time behavior depending on the competition parameters a_1 and a_2. In case of weak competition, namely $a_1, a_2 \in [0,1)$, both species coexist in the sense that $(u(t), v(t))$ converges to the positive steady state $(u^*, v^*) := (\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$ as $t\to\infty$. In case of $a_1 > 1 > a_2 \geq 0$, v has a stronger influence on u than u on v so that v outcompetes u in the sense that $(u(t), v(t)) \to (0,1)$ as

$t \to \infty$, a phenomenon called *competitive exclusion* as u converges to zero, but v converges to a positive steady state. By symmetry, u outcompetes v in case of $0 \leq a_1 < 1 < a_2$. Finally, in case of $a_1, a_2 > 1$ in (4.12) the steady state $(0,0)$ is unstable, $(1,0)$ and $(0,1)$ are locally asymptotically stable, and (u^*, v^*) is a saddle. These results for (4.12) can be found e.g. in [16, 82, 102].
One basic question is in how far chemotaxis influences the competition behavior as compared to (4.12). For mathematical models containing both chemotaxis and competition, results on the qualitative behavior of solutions including global existence were obtained e.g. in [70, 105]. Furthermore, the existence and stability of steady states for coexistence or competitive exclusion were studied e.g. in [33, 34, 59, 98, 105]. However, before 2012 there was no result concerning the qualitative behavior of solutions in the case of competitive exclusion, when chemotaxis and competition involving both species are present. For systems like (4.11) fulfilling the latter conditions, a few recent results are known. Namely, (4.11) was studied in [97] in the coexistence case $a_1, a_2 \in [0, 1)$ and it was shown that if $k = 1$ and the proliferation rates μ_i are large enough as compared to the competition parameters a_i and the chemotactic sensitivity rates χ_i, then for all positive and continuous initial data the solution to (4.11) satisfies $(u(\cdot,t), v(\cdot,t)) \to (u^*, v^*)$ as $t \to \infty$ so that coexistence is observed. A further coexistence result was proved in [84] for a system related to (4.11), which includes non-local competition terms, if the chemotactic sensitivity parameters are small as compared to the competition terms. In addition, the large time behavior for a system, where the right-hand side of the third equation in (4.11) is replaced by a given regular function $f(x,t)$ satisfying $\|f(\cdot,t) - \frac{1}{|\Omega|}\int_\Omega f(x,t)dx\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$, was recently studied in [85]. In particular, competitive exclusion for $a_1 > 1 > a_2 \geq 0$ as well as coexistence for $a_1, a_2 \in [0, 1)$ were proved without a smallness assumption on the chemotactic sensitivity parameters for positive initial data, but in this case the third equation for w is discoupled from the first two equations for u and v. The proofs of [84, 85, 97] rely on comparison arguments with solutions to ODEs and such arguments are also part of our method used in Article 6. For a general framework we refer to [85].
We study the coexistence case for (4.11) in Article 6. Assuming that $a_1 > 1 > a_2 \geq 0$ and that the chemotactic sensitivity parameters are small as compared to the proliferation rates in the sense that $\frac{\chi_1}{\mu_1}$ and $\frac{\chi_2}{\mu_2}$ are small enough, we prove that for any nonnegative and continuous initial data (u_0, v_0) with $v_0 \not\equiv 0$ there exists a unique global classical solution to (4.11) and competitive exclusion occurs such that $(u(\cdot,t), v(\cdot,t), w(\cdot,t))$ converges to $(0, 1, \frac{1}{\lambda})$ as $t \to \infty$. Again we need the smallness of the chemotactic sensitivity rates and it remains open whether a similar behavior can also be observed for larger chemotactic sensitivities. Possibly even blowup can occur for large enough chemotactic sensitivities, which has been observed in two-species chemotaxis systems without competition terms (see e.g. [36, 37]), but seems to be unknown for systems like (4.11) including competition.

4.2.1 Results

The results of Article 6 are given in the following theorem (see Theorem 6.1.1 and Remark 6.1.3).

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n \in \mathbb{N}$, $d_i, \mu_i, \lambda > 0$, $\chi_i \geq 0$ for $i \in \{1,2\}$ as well as*

$$a_1 > 1 > a_2 \geq 0$$

be fulfilled. Assume further that k, $q_1 := \frac{\chi_1}{\mu_1}$, and $q_2 := \frac{\chi_2}{\mu_2}$ are nonnegative such that

$$q_1 \leq a_1, \quad q_2 < \frac{1}{2}, \quad \text{and} \quad kq_1 + \max\left\{q_2 \ , \ \frac{a_2 - a_2 q_2}{1 - 2q_2} \ , \ \frac{kq_2 - a_2 q_2}{1 - 2q_2}\right\} < 1 \tag{4.13}$$

are satisfied. Then for any nonnegative $u_0, v_0 \in C^0(\bar{\Omega})$ such that $v_0 \not\equiv 0$ there exists a unique global-in-time classical solution (u, v, w) to (4.11) satisfying $v > 0$, $w > 0$ in $\bar{\Omega} \times (0, \infty)$, and either $u \equiv 0$ in $\bar{\Omega} \times [0, \infty)$ or $u > 0$ in $\bar{\Omega} \times (0, \infty)$ as well as

$$u(\cdot, t) \to 0, \quad v(\cdot, t) \to 1, \quad \text{and} \quad w(\cdot, t) \to \frac{1}{\lambda} \qquad \text{as } t \to \infty,$$

uniformly with respect to $x \in \Omega$.
Furthermore, for the existence of a unique global-in-time solution (u, v, w) to (4.11) such that u, v, and w are bounded in $\bar{\Omega} \times [0, \infty)$ it is sufficient to require $kq_1 + q_2 < 1$ instead of (4.13).

Remark 6.1.2 provides particular examples for which (4.13) becomes easier to handle. Although it remains open whether (4.13) is optimal in general, in the special case $k = a_2 = 0$, where (4.11) reduces to the single-species parabolic-elliptic Keller-Segel system with logistic source (for v and w), (4.13) becomes $q_2 < \frac{1}{2}$, which coincides with the condition from [96] for the latter single-species chemotaxis system.

4.2.2 Methods

In order to prove the result of global existence and uniqueness from Theorem 4.5, in Article 6 we first show the local existence of a solution by using a usual fixed point argument. For the global existence it then remains to show the boundedness of u and v. This is done by proving that (u, v) is a subsolution to a cooperative parabolic system, for which on the one hand the comparison principle is valid and on the other hand appropriate constant supersolutions in case of $kq_1 + q_2 < 1$ can be constructed (see Lemma 6.2.2). In order to prove the competitive exclusion behavior, we use comparison arguments with solutions to ODEs in a different framework as compared to [84, 85, 97], where appropriate ODE systems were constructed from the original PDE model. In Article 6, we show for instance that u is a subsolution to a suitable scalar parabolic equation, for which a spatially homogeneous supersolution solving a corresponding ODE exists. By comparing u with this supersolution and using similar arguments for v, we obtain inequalities containing upper or lower bounds for

$$L_1 := \limsup_{t \to \infty} \Big(\max_{x \in \bar{\Omega}} u(x,t) \Big), \ \ L_2 := \limsup_{t \to \infty} \Big(\max_{x \in \bar{\Omega}} v(x,t) \Big), \ \ l_2 := \liminf_{t \to \infty} \Big(\min_{x \in \bar{\Omega}} v(x,t) \Big)$$

in terms of some of these quantities and the constants k, q_i, a_i, $i = 1, 2$, defined in Theorem 4.5. These inequalities are developed in Lemmas 6.3.3–6.3.5. By just using these inequalities irrespective of their origin in conjunction with (4.13), we finally show that $L_1 = 0$ and $L_2 = l_2 = 1$ are fulfilled (see Lemma 6.3.6 and Section 6.4 in Article 6). In view of Lemma 6.3.1, this proves the claimed asymptotic behavior of (u, v, w).

The main part of Article 6 was jointly developed by José Ignacio Tello, Michael Winkler, and myself in May 2012 during a visit of J.I. Tello to the Universität Paderborn. Article 6 was then completed by exchange of emails, where again each of us contributed comparably.

5. Contributions to multiscale models for tumor invasion

This chapter is devoted to the description of results and methods from [7, 13] (Articles 7 and 8). Both works are concerned with multiscale models for cancer cell migration and, unlike in the previous articles, one of the issues is the derivation of appropriate models. Tumor cell migration is influenced by a plethora of processes taking place at different spatial scales which range from the subcellular level (microscopic scale) via the mesoscopic scale of cell interactions and up to the macroscopic scale of cell and tissue populations. Related mathematical descriptions usually focus on specific aspects of cancer cell migration which are either modeled on one of these scales or in multiscale settings involving two or all three scales. We refer to the introduction of Articles 7 and 8 for a review of monoscale models related to the settings therein. For couplings of subcellular processes with macroscopic population behaviors, several types of multiscale models are used in the context of tumor cell migration. Individual- and force-based models can be found e.g. in [90, 91]. Moreover, hybrid models relying on cellular automata or agent-based approaches provide a framework for coupling individual events with macroscopic features (see e.g. [27, 51, 66, 104]). On the one hand these models allow for a very detailed modeling, but on the other hand they become computationally very expensive for biologically realistic numbers of cells. In contrast to this, continuum models allow to describe the evolution of averaged phenomena by means of differential equations and are more efficient from the numerical point of view. Thereby, mesoscopic interactions are usually described with the help of kinetic transport equations of Boltzmann type from which macroscopic parabolic or hyperbolic PDEs can be derived by appropriate scalings and limits. In this context a general framework which allows to include subcellular processes was provided in [25]. Two particular examples of this model class can be found in [75], where a general global well-posedness result for weak solutions was established, and [35], where the inclusion of subcellular processes in a micro-meso-macro model for glioma invasion was crucial for the observation of fingering patterns in numerical simulations. Since these micro-meso-macro models are very challenging for the numerics and the rigorous derivation of macroscopic PDEs from mesoscopic kinetic equations is only available in particular cases, micro-macro models provide a simplified multiscale approach which avoids these difficulties and concentrates on the evolution of macroscopic populations, but still allows to take into account subcellular dynamics.

The latter approach is rather new in the context of cancer cell migration and consists of coupling a system of PDEs on the macroscale with ODEs modeling particular aspects of the subcellular dynamics. In such a way for instance the influence of intra- and extracellular acidity (see [99] for a model only relying on ODEs), heat shock proteins (see [78, 94]), and glycolysis (see [67]) on tumor invasion as well as the influence of integrin dynamics on haptotactic invasion (see [76]) were modeled. In these works, apart from the modeling the solution behavior is illustrated by numerical simulations, whereas only [78] provides a proof of the mathematical (local) well-posedness. The latter is a nontrivial issue since these multiscale models contain quite different types of equations which are highly nonlinearly coupled and the biologically motivated regularity assumptions are rather modest. We use such micro-macro models as well in Article 7, where we focus on the influence of cell contractivity on tumor invasion, and in Article 8, where the influence of intra- and extracellular acidity on cancer cell invasion is studied. For the contractivity model we prove the local well-posedness in the context of weak solutions in Article 7, while the global existence was later proved in [14] in a slightly more specific framework. In Article 8, we prove the global well-posedness of the acidity model in an appropriate setting of weak solutions. In both articles we also perform numerical simulations in order to illustrate the behavior of the solutions.

5.1 Results

In this section we present the analytical results of Articles 7 and 8, along with a short description of the modeling. We refer to Articles 7 and 8 for more details about the models as well as for the numerical simulations.

In Article 7 we focus on the influence of cell contractivity on tumor invasion. Thereby, the cell contractivity describes the ability of the cancer cell to modify its shape according to its environment. As cancer cells use adhesion to tissue fibers of the extracellular matrix (ECM) for migration, they preferably move toward increasing fiber concentrations by means of haptotaxis. Thereby, the contractivity influences the cell motility by enabling them to drastically change their shape and hence to squeeze through the network of tissue fibers. In addition, when the tissue is too dense, cancer cells are prone to degrade the tissue fibers by proteolysis. The resulting small proteolytic rests are soluble, and thus can diffuse and serve as a chemoattractant for the cancer cells. Altogether, we propose the multiscale model

$$\begin{cases} c_t = \nabla\cdot(\varphi(\kappa,c,v)\nabla c) - \nabla\cdot(\psi(\kappa,v)c\nabla v) - \nabla\cdot(f(c,l)c\nabla l) \\ \qquad +\mu_c c\left(1-\frac{c}{K_c}-\eta_1\frac{v}{K_v}\right), \\ v_t = -\delta_v cv + \mu_v v\left(1-\eta_2\frac{c}{K_c}-\frac{v}{K_v}\right), \\ l_t = \alpha\Delta l + \delta_l cv - \beta l, \\ \boldsymbol{y}_t = \boldsymbol{G}(v,l,\boldsymbol{y}), \\ \kappa_t = -q\kappa + H(\boldsymbol{y}(t-\tau)). \end{cases} \tag{5.1}$$

in $(0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth enough boundary and

$n \in \{1, 2, 3\}$. We further assume the boundary conditions

$$\frac{\partial c}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial l}{\partial \nu} = 0 \qquad \text{on } (0,T) \times \partial\Omega, \tag{5.2}$$

where ν denotes the outward unit normal vector on $\partial\Omega$, and the initial conditions

$$\begin{matrix} c(0,x) = c_0(x), & v(0,x) = v_0(x), & l(0,x) = l_0(x), \\ \kappa(0,x) = \kappa_0(x), & \boldsymbol{y}(t,x) = \boldsymbol{y}_0(x), & \end{matrix} \qquad t \in (-\infty, 0],\, x \in \Omega. \tag{5.3}$$

Here, accounting for the cancer cell density c, the density v of tissue fibers in the ECM, and the concentration l of proteolytic rests on the macroscopic scale, the PDE for c in (5.1) includes diffusion, cross diffusion terms modeling hapto- and chemotaxis as well as a Lotka-Volterra competition term describing the proliferation of cancer cells limited by the presence of the ECM. As the ECM is not moving, v satisfies an ODE including the degradation by the cancer cells as well as the remodeling of the tissue in presence of competition for space with the cancer cells. The PDE for the concentration l of proteolytic residuals includes diffusion, production due to the degradation of tissue by the cancer cells as well as decay. A new feature in these macroscopic equations is the dependence of the diffusivity φ and the haptotactic sensitivity ψ of cancer cells on the cell contractivity κ carrying the information about microscopic dynamics to the macroscopic level. On the microscale, we see the binding of cell surface receptors (called integrins) to tissue fibers and proteolytic residuals as the onset of a number of processes which finally lead to changes in the contractivity. Denoting by y_1 and y_2 the respective concentrations of integrins bound to ECM and proteolytic rests, the integrin dynamics is described by an ODE system for $\boldsymbol{y} = (y_1, y_2)$. Finally, as changes in the contractivity are the outcome of several processes which are initiated by the integrin dynamics, we propose an ODE for κ including a time delay τ. More details about the derivation of the resulting model (5.1), as well as examples for the coefficient functions are provided in Article 7 (Section 7.2).

As the diffusivity and the haptotactic sensitivity of cancer cells depend on the contractivity κ, which in turn is influenced by the subcellular integrin dynamics, where the latter depends on the maroscopic quantities v and l, (5.1) contains several strong couplings between the micro- and macroscales, which affect the analysis. In order to present the general well-posedness result, we define

$$Y := \{(y_1, y_2) \in (0, R_0)^2 \mid y_1 + y_2 < R_0\}$$

and the mapping $\boldsymbol{G} : Y \times [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\boldsymbol{G}(\boldsymbol{y}, v, l) := \begin{pmatrix} k_1(R_0 - y_1 - y_2)v - k_{-1}y_1 \\ k_2(R_0 - y_1 - y_2)l - k_{-2}y_2 \end{pmatrix}, \tag{5.4}$$

where R_0 is the total integrin concentration of each cancer cell, k_1, k_2 are respective binding rates for the binding of integrins to ECM and proteolytic residuals, and k_{-1}, k_{-2} the corresponding detaching rates. Then Y is a positive invariant set for the ODE system forming the fourth equation of (5.1). We have the following result of local well-posedness in Article 7 (see Theorems 7.3.1 and 7.3.2) which particularly applies to the coefficient functions given in (7.2.8):

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n \in \{1,2,3\}$, $p \in (\frac{n+2}{2}, \infty)$, and define the spaces*

$$X := \left\{u \in L^p(0,T;W^{2,p}(\Omega)) \;:\; u_t \in L^p(0,T;L^p(\Omega))\right\},$$
$$Z := L^{2p}(0,T;W^{1,2p}(\Omega)), \qquad V := C^1(0,T;C^0(\bar{\Omega})).$$

Furthermore, we fix a time lag $\tau \geq 0$ and assume that

$$c_0, v_0, l_0 \in W^{2,p}(\Omega),\ \kappa_0 \in W^{1,2p}(\Omega),\ \boldsymbol{y}_0 \in (W^{1,2p}(\Omega))^2,$$
$$\frac{\partial c_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial l_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad 0 < c_0 < K_c,$$
$$0 < v_0 < K_v,\ l_0 > 0,\ \kappa_0 > 0 \text{ and } \boldsymbol{y}_0 \in Y \text{ for all } x \in \bar{\Omega}.$$

Moreover, suppose that (5.4) *is fulfilled, all constants in* (5.1) *are positive with $\eta_1, \eta_2 \in (0,1)$, and let*

$$H \in C^1(\bar{Y}), f \in C^1([0,\infty)^2), \varphi \in C^1([0,\infty)^3), \psi \in C^1([0,\infty)^2) \text{ be}$$
$$\text{nonnegative such that for any } 0 < a < b < \infty \text{ there exists } \delta_{a,b} > 0 \text{ with}$$
$$\varphi(\kappa, c, v) \geq \delta_{a,b} \text{ for all } (\kappa, c, v) \in [a,b] \times [0,b]^2.$$

Then there is $T > 0$ such that there exists a unique weak solution to (5.1)–(5.3) *satisfying*

$$c, l \in X,\ v \in X \cap V,\ \kappa \in Z \cap V,\ \boldsymbol{y} \in Z^2 \cap V^2 \text{ such that } 0 \leq c \leq K_c,$$
$$0 < v \leq K_v,\ l \geq 0,\ \kappa > 0 \text{ and } \boldsymbol{y} \in Y \text{ for all } (t,x) \in [0,T) \times \bar{\Omega}.$$

Furthermore, for the specific parameter choices given in (7.2.8) we have proved the global existence of a weak solution in [14] together with M. Winkler by establishing a certain quasi-dissipative functional for regularizations of (5.1) (stemming to a large extent from his contribution).

In Article 8 we focus on the influence of intra- and extracellular acidity on cancer cell migration. As normal cells cannot survive in an acidic surrounding, the latter provides space for cancer cell invasion. An acidic tumor environment is generated when cancer cells regulate their intracellular acidity for instance by increasing extrusion of intracellular protons through membrane transporters. Hence, we account for the densities c of cancer cells and n of normal cells and the concentration h of extracellular protons on the macroscopic scale, as well as for the intracellular proton concentration y on the microscopic scale. As compared to the ODE model in [99] and the multiscale model proposed in [57] including stochasticity, in particular pH taxis as well as time varying carrying capacities are new features in our model. As cancer cells prefer an acidic environment they follow extracellular pH gradients. This so-called pH taxis is a mechanism by which tumor cells follow the pH gradient available in their surroundings. Since tumor cells have more space in an acidic environment due to apoptosis of normal cells, we propose that their carrying capacitity depends on the extracellular acidity h and that it is time delayed, as its adaptation to the acidosis is not instantaneous. Altogether, in Article 8 we propose the micro-macro model

$$\begin{cases} \partial_t c = \nabla\cdot(\varphi(c,n)\nabla c) - \nabla\cdot(f(h,c)\nabla h) & \\ \qquad +\mu_c(y)c\left(1-\frac{c}{K_c(h(\cdot,t-\tau))}-\eta_1\frac{n}{K_n}\right) & \text{in } \Omega\times(0,T), \\ \partial_t n = -\delta_n hn + \mu_n n\left(1-\eta_2\frac{c}{K_c(h(\cdot,t))}-\frac{n}{K_n}\right) & \text{in } \Omega\times(0,T), \\ \partial_t h = D_h\Delta h + R(y,h) & \text{in } \Omega\times(0,T), \\ \partial_t y = -R(y,h) - \alpha y + g(c) & \text{in } \Omega\times(0,T), \end{cases} \tag{5.5}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary and $N\leq 3$. We further endow (5.5) with the boundary conditions

$$\partial_\nu c = \partial_\nu h = 0 \qquad \text{on } \partial\Omega\times(0,T) \tag{5.6}$$

and initial conditions

$$\begin{aligned} c(x,0)=c_0(x),\quad n(x,0)=n_0(x),\quad y(x,0)=y_0(x) \qquad & \text{for } x\in\Omega, \\ h(x,t)=h_0(x,t) \qquad & \text{for } x\in\Omega,\ t\in[-\tau,0]. \end{aligned} \tag{5.7}$$

The PDE for the cancer cells c includes diffusion and pH taxis with both nonlinear diffusivity and pH-tactic sensitivity as well as the proliferation of cancer cells in presence of competition with normal cells for space. Here, as described above, the carrying capacity K_c of cancer cells is time varying and delayed, while their proliferation rate depends on the intracellular acidity described by y. Furthermore, the normal cells are not diffusing, so that n solves an ODE involving their degradation by extracellular acidity and their proliferation in presence of competition with cancer cells. Since extracellular protons diffuse, h solves a PDE including diffusion and exchange of extra- and intracellular protons. The latter is described by the function $R(y,h)$. Correspondingly, $-R(y,h)$ describes the corresponding loss term in the microscopic ODE solved by the intracellular proton concentration y. In addition, the latter equation accounts for degradation of y as well as its production by the cancer cells. Details on the derivation of (5.5) are presented in Article 8 (Section 8.2).

In order to obtain the global existence of a weak solution to (5.5), we require the following assumptions. These are motivated by the coefficient functions proposed in (8.2.6), which indeed satisfy (5.8)–(5.11) below. Namely, let

$$\begin{gathered} \varphi,f,R\in C^1([0,\infty)^2),\quad \mu_c,K_c,g\in C^1([0,\infty)) \text{ such that } g\in L^\infty((0,\infty)) \\ \text{and}\quad g\geq 0,\ \mu_c>0,\ K_c>0 \quad \text{on } [0,\infty). \end{gathered} \tag{5.8}$$

Moreover, we assume that there exist $H_0, Y_0\in(0,\infty)$ such that

$$\begin{gathered} R(y,0)\geq 0,\ R(y,H_0)\leq 0 \quad \text{for all } y\in[0,Y_0],\quad R(0,h)\leq 0 \quad \text{for all } h\in[0,H_0], \\ -R(Y_0,h)-\alpha Y_0+\|g\|_{L^\infty((0,\infty))}\leq 0 \quad \text{for all } h\in[0,H_0]. \end{gathered} \tag{5.9}$$

H_0 and Y_0 are upper bounds for the concentrations of the extra- and intracellular protons, respectively. R describes the effect of the proton exchange between the interior of the cancer cell and its environment. Here the first two conditions in (5.9) mean for instance that

there is no proton transport into the tumor cell if there are no extracellular protons, while protons cannot leave the cell if the extracellular proton concentration is at its maximal value.
With H_0 and Y_0 as defined above, we further assume that there exist positive constants C_1 and C_2 such that

$$0 \le f(h,c) \le C_1(1+c), \quad \frac{C_2}{1+c} \le \varphi(c,n) \le C_1 \;\; \forall (c,n,h) \in [0,\infty)\times[0,K_n]\times[0,H_0], \tag{5.10}$$

and that for any $a \in (0,H_0)$ there is $C_a > 0$ such that

$$f(h,c) \le C_a \quad \text{for all } (c,h) \in [0,\infty)\times[a,H_0]. \tag{5.11}$$

Concerning the initial data suppose that

$$\begin{gathered} c_0, n_0, y_0 \in C^0(\bar\Omega), \quad h_0 \in C^0([-\tau,0];W^{1,q}(\Omega)), \\ c_0 \ge 0, \quad 0 \le n_0 \le K_n, \quad 0 \le y_0 \le Y_0 \quad \text{in } \bar\Omega, \quad \delta \le h_0 \le H_0 \quad \text{in } \bar\Omega \times [-\tau,0]. \end{gathered} \tag{5.12}$$

with some $q \in (N+2,\infty)$ and $\delta > 0$. Then we have the following global existence result in Article 8 (see Theorem 8.3.2 and Remark 8.3.7):

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $N \in \mathbb{N}$, $\alpha > 0$, $\eta_1, \eta_2 \in (0,1)$, $\tau > 0$, and assume that (5.8)–(5.12) are fulfilled. Then there exists a global weak solution to (5.5)–(5.7) in the sense of Definition 8.3.1 satisfying*

$$\begin{gathered} c \in L^\infty_{loc}(\bar\Omega \times [0,\infty)), \quad 0 \le n \le K_n \quad \text{and} \quad 0 \le y \le Y_0 \quad \text{in } \Omega\times(0,\infty), \\ h \in L^\infty((0,\infty);W^{1,q}(\Omega)), \quad 0 \le h \le H_0 \quad \text{in } \Omega \times (-\tau,\infty). \end{gathered} \tag{5.13}$$

If in addition $c_0 \in C^\beta(\bar\Omega)$ is satisfied with some $\beta \in (\frac{1}{N+2},1)$, then there is a unique global weak solution within the class of functions satisfying the conditions of Definition 8.3.1 and $h \in L^r_{loc}([0,\infty);W^{1,r}(\Omega))$ for some $r > N+2$.
If the conditions on h_0 are replaced by $h_0 \in C^0([-\tau,0];W^{1,\infty}(\Omega))$ with $h_0 \ge 0$ in $\bar\Omega\times[-\tau,0]$ and instead of (5.10) and (5.11) we assume

$$0 \le f(h,c) \le C_1(1+c)^{m_1}, \quad C_2(1+c)^{-m_2} \le \varphi(c,n) \le C_1(1+c)^{m_3}$$

for all $(c,n,h) \in [0,\infty)\times[0,K_n]\times[0,H_0]$ with some real numbers m_j, $j=1,2,3$, satisfying $2m_1+m_2 < 3$, then there exists a global weak solution to (5.5)–(5.7) fulfilling $c \in L^\infty(\bar\Omega \times [0,\infty))$ in addition to (5.13).

One particular intention of Theorems 5.1 and 5.2 was to allow on the one hand a preferably general class of coefficient functions including the respective examples (7.2.8) and (8.2.6) and to require on the other hand fairly weak regularity of the data so that both well-posedness results can be applied to rather general classes of micro-macro models.

5.2 Methods

The local existence result of Theorem 5.1 is proved by defining an iterative sequence of solutions to equations which approximate (5.1), but are decoupled from each other, hence allowing to circumvent the strong couplings of the original system (see (7.3.5)–(7.3.9)). A fixed point argument in conjunction with compactness properties then imply the convergence of this sequence of solutions to a local weak solution of the original problem. Its uniqueness is shown with the help of Gronwall's lemma. Thereby, in particular the haptotaxis term requires very detailed estimates as on the one hand the spatial derivatives of v are present in the equation for c, but on the other hand the ODE satisfied by v does not provide any gain of spatial regularity as compared to the initial data. Moreover, Theorem 5.1 is first proved for the case $\tau = 0$ (see Theorem 7.3.1) and then extended to the case of a constant temporal delay $\tau > 0$ with the help of the method of steps (see Theorem 7.3.2).
For proving the global existence result of Theorem 5.2, we approximate (5.5) by appropriate regularizations (see (8.3.11)), for which the local existence of a classical solution is verified by a fixed point argument involving parabolic Schauder theory. The global existence for any of these approximate solutions is further shown by providing suitable uniform bounds on ∇c_ε and ∇h_ε, which in conjunction with bounds on $\|c_\varepsilon\|_{L^\infty_{loc}((0,\infty),L^p(\Omega))}$ for any finite p also allow to obtain the latter uniform bound for $p = \infty$. Thereby, we adapt arguments which are known for purely macroscopic Keller-Segel chemotaxis systems (see Lemmas 8.3.4–8.3.6). Appropriate compactness arguments finally imply the convergence of these approximate solutions to a global weak solution to (5.5)–(5.7). The uniqueness part of Theorem 5.2 is obtained by combining results on maximal parabolic regularity with Gronwall's lemma and applying the proof of uniqueness from Theorem 5.1. Although (5.5) contains pH taxis, which is a kind of chemotaxis, the absence of haptotaxis allows to obtain stronger results as compared to (5.1).

In Articles 7 and 8 the analysis was done by myself, while Christina Surulescu mainly contributed the modeling and Gülnihal Meral was responsible for the implementation of the numerical schemes. However, these three parts influence each other, since e.g. discussions on the analytic and numerical results induced changes in the model, hence our contributions cannot be completely separated.

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Part II

Selected Articles

Article 1:
Very slow convergence to zero for a supercritical semilinear parabolic equation

by Christian Stinner

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Abstract

We study the asymptotic behavior of nonnegative solutions to the Cauchy problem for a semilinear parabolic equation with a supercritical nonlinearity. It is known that there are initial data such that the corresponding solution decays to zero with an algebraic rate. Furthermore, any algebraic rate which is slower than the self-similar rate occurs as decay rate for some solution. In this paper we prove that the convergence to zero can take place with an "arbitrarily" slow rate, if the initial data are chosen properly.

Key words: convergence to zero, semilinear parabolic equation, Cauchy problem
MSC 2010: 35K15, 35B40, 35K57

1.1 Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1.1}$$

where $u = u(x,t)$, $p > 1$, Δ denotes the Laplacian operator with respect to x and the function u_0 is nonnegative and continuous in $\mathbb{R}^N$. In spite of its simple structure, problem

(1.1.1) offers a rich variety of mathematical phenomena and has been studied intensively by several authors. The monograph [A1.20] and the references given therein provide a broad overview.
Concerning the large-time behavior of solutions to (1.1.1), the Fujita exponent

$$p_F := \frac{N+2}{N}$$

is one of the critical exponents. It is well-known that in case of $1 < p \leq p_F$ each positive solution of (1.1.1) blows up in finite time, whereas there are positive global solutions in case of $p > p_F$ (see [A1.12]). With regard to global solutions converging to zero, different decay rates have been proved by several authors. In [A1.16] it has been proved that, for $p > p_F$ and initial data u_0 satisfying

$$k_1(1+|x|)^{-l} \leq u_0(x) \leq k_2(1+|x|)^{-l}, \quad x \in \mathbb{R}^N,$$

with positive and small constants k_1 and k_2 the corresponding solution u of (1.1.1) exists globally in time and decays to zero at the same rate as the solution of the linear heat equation with the same initial data. Namely, u fulfills

$$K_1\, g(t) \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq K_2\, g(t), \qquad \text{for } t \geq t_1 > 1,$$

where g is given by

$$g(t) := \begin{cases} t^{-\frac{N}{2}} & \text{if } l > N, \\ t^{-\frac{N}{2}} \ln t & \text{if } l = N, \\ t^{-\frac{l}{2}} & \text{if } \frac{2}{p-1} \leq l < N. \end{cases}$$

Moreover, it is shown in [A1.4] that for $l > \frac{2}{p-1}$ this behavior of u occurs for a larger class of initial data without a smallness condition on k_2. We remark that the slowest of these decay rates is $t^{-\frac{1}{p-1}}$.
Furthermore, several conditions have been found which imply that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\, t^{-\frac{1}{p-1}} \qquad \text{for } t > 0$$

holds with some positive constant C (see e.g. [A1.13], [A1.15], [A1.17], [A1.21], [A1.22]). Additionally, there are solutions which decay to zero at a rate which is slower than $t^{-\frac{1}{p-1}}$. In order to present these results, we define another critical exponent

$$p_c := \begin{cases} \infty & \text{for } N \leq 10, \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N \geq 11, \end{cases}$$

which satisfies $p_c > \frac{N}{N-2} > 1$ for $N \geq 11$. Moreover, let $\varphi_\infty = \varphi_\infty(|x|)$ denote the singular steady state of (1.1.1), which exists for $p > \frac{N}{N-2}$, $N > 2$, and is given by

$$\varphi_\infty(|x|) := L|x|^{-m}, \quad |x| > 0,$$

where $m := \frac{2}{p-1}$ and $L := \{m(N-2-m)\}^{\frac{1}{p-1}}$. Finally, in case of $p > p_c$ we set

$$\lambda_1 = \lambda_1(N,p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2-8(N-2-m)}}{2},$$

which is the smaller positive root of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0,$$

while λ_2 is defined to be the larger positive root of this equation. It was proved in [A1.15] that for $p > p_c$ with initial data u_0 fulfilling

$$0 \le u_0(x) < \varphi_\infty(|x|) \quad \text{for } |x| > 0 \tag{1.1.2}$$

and

$$\varphi_\infty(|x|) - \kappa_1|x|^{-l} \le u_0(x) \le \varphi_\infty(|x|) - \kappa_2|x|^{-l} \quad \text{for } |x| > R \tag{1.1.3}$$

with $l \in (m, m+\lambda_1)$ and some positive constants κ_1, κ_2 and R, the solution u of (1.1.1) is global in time and converges to zero in such a way that

$$t^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to \infty \quad \text{as } t \to \infty$$

is satisfied. In [A1.9] the exact decay rate of this slow convergence to zero was determined and it was shown that

$$K_1(t+1)^{-\frac{m(m+\lambda_1-l)}{2\lambda_1}} \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le K_2(t+1)^{-\frac{m(m+\lambda_1-l)}{2\lambda_1}} \quad \text{for all } t \ge 0 \tag{1.1.4}$$

holds with positive constants K_1 and K_2. Moreover, (1.1.4) is also valid in case of $p > p_c$, if the initial data satisfy (1.1.2) and (1.1.3) with some $l \in [m+\lambda_1, m+\lambda_2+2)$. This shows that solutions grow up for $l > m+\lambda_1$ while they remain bounded and bounded away from zero for $l = m+\lambda_1$. We refer to [A1.2] and [A1.7] for the grow-up rate when $l > m+\lambda_1$ and to [A1.8], [A1.14], [A1.18] and [A1.19] for the convergence to regular steady states if $l = m+\lambda_1$. Furthermore, the rates of convergence to singular steady states (see [A1.6]), the convergence to self-similar solutions (see [A1.10], [A1.11]) and the grow-up rate in the critical case $p = p_c$ (see [A1.3]) have been established.

We remark that any algebraic decay rate slower than the self-similar one occurs due to (1.1.4) for solutions converging to zero, if the initial data are chosen suitably. As the grow-up can take place with any arbitrarily slow rate and in particular with rates which are slower than any algebraic rate (see [A1.5]), we are concerned with the question whether the convergence to zero also occurs with an arbitrarily slow rate.
To this end, we assume in this paper $p > p_c$ with $N \ge 11$ as well as (1.1.2) and

$$\varphi_\infty(|x|) - c_1|x|^{-m-\lambda_1}\eta(|x|) \le u_0(x) \le \varphi_\infty(|x|) - c_2|x|^{-m-\lambda_1}\eta(|x|), \quad |x| > R, \tag{1.1.5}$$

where c_1, c_2 and R are positive constants. Here η is supposed to increase slowly at infinity like for example $\eta(z) = (\ln(z+z_0))^n$ for $n \in \mathbb{N}$ or $\eta(z) = \ln(\ln(\ldots(\ln(z+z_0))\ldots))$.

Actually, the conditions on η which are raised below are satisfied for any of these examples if z_0 is chosen large enough.
Throughout this paper, $\eta \in C^2([0,\infty))$ is supposed to fulfill

$$\eta(z) > 0, \quad \eta'(z) > 0 \quad \text{and} \quad \eta''(z) \le 0 \quad \text{for all } z \ge 0 \tag{1.1.6}$$

such that η increases slowly near infinity in the sense that

$$\frac{z\eta'(z)}{\eta(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.1.7}$$

Furthermore, we assume that

$$\left| \frac{z\eta''(z)}{\eta'(z)} \right| \le C_\eta \quad \text{for all } z \ge 0 \tag{1.1.8}$$

holds with a positive constant C_η. Finally, we suppose that for any $\alpha > 0$ and $\gamma > 0$ there is a positive constant $c_{\alpha,\gamma}$ such that

$$\eta(\gamma z^\alpha) \le c_{\alpha,\gamma}\eta(z) \quad \text{for all } z \ge 1 \tag{1.1.9}$$

is fulfilled. Indeed, condition (1.1.9) is not a consequence of (1.1.6), (1.1.7) and (1.1.8) which can be seen for example with the function $\eta(z) := e^{(\ln(z+2))^\varepsilon}$, where $\varepsilon > 0$ is a small constant. Now (1.1.7) and (1.1.8) imply

$$\frac{z^2\eta''(z)}{\eta(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.1.10}$$

Moreover, we obtain from (1.1.6) and (1.1.7) that for any $\alpha > 0$ there is a constant $C_\alpha > 0$ such that

$$\eta(z) \le C_\alpha z^\alpha \quad \text{for all } z \ge 1. \tag{1.1.11}$$

We are now able to state our main result which shows that the convergence to zero in (1.1.1) takes place with arbitrarily slow decay rates, if the initial data are chosen suitably. In particular, there are solutions converging to zero with decay rates that are slower than any algebraic rate.

Theorem 1.1.1 *Let $N \ge 11$, $p > p_c$ and assume that $u_0 \in C^0(\mathbb{R}^N)$ fulfills* (1.1.2) *and* (1.1.5)*, where η meets the conditions* (1.1.6)*,* (1.1.7)*,* (1.1.8) *and* (1.1.9)*. Then there are positive constants C_1 and C_2 such that the solution u of* (1.1.1) *satisfies*

$$C_1\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C_2\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \quad \textit{for all } t \ge 0.$$

We remark that there are also bounded functions η which fulfill the conditions raised above. In this case Theorem 1.1.1 gives another proof of (1.1.4) with $l = m + \lambda_1$. Although we are here only interested in the case where η is unbounded, we prove Theorem 1.1.1 for general functions η as we do not need the unboundedness of η in the proof.

This paper is organized in the following way. In Section 1.2 we shortly introduce the self-similar change of variables which transforms radially symmetric solutions of (1.1.1) decaying to zero with a very slow rate into solutions of another problem which grow up. In Sections 1.3 and 1.4 we prove an upper and lower bound of the corresponding grow-up rate by constructing suitable super- and subsolutions, respectively, and using comparison arguments. These super- and subsolutions for the transformed problem are more transparent than for (1.1.1) itself. Finally, we complete the proof of Theorem 1.1.1 in Section 1.5.

1.2 Self-similar change of variables

To prove Theorem 1.1.1 we make use of a suitable transformation which has been an important ingredient of [A1.9]. A radially symmetric solution of (1.1.1) with the behavior claimed in Theorem 1.1.1 will be transformed into a function which grows up. We will derive estimates for this grow-up rate which will imply the claimed behavior of solutions to (1.1.1). Here we shortly introduce the transformation and refer to [A1.9] for more details. If $u = u(r,t)$, $r = |x|$, is a radially symmetric solution of (1.1.1), it satisfies

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r} u_r + u^p, & r > 0, t > 0, \\ u(r,0) = u_0(r), & r > 0. \end{cases} \tag{1.2.1}$$

The self-similar change of variables

$$v(\rho, s) = (t+1)^{\frac{1}{p-1}} u(r,t), \quad \rho = \frac{r}{\sqrt{t+1}}, \quad s = \log(t+1) \tag{1.2.2}$$

transforms (1.2.1) into the problem

$$\begin{cases} v_s = v_{\rho\rho} + \frac{N-1}{\rho} v_\rho + v^p + \frac{\rho}{2} v_\rho + \frac{m}{2} v, & \rho > 0, s > 0, \\ v(\rho, 0) = v_0(\rho) \equiv u_0(\rho), & \rho > 0. \end{cases} \tag{1.2.3}$$

Now our aim is to show that a radially nonincreasing solution v of (1.2.3) grows up such that

$$v(0,s) \simeq e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0 \tag{1.2.4}$$

is satisfied. This will imply the claimed behavior of u.

1.3 Upper bound

We use an idea from [A1.9] to prove an upper estimate for the grow-up rate of solutions to (1.2.3) which corresponds to (1.2.4). For this purpose, we construct two supersolutions of (1.2.3), one of them in an inner region near $\rho = 0$ and the other one in a corresponding outer region which is bounded away from $\rho = 0$.

To this end, let ψ denote the classical solution of

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \psi^p = 0, & \xi > 0, \\ \psi(0) = 1, \psi_\xi(0) = 0. \end{cases} \tag{1.3.1}$$

Then for $p > p_c$, ψ satisfies the asymptotic expansion

$$\begin{array}{ll} \psi(\xi) & = L\xi^{-m} - a\xi^{-m-\lambda_1} + o(\xi^{-m-\lambda_1}), \\ \psi_\xi(\xi) & = -mL\xi^{-m-1} + a(m+\lambda_1)\xi^{-m-\lambda_1-1} + o(\xi^{-m-\lambda_1-1}), \end{array} \quad \xi \simeq \infty, \tag{1.3.2}$$

where a is a positive constant (see [A1.7], [A1.14]). This implies

$$L\xi^{-m} - a_1\xi^{-m-\lambda_1} \le \psi(\xi) \le L\xi^{-m} - a_2\xi^{-m-\lambda_1} \quad \text{for } \xi \ge 1 \tag{1.3.3}$$

with some positive constants a_1 and a_2. Furthermore, ψ has the following property.

Lemma 1.3.1 *Suppose $N \ge 11$, $p > p_c$ and ψ is the solution of* (1.3.1). *Then*

$$\psi(\xi) + \frac{1}{m}\xi\,\psi_\xi(\xi) \ge 0 \quad \textit{for } \xi \ge 0$$

is satisfied.

Proof. As the positive radially symmetric steady states of (1.1.1) are ordered in case of $p \ge p_c$, we obtain $\psi(\xi) < \varphi_\infty(\xi)$ for any $\xi > 0$ (see e.g. [A1.2] and the references given there).
We fix $\xi_0 > 0$ and set $B_\xi := B_\xi(0) \subset \mathbb{R}^N$ for $\xi > 0$. Due to the fact that $N \ge 11$ and $p > p_c > \frac{N}{N-2}$, we have

$$-m - 1 = -\frac{2}{p-1} - 1 > -\frac{2}{\frac{N}{N-2} - 1} - 1 = -(N-2) - 1 = -(N-1).$$

Adapting an idea used in [A1.14], we conclude by Green's identity (where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$) that

$$\begin{aligned} & N\omega_N\xi_0^{N-1}[\varphi_\infty(\xi_0)\psi_\xi(\xi_0) - \psi(\xi_0)\varphi_\infty'(\xi_0)] \\ &= N\omega_N\xi_0^{N-1}[\varphi_\infty(\xi_0)\psi_\xi(\xi_0) - \psi(\xi_0)\varphi_\infty'(\xi_0)] - \lim_{\xi\searrow 0} N\omega_N\xi^{N-1}[\varphi_\infty(\xi)\psi_\xi(\xi) - \psi(\xi)\varphi_\infty'(\xi)] \\ &= \lim_{\xi\searrow 0} \int\limits_{\partial(B_{\xi_0}\setminus B_\xi)} \left(\varphi_\infty\frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\varphi_\infty}{\partial\nu}\right)\mathrm{d}S = \lim_{\xi\searrow 0} \int\limits_{B_{\xi_0}\setminus B_\xi} (\varphi_\infty\Delta\psi - \psi\Delta\varphi_\infty)\mathrm{d}x \\ &= \lim_{\xi\searrow 0} \int\limits_{B_{\xi_0}\setminus B_\xi} (-\varphi_\infty\psi^p + \psi\varphi_\infty^p)\mathrm{d}x = \lim_{\xi\searrow 0} \int\limits_{B_{\xi_0}\setminus B_\xi} \varphi_\infty\psi(\varphi_\infty^{p-1} - \psi^{p-1})\mathrm{d}x \ge 0. \end{aligned}$$

As this implies

$$\xi_0^{-m}\psi_\xi(\xi_0) + \psi(\xi_0)m\xi_0^{-m-1} \ge 0$$

and ξ_0 is positive, the claim is valid for ξ_0. Hence, the claim is proved, since $\xi_0 > 0$ is arbitrary and the claim is obvious for $\xi = 0$. ∎

In order to construct a supersolution of (1.2.3) in an inner region, we let Ψ denote the solution of

$$\begin{cases} \Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \frac{m+\lambda_1-l}{l-m}\left(\psi + \frac{1}{m}\xi\,\psi_\xi\right) + \frac{A}{1+\xi^{m+\lambda_1}}, \quad \xi > 0, \\ \Psi(0) = 0, \Psi_\xi(0) = 0, \end{cases} \tag{1.3.4}$$

where $l \in (m, m+\lambda_1)$ is fixed. Moreover, due to Lemma 3.1 in [A1.9], we are able to choose $A > 0$ such that

$$\begin{aligned} \Psi(\xi) &= K\xi^{2-m-\lambda_1} + o(\xi^{2-m-\lambda_1}), \\ \Psi_\xi(\xi) &= -K(m+\lambda_1-2)\xi^{1-m-\lambda_1} + o(\xi^{1-m-\lambda_1}), \end{aligned} \quad \xi \simeq \infty, \tag{1.3.5}$$

and

$$|\Psi(\xi)| + |\xi\,\Psi_\xi(\xi)| \le C_\Psi(1+\xi)^{2-m-\lambda_1} \quad \text{for all } \xi \ge 0 \tag{1.3.6}$$

is satisfied with positive constants K and C_Ψ.
Now by an adaption of the idea used to prove Lemma 3.2 of [A1.9] we obtain a suitable supersolution in an inner region near $\rho = 0$.

Lemma 1.3.2 *Suppose $N \ge 11$ and $p > p_c$. For $\rho \ge 0$, $s \ge 0$, $M > 0$, $\beta > 0$ and $\mu > 0$ we define*

$$\sigma(s) := Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}), \qquad \xi(\rho, s) := \sigma^{\frac{1}{m}}(s)\,\rho$$

and

$$v_{in}(\rho, s) := \sigma\left(\psi(\xi) - \frac{\sigma_s}{\sigma^p}\Psi(\xi)\right).$$

Then for any $\beta > 0$ there are $\mu > 0$, $M_0 > 0$ and $\rho_0 > 0$ such that v_{in} is a supersolution of (1.2.3) *for $0 < \rho < \rho_0$ and $s > 0$ and*

$$\frac{\sigma_s}{\sigma^p}\frac{|\Psi(\xi)|}{\psi(\xi)} \le \frac{1}{2} \quad \textit{for } \rho > 0 \textit{ and } s > 0 \tag{1.3.7}$$

is fulfilled, whenever $M \ge M_0$ holds.

Proof. We fix $\beta > 0$. First we compute

$$\sigma_s(s) = \frac{m}{2}Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)})\left(1 - \frac{2\beta}{\lambda_1}\cdot\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})}\right) \quad \text{for } s \ge 0$$

and define

$$\varepsilon := \frac{m+\lambda_1-l}{l-m} > 0,$$

where l is chosen in (1.3.4). Due to (1.1.6) and (1.1.7) there is $\mu > 0$ such that

$$\frac{m}{2(1+\varepsilon)}\sigma(s) \le \sigma_s(s) \le \frac{m}{2}\sigma(s) \quad \text{for all } s \ge 0 \tag{1.3.8}$$

is satisfied for any $M > 0$. We fix $\mu > 0$ with this property.
Next, (1.3.3), (1.3.6) and $\lambda_1 > 2$ imply the existence of $\xi_0 \geq 1$ such that

$$\frac{|\Psi(\xi)|}{\psi(\xi)} \leq \frac{C_\Psi}{\frac{L}{2}}\xi^{2-\lambda_1} \leq \frac{2C_\Psi}{L} \qquad \text{for } \xi \geq \xi_0.$$

Hence, due to the positivity of ψ there is $c_0 > 0$ such that $\frac{|\Psi(\xi)|}{\psi(\xi)} \leq c_0$ holds for all $\xi > 0$. Therefore, due to (1.1.11) we obtain

$$\begin{aligned}\frac{\sigma_s}{\sigma^p}\frac{|\Psi(\xi)|}{\psi(\xi)} &\leq c_0\frac{m}{2}\sigma^{1-p} = c_0\frac{m}{2}M^{-\frac{2}{m}}e^{-s}\eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)}) \\ &\leq c_0\frac{m}{2}M^{-\frac{2}{m}}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^{\mu} \leq \frac{1}{2} \qquad \text{for } \rho > 0, s > 0,\end{aligned}$$

whenever

$$M \geq M_1 := \left(c_0\, m(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^{\mu}\right)^{\frac{m}{2}} \tag{1.3.9}$$

is fulfilled. Thus, (1.3.7) is satisfied for sufficiently large M.
As $m + \lambda_1 > \lambda_1 > 2$, we can fix $\vartheta \in (\frac{2}{m+\lambda_1}, 1)$ with $\vartheta < \frac{2}{m+\lambda_1-2}$ and obtain $c_\vartheta > 0$ such that

$$(1-z)^p \leq 1 - pz + c_\vartheta|z|^{1+\vartheta} \qquad \text{for } |z| \leq \frac{1}{2}$$

holds. Hence, for $M \geq M_1$, we conclude

$$\left(\psi - \frac{\sigma_s}{\sigma^p}\Psi\right)^p \leq \psi^p - \frac{\sigma_s}{\sigma^p}p\psi^{p-1}\Psi + c_\vartheta\left(\frac{\sigma_s}{\sigma^p}\right)^{1+\vartheta}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} \tag{1.3.10}$$

for all $\rho > 0$ and $s > 0$ by (1.3.7).
Now, we let $\mathcal{P}$ denote the operator defined by

$$\mathcal{P}w := w_s - w_{\rho\rho} - \frac{N-1}{\rho}w_\rho - w^p - \frac{\rho}{2}w_\rho - \frac{m}{2}w.$$

Due to $\sigma\xi_s = \frac{1}{m}\sigma_s\xi$, $1 + \frac{2}{m} = p$, (1.3.1), (1.3.4) and (1.3.10) we compute for $M \geq M_1$

$$\begin{aligned}Pv_{in} &= \sigma_s\psi + \sigma\psi_\xi\xi_s - \left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s - \sigma^{1+\frac{2}{m}}\psi_{\xi\xi} + \frac{\sigma_s}{\sigma^{p-1-\frac{2}{m}}}\Psi_{\xi\xi} - \frac{N-1}{\rho}\sigma^{1+\frac{1}{m}}\psi_\xi \\ &\quad + \frac{N-1}{\rho}\frac{\sigma_s}{\sigma^{p-1-\frac{1}{m}}}\Psi_\xi - \sigma^p\left(\psi - \frac{\sigma_s}{\sigma^p}\Psi\right)^p - \frac{\rho}{2}\sigma^{1+\frac{1}{m}}\psi_\xi + \frac{\rho}{2}\frac{\sigma_s}{\sigma^{p-1-\frac{1}{m}}}\Psi_\xi \\ &\quad - \frac{m}{2}\sigma\psi + \frac{m}{2}\frac{\sigma_s}{\sigma^{p-1}}\Psi \\ &= \sigma_s\left(\psi + \frac{1}{m}\xi\,\psi_\xi\right) - \left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s - \sigma^p\left(\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi\right) \\ &\quad + \sigma_s\left(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi\right) - \sigma^p\left(\psi - \frac{\sigma_s}{\sigma^p}\Psi\right)^p - \sigma\left(\frac{\xi}{2}\psi_\xi + \frac{m}{2}\psi\right) \\ &\quad + \frac{\sigma_s}{\sigma^{p-1}}\left(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\right)\end{aligned}$$

$$
\begin{aligned}
&= \sigma_s\left(\Psi_{\xi\xi}+\frac{N-1}{\xi}\Psi_\xi+\psi+\frac{1}{m}\xi\,\psi_\xi\right)-\left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s \\
&\quad +\sigma^p\psi^p-\sigma^p\left(\psi-\frac{\sigma_s}{\sigma^p}\Psi\right)^p-\sigma\left(\frac{\xi}{2}\psi_\xi+\frac{m}{2}\psi\right)+\frac{\sigma_s}{\sigma^{p-1}}\left(\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right) \\
&\geq \sigma_s\left(\Psi_{\xi\xi}+\frac{N-1}{\xi}\Psi_\xi+p\psi^{p-1}\Psi+\psi+\frac{1}{m}\xi\,\psi_\xi\right)-\left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s \\
&\quad -c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta}-\frac{m}{2}\sigma\left(\psi+\frac{1}{m}\xi\,\psi_\xi\right)+\frac{\sigma_s}{\sigma^{p-1}}\left(\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right) \\
&= \sigma_s\left(\frac{A}{1+\xi^{m+\lambda_1}}+(1+\varepsilon)\big(\psi+\frac{1}{m}\xi\,\psi_\xi\big)\right)-\left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s \\
&\quad -c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta}-\frac{m}{2}\sigma\left(\psi+\frac{1}{m}\xi\,\psi_\xi\right)+\frac{\sigma_s}{\sigma^{p-1}}\left(\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right) \\
&\geq \frac{A\sigma_s}{1+\xi^{m+\lambda_1}}+\frac{\sigma_s}{\sigma^{p-1}}\left(\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right)-c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta}-\left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s \\
&=: I_1+I_2-I_3-I_4, \qquad (1.3.11)
\end{aligned}
$$

for $\rho > 0$ and $s > 0$, where the last inequality is valid thanks to (1.3.8) and Lemma 1.3.1. Next, we show that I_2, I_3 and I_4 are small as compared to I_1, if $\rho \leq \rho_0$, $M \geq M_0$ and ρ_0, M_0 are chosen suitably.

For $\xi \geq 1$, we obtain by (1.3.6) (as $m+\lambda_1 > 2$)

$$
\begin{aligned}
\frac{|I_2|}{\frac{1}{3}I_1} &= \frac{3}{A}\sigma^{-\frac{2}{m}}(1+\xi^{m+\lambda_1})\left|\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right| \leq \frac{3}{A}\sigma^{-\frac{2}{m}}2\xi^{m+\lambda_1}\left(\frac{1}{2}+\frac{m}{2}\right)C_\Psi\xi^{2-m-\lambda_1} \\
&= \frac{3(m+1)C_\Psi}{A}\sigma^{-\frac{2}{m}}\xi^2 = \frac{3(m+1)C_\Psi}{A}\rho^2 \leq 1 \qquad (1.3.12)
\end{aligned}
$$

provided that

$$
\rho \leq \rho_1 := \left(\frac{3(m+1)C_\Psi}{A}\right)^{-\frac{1}{2}}. \qquad (1.3.13)
$$

Furthermore, if $\xi < 1$, (1.1.11), (1.3.6) and $m+\lambda_1 > 2$ imply

$$
\begin{aligned}
\frac{|I_2|}{\frac{1}{3}I_1} &= \frac{3}{A}\sigma^{-\frac{2}{m}}(1+\xi^{m+\lambda_1})\left|\frac{\xi}{2}\Psi_\xi+\frac{m}{2}\Psi\right| \leq \frac{3}{A}M^{-\frac{2}{m}}e^{-s}\eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)})\cdot 2\left(\frac{1}{2}+\frac{m}{2}\right)C_\Psi \\
&\leq \frac{3(m+1)C_\Psi(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^\mu}{A}M^{-\frac{2}{m}} \leq 1, \qquad (1.3.14)
\end{aligned}
$$

whenever

$$
M \geq M_2 := \left(\frac{3(m+1)C_\Psi(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^\mu}{A}\right)^{\frac{m}{2}}. \qquad (1.3.15)
$$

Next, the choice of ϑ yields

$$
\frac{2}{m+\lambda_1} < \vartheta \leq \frac{2}{m+\lambda_1-2} < \frac{2}{m} = p-1
$$

since $\lambda_1 > 2$. As $\psi \leq 1$, we obtain

$$\frac{|I_3|}{\frac{1}{3}I_1} = \frac{3}{A}(1+\xi^{m+\lambda_1})c_\vartheta \frac{\sigma_s^\vartheta}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} \leq \frac{3c_\vartheta}{A}\left(\frac{\sigma_s}{\sigma^p}\right)^\vartheta (1+\xi^{m+\lambda_1})|\Psi|^{1+\vartheta}.$$

Thus, if $\xi \geq 1$ and $\rho \leq 1$, we conclude by (1.1.11), (1.3.6), (1.3.8) and the choice of ϑ

$$\begin{aligned}
\frac{|I_3|}{\frac{1}{3}I_1} &\leq \frac{3c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta \sigma^{-\frac{2\vartheta}{m}} 2\xi^{m+\lambda_1} C_\Psi^{1+\vartheta}\xi^{(1+\vartheta)(2-m-\lambda_1)} \\
&= \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}\sigma^{-\frac{2\vartheta}{m}}\xi^{2-(m+\lambda_1-2)\vartheta} \\
&= \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}\sigma^{-\frac{(m+\lambda_1)\vartheta-2}{m}}\rho^{2-(m+\lambda_1-2)\vartheta} \\
&\leq \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta} M^{-\frac{(m+\lambda_1)\vartheta-2}{m}} e^{-\frac{(m+\lambda_1)\vartheta-2}{2}s}\eta^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}}(e^{\beta(s+\mu)}) \\
&\leq \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}} e^{\frac{(m+\lambda_1)\vartheta-2}{2}\mu} M^{-\frac{(m+\lambda_1)\vartheta-2}{m}} \\
&\leq 1,
\end{aligned} \tag{1.3.16}$$

if

$$M \geq M_3 := \left(\frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}} e^{\frac{(m+\lambda_1)\vartheta-2}{2}\mu}\right)^{\frac{m}{(m+\lambda_1)\vartheta-2}}. \tag{1.3.17}$$

Similarly, if $\xi < 1$ we conclude

$$\begin{aligned}
\frac{|I_3|}{\frac{1}{3}I_1} &\leq \frac{3c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta \sigma^{-\frac{2\vartheta}{m}} 2C_\Psi^{1+\vartheta} = \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta} M^{-\frac{2\vartheta}{m}} e^{-\vartheta s}\eta^{\frac{2\vartheta}{\lambda_1}}(e^{\beta(s+\mu)}) \\
&\leq \frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2\vartheta}{\lambda_1}} e^{\vartheta\mu} M^{-\frac{2\vartheta}{m}} \leq 1,
\end{aligned} \tag{1.3.18}$$

provided that

$$M \geq M_4 := \left(\frac{6c_\vartheta}{A}\left(\frac{m}{2}\right)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2\vartheta}{\lambda_1}} e^{\vartheta\mu}\right)^{\frac{m}{2\vartheta}}. \tag{1.3.19}$$

Concerning I_4, we compute, using $\sigma\xi_s = \frac{1}{m}\sigma_s\xi$ and (1.3.8),

$$\begin{aligned}
|I_4| &= \left|\left(\frac{\sigma_s}{\sigma^{p-1}}\Psi\right)_s\right| = \left|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi - (p-1)\frac{\sigma_s^2}{\sigma^p}\Psi + \frac{\sigma_s}{\sigma^{p-1}}\xi_s\Psi_\xi\right| \\
&= \left|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi - (p-1)\frac{\sigma_s^2}{\sigma^p}\Psi + \frac{\sigma_s^2}{m\sigma^p}\xi\,\Psi_\xi\right| \\
&\leq \left|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi\right| + \frac{m(p-1)}{2}\frac{\sigma_s}{\sigma^{p-1}}|\Psi| + \frac{1}{2}\frac{\sigma_s}{\sigma^{p-1}}|\xi\,\Psi_\xi|.
\end{aligned}$$

Moreover, as μ satisfies (1.3.8), we have

$$0 \leq \frac{2\beta}{\lambda_1}\cdot\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \leq \frac{\varepsilon}{1+\varepsilon} < 1 \quad \text{for all } s \geq 0.$$

Hence, we obtain by (1.1.8) and (1.3.8)

$$\begin{aligned}
|\sigma_{ss}| &= \left| \left[\frac{m}{2} M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \left(1 - \frac{2\beta}{\lambda_1} \cdot \frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \right) \right]_s \right| \\
&= \left| \left(\frac{m}{2}\right)^2 M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \left(1 - \frac{2\beta}{\lambda_1} \cdot \frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \right)^2 \right. \\
&\quad - \frac{m}{2} M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \frac{2\beta}{\lambda_1} \left(\beta \frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} + \beta \frac{e^{2\beta(s+\mu)}\eta''(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \right. \\
&\quad \left. \left. - \beta \left(\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \right)^2 \right) \right| \\
&\le \frac{m}{2}\sigma_s + \frac{m}{2}\sigma\left(\beta + \beta C_\eta + \frac{\lambda_1}{2}\right) \le \left(\frac{m}{2} + (1+\varepsilon)\Big(\beta(1+C_\eta) + \frac{\lambda_1}{2}\Big)\right)\sigma_s \\
&=: \tilde{C}\sigma_s.
\end{aligned}$$

If $\xi \ge 1$, this implies due to (1.3.6)

$$\begin{aligned}
\frac{|I_4|}{\frac{1}{3}I_1} &\le \frac{3}{A}(1+\xi^{m+\lambda_1})\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)\sigma^{-\frac{2}{m}}(|\Psi| + |\xi\,\Psi_\xi|) \\
&\le \frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)\xi^{m+\lambda_1}\sigma^{-\frac{2}{m}}C_\Psi \xi^{2-m-\lambda_1} \\
&= \frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)C_\Psi\rho^2 \\
&\le 1, \qquad (1.3.20)
\end{aligned}$$

provided that

$$\rho \le \rho_2 := \left(\frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)C_\Psi\right)^{-\frac{1}{2}}. \qquad (1.3.21)$$

If $\xi < 1$, by (1.3.6) and (1.1.11) we obtain

$$\begin{aligned}
\frac{|I_4|}{\frac{1}{3}I_1} &\le \frac{3}{A}(1+\xi^{m+\lambda_1})\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)\sigma^{-\frac{2}{m}}(|\Psi| + |\xi\,\Psi_\xi|) \\
&\le \frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)\sigma^{-\frac{2}{m}}C_\Psi \\
&= \frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)C_\Psi M^{-\frac{2}{m}}e^{-s}\eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)}) \\
&\le \frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)C_\Psi (C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^{\mu}M^{-\frac{2}{m}} \\
&\le 1 \qquad (1.3.22)
\end{aligned}$$

under the additional restriction

$$M \ge M_5 := \left(\frac{6}{A}\left(\tilde{C} + \frac{m(p-1)+1}{2}\right)C_\Psi (C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^{\mu}\right)^{\frac{m}{2}}. \qquad (1.3.23)$$

Finally, we conclude by (1.3.11)-(1.3.23) that v_{in} is a supersolution of (1.2.3) for $s > 0$ and $0 < \rho \leq \rho_0 := \min\{\rho_1, \rho_2, 1\}$, if

$$M \geq M_0 := \max\{M_1, M_2, M_3, M_4, M_5\}$$

is fulfilled, and hence the claim is proved. ∎

Next, we construct a suitable supersolution of (1.2.3) in a region where ρ is bounded away from zero. We let $W = W(\rho)$ denote the solution of the problem

$$\begin{cases} W_{\rho\rho} + \frac{N-1}{\rho} W_\rho + \frac{\rho}{2} W_\rho + \frac{m+\lambda_1}{2} W = 0, & \rho > 0, \\ W(0) = 1, \quad W_\rho(0) = 0. \end{cases} \tag{1.3.24}$$

By Lemma 3.1 in [A1.2] this problem has a positive, decreasing solution W fulfilling

$$c_- \rho^{-m-\lambda_1} \leq W(\rho) \leq c_+ \rho^{-m-\lambda_1} \qquad \text{for } \rho \geq 1. \tag{1.3.25}$$

Furthermore, W is given by

$$W(\rho) = e^{-\frac{\rho^2}{4}} \mathcal{M}\left(\frac{N-m-\lambda_1}{2}, \frac{N}{2}, \frac{\rho^2}{4} \right), \quad \rho \geq 0,$$

where $\mathcal{M}$ denotes Kummer's function

$$\mathcal{M}(a,b,z) := 1 + \frac{az}{b} + \cdots + \frac{a(a+1)\cdots(a+n-1)z^n}{b(b+1)\cdots(b+n-1)n!} + \cdots$$

which can be found in [A1.1]. Now we give a suitable supersolution in an outer region.

Lemma 1.3.3 *Let $N \geq 11$ and $p > p_c$. Then there are $\alpha > 0$ and $\beta \in (0, \frac{1}{2}]$ such that for any $\mu > 0$ there exists $b_0 > 0$ with the property that*

$$v_{out}(\rho, s) := L\rho^{-m} - b\, e^{-\frac{\lambda_1}{2}s} \eta\left(e^{\beta(s+\mu)} \rho^\alpha \right) W(\rho)$$

is a positive supersolution of (1.2.3) *for $\rho > 0$ and $s > 0$, whenever $b \in (0, b_0)$ holds.*

Proof. Recalling the definition of C_η in (1.1.8), we fix $\alpha > 0$ such that $\alpha(C_\eta - 1) \leq N - 2$ is satisfied. Then, we fix $\beta \in (0, \frac{1}{2}]$ such that $\beta \leq \frac{\alpha}{N}$ holds. As $\frac{\partial}{\partial z}\mathcal{M}(a,b,z) \geq \frac{a}{b}\mathcal{M}(a,b,z)$ holds for any $z \geq 0$ in case of $0 < a < b$ (since $\frac{a+n-1}{b+n-1} \geq \frac{a}{b}$ for $n \in \mathbb{N}$ in this case), we have

$$\begin{aligned} W_\rho(\rho) &\geq -\frac{\rho}{2} W(\rho) + \frac{\rho}{2} \frac{N-m-\lambda_1}{N} W(\rho) = \frac{-m-\lambda_1}{N} \frac{\rho}{2} W(\rho) \\ &\geq -\frac{\frac{N-2}{2}}{N} \frac{\rho}{2} W(\rho) = -\left(1 - \frac{2}{N}\right) \frac{\rho}{4} W(\rho) \end{aligned} \tag{1.3.26}$$

due to the fact that $m + \lambda_1 \leq \frac{N-2}{2}$.

Now let $\mu > 0$ be given. By (1.1.11) and (1.3.25) there is $b_0 > 0$ such that v_{out} is positive for any $\rho > 0$ and $s > 0$ in case of $b \in (0, b_0)$. As moreover W is nonnegative, by (1.3.24), (1.3.26) and (1.1.8) we obtain for any $b \in (0, b_0)$ (omitting the argument $e^{\beta(s+\mu)}\rho^\alpha$ of η, η' and η'')

$$
\begin{aligned}
\mathcal{P}v_{out} &= (v_{out})_s - (v_{out})_{\rho\rho} - \frac{N-1}{\rho}(v_{out})_\rho - (v_{out})^p - \frac{\rho}{2}(v_{out})_\rho - \frac{m}{2}v_{out} \\
&= -(L\rho^{-m})_{\rho\rho} - \frac{N-1}{\rho}(L\rho^{-m})_\rho - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta W\Big)^p \\
&\quad -\frac{\rho}{2}(L\rho^{-m})_\rho - \frac{m}{2}L\rho^{-m} + b\frac{\lambda_1}{2}e^{-\frac{\lambda_1}{2}s}\eta W - b\beta e^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\rho^\alpha\eta' W \\
&\quad +be^{-\frac{\lambda_1}{2}s}\eta\left[W_{\rho\rho} + \frac{N-1}{\rho}W_\rho + \frac{\rho}{2}W_\rho + \frac{m}{2}W\right] \\
&\quad +be^{-\frac{\lambda_1}{2}s}\Bigg[e^{2\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}\eta'' W + e^{\beta(s+\mu)}\alpha(\alpha-1)\rho^{\alpha-2}\eta' W \\
&\quad +2e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W_\rho + \left(\frac{N-1}{\rho} + \frac{\rho}{2}\right)e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W\Bigg] \\
&= (L\rho^{-m})^p - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta W\Big)^p \\
&\quad +be^{-\frac{\lambda_1}{2}s}\eta\left[W_{\rho\rho} + \frac{N-1}{\rho}W_\rho + \frac{\rho}{2}W_\rho + \frac{m+\lambda_1}{2}W\right] \\
&\quad +be^{-\frac{\lambda_1}{2}s}\Bigg[-\beta e^{\beta(s+\mu)}\rho^\alpha\eta' W + e^{2\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}\eta'' W \\
&\quad +e^{\beta(s+\mu)}\alpha(\alpha-1)\rho^{\alpha-2}\eta' W + 2e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W_\rho \\
&\quad +\left(\frac{N-1}{\rho} + \frac{\rho}{2}\right)e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W\Bigg] \\
&\geq be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\Bigg[-\beta\rho^\alpha - e^{\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}C_\eta e^{-\beta(s+\mu)}\rho^{-\alpha} \\
&\quad +\alpha(\alpha-1)\rho^{\alpha-2} - 2\Big(1 - \frac{2}{N}\Big)\frac{\rho}{4}\alpha\rho^{\alpha-1} + \left(\frac{N-1}{\rho} + \frac{\rho}{2}\right)\alpha\rho^{\alpha-1}\Bigg] \\
&= be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\Big[-\beta\rho^\alpha - C_\eta\alpha^2\rho^{\alpha-2} + \alpha(\alpha-1)\rho^{\alpha-2} + \frac{\alpha}{N}\rho^\alpha \\
&\quad +(N-1)\alpha\rho^{\alpha-2}\Big] \\
&= be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\left[\left(\frac{\alpha}{N} - \beta\right)\rho^\alpha + \alpha(N-2-\alpha(C_\eta-1))\rho^{\alpha-2}\right] \\
&\geq 0 \qquad \text{for } \rho > 0 \text{ and } s > 0,
\end{aligned}
$$

where we have used the choices of α and β. ∎

We now use the functions v_{in} and v_{out} to obtain a supersolution of (1.2.3) for $\rho > 0$ and $s > 0$ which does not grow up faster than at the rate claimed in (1.2.4).

Lemma 1.3.4 *Let $N \geq 11$, $p > p_c$ and $v_0 = v_0(\rho)$ be a nonnegative and nonincreasing continuous function of $\rho \geq 0$ fulfilling*

$$v_0(\rho) < L\rho^{-m} \quad \textit{for } \rho > 0$$

and

$$v_0(\rho) \leq L\rho^{-m} - b_1\rho^{-m-\lambda_1}\eta(\rho) \quad \textit{for } \rho \geq R \tag{1.3.27}$$

with some positive constants b_1 and R. Moreover, let $v = v(\rho, s)$ denote the nonnegative solution of (1.2.3)*. Then there is a positive constant c such that*

$$v(\rho, s) \leq c\, e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \tag{1.3.28}$$

holds for $\rho \geq 0$ and $s \geq 0$.

Proof. We fix $\alpha > 0$ and $\beta \in (0, \frac{1}{2}]$ as in Lemma 1.3.3. Then, we choose $\mu > 0$, $\rho_0 > 0$, $M_0 > 0$ and v_{in} as in Lemma 1.3.2. Furthermore, let b_0 and v_{out} be chosen as in Lemma 1.3.3. As v_0 is bounded, there is $\rho_1 \in (0, \rho_0)$ such that $v_0(\rho) \leq \frac{1}{4}L\rho_1^{-m}$ is satisfied for $0 \leq \rho \leq \rho_1$. Next, we set

$$M_6 := \max\left\{M_0, \left(\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\right)^{-m}, \left(\frac{L\rho_1^{\lambda_1}}{2a_1\eta(e^{\beta\mu})}\right)^{-\frac{m}{\lambda_1}}\right\},$$

where a_1 is defined in (1.3.3).
Since ψ is decreasing, we thus obtain by (1.3.3) and Lemma 1.3.2

$$\begin{aligned}
v_{in}(\rho, 0) &= \sigma(0)\left(\psi(\xi) - \frac{\sigma_s(0)}{\sigma^p(0)}\Psi(\xi)\right) \geq \frac{1}{2}\sigma(0)\psi(\xi)\\
&= \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\,\psi\left(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho\right)\\
&\geq \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\,\psi\left(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\right)\\
&\geq \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\left(L\left(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\right)^{-m} - a_1\left(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\right)^{-m-\lambda_1}\right)\\
&= \frac{1}{2}L\rho_1^{-m} - \frac{a_1}{2}M^{-\frac{\lambda_1}{m}}\eta(e^{\beta\mu})\rho_1^{-m-\lambda_1} \geq \frac{1}{4}L\rho_1^{-m}\\
&\geq v_0(\rho) \qquad \text{for } \rho \leq \rho_1,
\end{aligned} \tag{1.3.29}$$

provided that $M \geq M_6$ holds.
Now, we show that $v_{out} \geq v_0$ holds for $\rho > 0$, if $b > 0$ is chosen suitably small. In case of $\rho \geq \max\{1, R\}$, due to (1.3.25), (1.3.27) and (1.1.9) we conclude

$$v_{out}(\rho, 0) = L\rho^{-m} - b\,\eta(e^{\beta\mu}\rho^{\alpha})W(\rho) \geq L\rho^{-m} - b\,(c_{\alpha,e^{\beta\mu}})\eta(\rho)c_+\rho^{-m-\lambda_1} \geq v_0(\rho),$$

if $b \le b_2 := \frac{b_1}{(c_{\alpha,e^{\beta\mu}})c_+}$ is fulfilled. As v_0 is continuous with $v_0(\rho) < L\rho^{-m}$ for $\rho > 0$, there is $b_3 > 0$ such that $v_0(\rho) \le L\rho^{-m} - b_3$ holds for $\rho \le \max\{1, R\}$. Since W is nonincreasing and thus satisfies $W \le W(0) = 1$ in $[0,\infty)$, we obtain

$$v_{out}(\rho, 0) \ge L\rho^{-m} - b\,\eta(e^{\beta\mu}(R^\alpha + 1)) \ge v_0(\rho) \quad \text{for } \rho \le \max\{1, R\},$$

if $b \le b_4 := \frac{b_3}{\eta(e^{\beta\mu}(R^\alpha+1))}$. Accordingly, we fix $b \in (0, \min\{b_0, b_2, b_4\})$ and have

$$v_{out}(\rho, 0) \ge v_0(\rho) \quad \text{for all } \rho \ge 0. \tag{1.3.30}$$

Keeping this value of b fixed, we now claim that for any M sufficiently large

$$\rho_M(s) := \inf\{\rho > 0 \ : \ v_{out}(\rho, s) < v_{in}(\rho, s)\}$$

is well-defined for all $s \ge 0$ and fulfills

$$\rho_M(s) \le \rho_1 \quad \text{for } s \ge 0. \tag{1.3.31}$$

Once this has been shown, we will obtain from Lemma 1.3.2 and Lemma 1.3.3 that

$$v_{sup}(\rho, s) := \begin{cases} v_{in}(\rho, s) & \text{for } s \ge 0, \rho \le \rho_M(s), \\ v_{out}(\rho, s) & \text{for } s \ge 0, \rho > \rho_M(s), \end{cases}$$

is a supersolution of (1.2.3) which moreover satisfies $v_{sup}(\rho, 0) \ge v_0(\rho)$ for $\rho \ge 0$ by (1.3.29) and (1.3.30). As $v_\rho \le 0$ holds due to the properties of v_0, the comparison principle implies

$$\begin{aligned} v(\rho, s) &\le v(0, s) \le v_{sup}(0, s) = v_{in}(0, s) = \sigma(s) = Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \\ &\le Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta s}) \le M(c_{\frac{1}{2\beta},1})^{\frac{m}{\lambda_1}}e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } \rho \ge 0 \text{ and } s \ge 0, \end{aligned}$$

where we have used (1.1.6) and (1.1.9). Consequently, the lemma is proved if we show (1.3.31).
To this end, we prove that

$$v_{out}(\rho_1, s) < v_{in}(\rho_1, s) \quad \text{for } s \ge 0 \tag{1.3.32}$$

holds if M is chosen large enough. By (1.3.3), (1.3.6) and (1.3.8) we have

$$\begin{aligned} v_{in}(\rho_1, s) &= \sigma\left(\psi(\xi) - \frac{\sigma_s}{\sigma^p}\Psi(\xi)\right) \\ &\ge \sigma\left(L(\sigma^{\frac{1}{m}}\rho_1)^{-m} - a_1(\sigma^{\frac{1}{m}}\rho_1)^{-m-\lambda_1} - \frac{m}{2}\sigma^{-\frac{2}{m}}C_\Psi(\sigma^{\frac{1}{m}}\rho_1)^{2-m-\lambda_1}\right) \\ &= L\rho_1^{-m} - \left(a_1\rho_1^{-m-\lambda_1} + \frac{m}{2}C_\Psi\rho_1^{2-m-\lambda_1}\right)\sigma^{-\frac{\lambda_1}{m}} \quad \text{for } s \ge 0 \end{aligned}$$

whenever

$$M \ge M_7 := \max\left\{M_6, \left((C_{\lambda_1})^{-\frac{1}{\lambda_1}}e^{-\beta\mu}\rho_1\right)^{-m}\right\},$$

because
$$\sigma^{\frac{1}{m}}(s)\rho_1 \geq M^{\frac{1}{m}} e^{\frac{1}{2}s}(C_{\lambda_1})^{-\frac{1}{\lambda_1}} e^{-\beta(s+\mu)}\rho_1 \geq M^{\frac{1}{m}}(C_{\lambda_1})^{-\frac{1}{\lambda_1}} e^{-\beta\mu}\rho_1$$
is satisfied due to (1.1.11) and $\beta \in (0, \frac{1}{2}]$. As moreover (1.1.9) implies

$$\begin{aligned} v_{out}(\rho_1, s) &= L\rho_1^{-m} - b\, e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta(s+\mu)}\rho_1^{\alpha})W(\rho_1) \\ &\leq L\rho_1^{-m} - b\, e^{-\frac{\lambda_1}{2}s}(c_{1,\rho_1^{-\alpha}})^{-1}\eta(e^{\beta(s+\mu)})W(\rho_1) \\ &= L\rho_1^{-m} - b\,(c_{1,\rho_1^{-\alpha}})^{-1}W(\rho_1)M^{\frac{\lambda_1}{m}}\sigma^{-\frac{\lambda_1}{m}} \qquad \text{for } s \geq 0, \end{aligned}$$

(1.3.32) is valid for any

$$M > M_8 := \max\left\{ M_7, \left(\frac{a_1\rho_1^{-m-\lambda_1} + \frac{m}{2}C_\Psi \rho_1^{2-m-\lambda_1}}{b\,(c_{1,\rho_1^{-\alpha}})^{-1}W(\rho_1)} \right)^{\frac{m}{\lambda_1}} \right\}.$$

Finally, $\rho_M(s)$ is well-defined for any $s \geq 0$, since $\lim_{\rho\searrow 0} v_{in}(\rho, s) = \sigma(s) < \infty$ and $v_{out}(\rho, s) \to \infty$ as $\rho \searrow 0$ holds for any $s \geq 0$. Thus, (1.3.31) is fulfilled due to (1.3.32) if we choose $M > M_8$, and the proof is complete. ∎

1.4 Lower bound

In this section, we derive the corresponding lower bound for v and adapt an idea from [A1.9].
The function
$$\bar{W}(\rho) := \rho^{-m-\lambda_1}, \qquad \rho > 0,$$
is a positive solution of the equation

$$\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W} + \frac{pL^{p-1}}{\rho^2}\bar{W} = 0 \quad \text{for } \rho > 0. \tag{1.4.1}$$

Now we construct a suitable subsolution of (1.2.3).

Lemma 1.4.1 *Suppose $N \geq 11$ and $p > p_c$. Then for any $\beta \geq 2N$ and each $b > 0$ the function*

$$v_{sub}(\rho, s) := \max\left\{0, L\rho^{-m} - b\, e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)\right\}, \quad \rho > 0, s \geq 0,$$

is a subsolution of (1.2.3) for all $\rho > 0$ and $s > 0$.

Proof. Fixing $\beta \geq 2N$ and $b > 0$, we choose $\rho > 0$ and $s > 0$ such that $v_{sub}(\rho, s)$ is positive.
As $be^{-\frac{\lambda_1}{2}}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)$ is positive and $p > 1$, the mean value theorem implies

$$(L\rho^{-m})^p - \left(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)\right)^p$$

$$
\begin{aligned}
&\leq \; p(L\rho^{-m})^{p-1} b e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho) \\
&= \; b e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\frac{pL^{p-1}}{\rho^2}\bar{W}(\rho).
\end{aligned}
$$

Thus, we obtain due to (1.1.6) and (1.4.1) (suppressing the argument $e^{\beta s}(1+\rho^2)$ of η, η' and η'')

$$
\begin{aligned}
\mathcal{P}v_{sub} &= (v_{sub})_s - (v_{sub})_{\rho\rho} - \frac{N-1}{\rho}(v_{sub})_\rho - (v_{sub})^p - \frac{\rho}{2}(v_{sub})_\rho - \frac{m}{2}v_{sub} \\
&= -(L\rho^{-m})_{\rho\rho} - \frac{N-1}{\rho}(L\rho^{-m})_\rho - \left(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta\bar{W}\right)^p \\
&\quad -\frac{\rho}{2}(L\rho^{-m})_\rho - \frac{m}{2}L\rho^{-m} + b\frac{\lambda_1}{2}e^{-\frac{\lambda_1}{2}s}\eta\bar{W} - b\beta e^{-\frac{\lambda_1}{2}s}e^{\beta s}(1+\rho^2)\eta'\bar{W} \\
&\quad +be^{-\frac{\lambda_1}{2}s}\eta\left[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m}{2}\bar{W}\right] \\
&\quad +be^{-\frac{\lambda_1}{2}s}\left[e^{2\beta s}4\rho^2\eta''\bar{W} + 2e^{\beta s}\eta'\bar{W} + 2e^{\beta s}2\rho\eta'\bar{W}_\rho + \left(\frac{N-1}{\rho} + \frac{\rho}{2}\right)e^{\beta s}2\rho\eta'\bar{W}\right] \\
&= (L\rho^{-m})^p - \left(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta\bar{W}\right)^p \\
&\quad +be^{-\frac{\lambda_1}{2}s}\eta\left[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W}\right] \\
&\quad +be^{-\frac{\lambda_1}{2}s}\Bigg[-\beta e^{\beta s}(1+\rho^2)\eta'\bar{W} + 4e^{2\beta s}\rho^2\eta''\bar{W} + 2e^{\beta s}\eta'\bar{W} + 4e^{\beta s}\rho\eta'\bar{W}_\rho \\
&\quad +\left(\frac{N-1}{\rho} + \frac{\rho}{2}\right)2e^{\beta s}\rho\eta'\bar{W}\Bigg] \\
&\leq be^{-\frac{\lambda_1}{2}s}\eta\left[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W} + \frac{pL^{p-1}}{\rho^2}\bar{W}\right] \\
&\quad +be^{-\frac{\lambda_1}{2}s}e^{\beta s}\eta'\bar{W}\left[-\beta(1+\rho^2) + 2 + 2(N-1) + \rho^2\right] \\
&\leq be^{-\frac{\lambda_1}{2}s}e^{\beta s}\eta'\bar{W}(-\beta + 2N)(1+\rho^2) \leq 0
\end{aligned}
$$

due to the choice of β. Now the claim is proved since $v_1 \equiv 0$ as well is a subsolution of (1.2.3). ■

Now we are in position to prove the lower bound for the grow-up rate of solutions to (1.2.3) which corresponds to the rate claimed in (1.2.4).

Lemma 1.4.2 *Assume $N \geq 11$ and $p > p_c$. Moreover, let $v_0 = v_0(\rho)$ be radially symmetric and nonnegative satisfying*

$$
v_0(\rho) \geq L\rho^{-m} - b_2\,\rho^{-m-\lambda_1}\eta(\rho) \quad \textit{for } \rho > 0 \tag{1.4.2}
$$

with some $b_2 > 0$. Then the solution v of (1.2.3) fulfills

$$\sup_{\rho>0} v(\rho,s) \geq c\, e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \textit{for all } s \geq 0$$

with some constant $c > 0$.

Proof. We fix $\beta \geq 2N$, define

$$b := \max\left\{b_2, \frac{L}{\eta(1)}\right\}$$

and take the function v_{sub} from Lemma 1.4.1. Due to (1.1.6) and the choice of b, we obtain

$$L\rho^{-m} - b\,\eta(1+\rho^2)\rho^{-m-\lambda_1} \leq L\rho^{-m} - b\,\eta(1)\rho^{-m-\lambda_1} \leq L\rho^{-m} - L\rho^{-m-\lambda_1} \leq 0$$

for $\rho \leq 1$. Thus, the definition of $\bar{W}$ implies

$$v_{sub}(\rho,0) = 0 \leq v_0(\rho) \quad \text{for } \rho \leq 1.$$

Furthermore, by (1.1.6) and (1.4.2) we have

$$\begin{aligned} v_{sub}(\rho,0) &= L\rho^{-m} - b\,\eta(1+\rho^2)\rho^{-m-\lambda_1} \leq L\rho^{-m} - b\,\eta(\rho)\rho^{-m-\lambda_1} \\ &\leq L\rho^{-m} - b_2\eta(\rho)\rho^{-m-\lambda_1} \leq v_0(\rho) \quad \text{for all } \rho \geq 1 \text{ where } v_{sub}(\rho,0) > 0. \end{aligned}$$

Altogether, we conclude $v_{sub}(\rho,0) \leq v_0(\rho)$ for all $\rho \geq 0$. Hence, the comparison principle yields $v \geq v_{sub}$ for all $\rho \geq 0$ and $s \geq 0$. Defining

$$\rho(s) := \left(\frac{L}{2b(c_{2\beta,2})}\right)^{-\frac{1}{\lambda_1}} e^{-\frac{1}{2}s}\eta^{\frac{1}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0,$$

where $c_{2\beta,2}$ is defined in (1.1.9), we find $s_0 \geq 0$ such that $\rho(s) \leq 1$ is satisfied for all $s \geq s_0$ by (1.1.11). Hence, due to (1.1.6) and (1.1.9) we obtain

$$\begin{aligned} \sup_{\rho>0} v(\rho,s) &\geq v_{sub}(\rho(s),s) \\ &= L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2(s)))\bar{W}(\rho(s)) \\ &\geq L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}\eta(2e^{\beta s})\rho^{-m-\lambda_1}(s) \\ &\geq L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}(c_{2\beta,2})\eta(e^{\frac{1}{2}s})\rho^{-m-\lambda_1}(s) \\ &= \rho^{-m}(s)\left(L - b(c_{2\beta,2})e^{-\frac{\lambda_1}{2}s}\eta(e^{\frac{1}{2}s})\rho^{-\lambda_1}(s)\right) \\ &= \frac{L}{2}\rho^{-m}(s) \\ &= \frac{L}{2}\left(\frac{L}{2b(c_{2\beta,2})}\right)^{\frac{m}{\lambda_1}} e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq s_0. \end{aligned}$$

This implies the claim as v is continuous and

$$\sup_{\rho>0} v(\rho,s) \geq \sup_{\rho>0} v_{sub}(\rho,s) > 0 \quad \text{for any } s \in [0,s_0]$$

is fulfilled due to the choice of v_{sub}. ∎

1.5 Proof of Theorem 1.1.1

In this section we complete the proof of Theorem 1.1.1 with the help of the estimates which are derived in Sections 1.3 and 1.4.
Let $u_0 \in C^0(\mathbb{R}^N)$ satisfy (1.1.2) and (1.1.5). Moreover, we let u denote the corresponding solution of (1.1.1) and define the radially symmetric functions

$$\underline{u}_0(r) := \min\{u_0(x) \ : \ x \in \mathbb{R}^N, |x| = r\} \quad \text{for } r \geq 0$$

and

$$\overline{u}_0(r) := \max\{u_0(x) \ : \ x \in \mathbb{R}^N, |x| \geq r\} \quad \text{for } r \geq 0.$$

Then the properties of u_0 imply that $\underline{u}_0(r)$ and $\overline{u}_0(r)$ are continuous in $r \geq 0$ and satisfy (1.1.5) (with a possibly larger constant R). Moreover, $\overline{u}_0(r)$ is nonincreasing for $r \geq 0$ and we have

$$0 \leq \underline{u}_0(|x|) \leq u_0(x) \leq \overline{u}_0(|x|) < \varphi_\infty(|x|) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

We define $\underline{u}(r,t)$ and $\overline{u}(r,t)$ to be the solutions of (1.2.1) corresponding to the initial data $\underline{u}_0(r)$ and $\overline{u}_0(r)$, respectively. Both solutions exist globally in time and $\overline{u}(r,t)$ is nonincreasing in r for any $t \geq 0$. Furthermore, let $\underline{v}(\rho,s)$ and $\overline{v}(\rho,s)$ denote the solutions of (1.2.3) which are obtained from $\underline{u}$ and $\overline{u}$, respectively, by the self-similar change of variables defined in (1.2.2). As the initial data $\underline{v}_0(\rho) = \underline{u}_0(\rho)$ and $\overline{v}_0(\rho) = \overline{u}_0(\rho)$ fulfill the conditions of Lemma 1.4.2 and Lemma 1.3.4, respectively, we conclude

$$\sup_{\rho \geq 0} \underline{v}(\rho,s) \geq C_1 \, e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0$$

by Lemma 1.4.2 and

$$\sup_{\rho \geq 0} \overline{v}(\rho,s) \leq C_2 \, e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0$$

by Lemma 1.3.4 with some positive constants C_1 and C_2. Hence, (1.2.2) implies

$$\|\underline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} = (t+1)^{-\frac{1}{p-1}} \sup_{\rho \geq 0} \underline{v}(\rho, \log(t+1)) \geq C_1 \, \eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}), \quad t \geq 0,$$

and

$$\|\overline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} = (t+1)^{-\frac{1}{p-1}} \sup_{\rho \geq 0} \overline{v}(\rho, \log(t+1)) \leq C_2 \, \eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}), \quad t \geq 0.$$

As the comparison principle yields

$$\|\underline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\overline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} \quad \text{for } t \geq 0,$$

the proof of Theorem 1.1.1 is complete.

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Article 2:
Convergence to steady states in a viscous Hamilton-Jacobi equation with degenerate diffusion

by Christian Stinner

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Abstract

This paper deals with weak solutions of the one-dimensional viscous Hamilton-Jacobi equation

$$u_t = (|u_x|^{p-2}u_x)_x + |u_x|^q \quad \text{in } (-R,R)\times(0,\infty)$$

with homogeneous Dirichlet boundary conditions, where $R>0$, $p>2$ and $1<q<p-1$. For these solutions we investigate the convergence to steady states via a Lyapunov functional, which is constructed with a technique developed by Zelenyak.

Key words: asymptotic behavior, diffusive Hamilton-Jacobi equation, degenerate diffusion
MSC 2010: 35B40, 35B35, 35K55, 35K65

2.1 Introduction

We consider the one-dimensional diffusive Hamilton-Jacobi equation

$$\begin{cases} u_t = (|u_x|^{p-2}u_x)_x + |u_x|^q, & x\in\Omega,\ t\in(0,\infty), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{2.1.1}$$

where $\Omega := (-R, R) \subset \mathbb{R}$ with $R > 0$,

$$u_0 \in C^1(\bar{\Omega}) \text{ with } u_0 = 0 \text{ on } \partial\Omega, \tag{2.1.2}$$

$$p > 2 \quad \text{and} \quad 1 < q < p - 1 \quad [1] \tag{2.1.3}$$

is assumed. As $(|u_x|^{p-2}u_x)_x$ is the one-dimensional variant of the well-known p-Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the differential equation in (2.1.1) is a special case of the equation

$$u_t = \Delta_p u + a\,|\nabla u|^q \quad \text{in } \Omega \times (0, \infty), \qquad \text{where } \Omega \subset \mathbb{R}^n,\ p > 2,\ q > 1,\ a \in \{-1, 1\} \tag{2.1.4}$$

and $n \in \mathbb{N}$ is arbitrary.
The corresponding semilinear equation

$$u_t = \Delta u + a\,|\nabla u|^q \quad \text{in } \Omega \times (0, \infty), \qquad \text{where } \Omega \subset \mathbb{R}^n,\ q > 0,\ a \in \{-1, 1\} \tag{2.1.5}$$

possesses many different qualitative behaviors and has been widely studied by several authors in view of a theoretical interest. In particular, the asymptotic behavior of nonnegative and integrable global solutions to the Cauchy problem has been of great interest. In case of $a = -1$, where the gradient term is an absorption term, solutions decay to zero as $t \to \infty$, and different decay rates and asymptotic profiles, depending on the value of q and the initial data, have been established (see e.g., [A2.5, A2.7, A2.8]). For $q \in (0, 1)$, in particular the phenomenon of extinction in finite time has been observed (see [A2.6, A2.9]). In case of $a = 1$, where the gradient term acts as a source term, the asymptotic behavior of solutions to the Cauchy problem depends as well strongly on the value of q and there are solutions tending to a nonzero state with infinite mass (see e.g., [A2.5, A2.9, A2.15, A2.16]). Moreover, gradient blow-up phenomena have been observed in [A2.2, A2.20, A2.21] for the Dirichlet problem in a bounded domain with suitably chosen boundary conditions and $a = 1$.
Concerning the large time behavior of global classical solutions to (2.1.5) in a bounded domain with homogeneous Dirichlet boundary conditions, it has been shown in [A2.4, A2.22] that they converge exponentially fast to zero in case of $q \geq 1$ and $a \in \{-1, 1\}$, while extinction in finite time occurs for nonnegative solutions in case of $q \in (0, 1)$ and $a = -1$. Furthermore, in space dimension one with $q \in (0, 1)$ and $a = 1$ there is a one parameter family of nonnegative steady states, and any solution evolving from sufficiently regular initial data converges uniformly to one of these stationary solutions (see [A2.13]). In particular, nonnegative solutions evolving from nonzero initial data converge to a nonzero steady state.
Now equation (2.1.4) is a quasilinear generalization of (2.1.5) which is of theoretical interest. As (2.1.4) degenerates at points where $\nabla u = 0$, one cannot expect to have classical solutions. Weak solutions of this equation are obtained by approximation with solutions of regularized equations. Often, this is done within the theory of viscosity solutions. The

[1]All results proved in this article remain valid for the larger regime $p > 2$ and $0 < q < p - 1$. In fact, even all the proofs persist without any change, as the positivity of q in conjunction with $q < p - 1$ is sufficient in all of their steps.

qualitative behavior of solutions to the Cauchy problem corresponding to (2.1.4) has been investigated recently.
In case of $a = -1$, nonnegative solutions tend to zero and asymptotic decay rates have been established in [A2.1, A2.3]. In particular, there are two critical exponents $q_1 = p-1$ and $q_* = p - \frac{n}{n+1}$ which separate different kinds of behavior (see [A2.3]). Moreover, for $1 < q < p-1$, in [A2.17] the convergence to a suitable self-similar solution was obtained as well as a detailed description of the evolution of the positivity set for solutions with compactly supported initial data. The latter is due to the fact that (2.1.4) allows finite speed of propagation in contrast to (2.1.5). Furthermore, the existence of self-similar solutions has been established in [A2.11, A2.18].
In case of $a = 1$, the asymptotic behavior of nonnegative solutions to the Cauchy problem has been investigated in [A2.14] for $1 < q < p$. In particular, for solutions which do not tend to zero, it is shown that they converge to a positive constant uniformly in compact subsets of $\mathbb{R}^n$, and the asymptotic profile is given.
To the best of our knowledge, no result implying the convergence to a nonzero state is known for (2.1.4) in a bounded domain with homogeneous Dirichlet boundary conditions. In this work, we consider the one-dimensional problem (2.1.1) in case of $1 < q < p-1$. We establish the existence of a one parameter family $(w_\vartheta)_{\vartheta\in[0,R]}$ of nonnegative steady states, where $w_\vartheta \not\equiv 0$ for $\vartheta \in [0, R)$. In particular, the family of steady states is ordered, w_0 is the maximal stationary solution of (2.1.1) and w_ϑ is flat in a subinterval of Ω for $\vartheta \in (0, R)$. This corresponds to the behavior observed in (2.1.5), but is a novelty in the quasilinear case (2.1.4). Moreover, we prove the existence of a global weak solution of (2.1.1) which converges to one nonnegative steady state w_ϑ as $t \to \infty$. This behavior is observed for all initial data u_0 satisfying (2.1.2). In particular, the limit w_ϑ fulfills $w_\vartheta \not\equiv 0$ in case of $u_0 \geq 0$ and $u_0 \not\equiv 0$, while the solution of (2.1.1) tends to zero for nonpositive initial data. In contrast to the semilinear case $p = 2$, it remains open if some solutions of (2.1.1) converge to a nonzero steady state in case of sign changing initial data. This corresponds to the fact that extinction in finite time for nonpositive solutions of (2.1.1) is not expected to occur in case of $q > 1$, while this phenomenon was an important help to investigate the behavior of solutions with sign changing initial data for $p = 2$.
To prove the large time behavior of solutions to (2.1.1), we establish the existence of a Lyapunov functional using a method developed in [A2.25]. Although this method has also been used in [A2.13] in the proof of the semilinear case $p = 2$, our proof of the large time behavior is different to the latter one. We use an idea which has been used for other degenerate parabolic equations (see e.g., [A2.24]). As our proof of the convergence to steady states strongly relies on the one-dimensional setting, the large time behavior of the Dirichlet problem corresponding to (2.1.4) in dimension $n \geq 2$ remains open here and has to be investigated elsewhere.
This paper is organized as follows. We classify the steady states of problem (2.1.1) in Section 2.2. In Section 2.3, we establish estimates for classical solutions of an approximative problem to (2.1.1) and their derivatives. In particular, we prove the existence of a Lyapunov functional for these regularized problems (see Lemma 2.3.3). Finally, in Section 2.4, we show the existence of a weak solution of (2.1.1) and prove our main result, namely the convergence of this weak solution to a stationary solution (see Theorem 2.4.4).

2.2 Stationary solutions

In this section, we classify the stationary solutions w of (2.1.1). More precisely, we call $w \in C^1([-R,R])$ a weak solution of

$$\begin{cases} (|w_x|^{p-2}w_x)_x + |w_x|^q = 0, & x \in (-R,R), \\ w(\pm R) = 0, \end{cases} \tag{2.2.1}$$

if w satisfies $w(\pm R) = 0$ and

$$\int_{-R}^{R} \Big(-(|w_x|^{p-2}w_x)(x)\,\xi_x(x) + |w_x|^q(x)\,\xi(x)\Big)\mathrm{d}x = 0 \quad \text{for any } \xi \in C_0^\infty((-R,R)). \tag{2.2.2}$$

Defining

$$\alpha := \frac{p-q}{p-1-q} > 1, \qquad c_0 := \left(\frac{p-1-q}{p-1}\right)^{\frac{1}{p-1-q}} \cdot \frac{p-1-q}{p-q}, \tag{2.2.3}$$

where $\alpha > 1$ due to $q < p-1$, we obtain the following lemma. In particular, any weak solution of (2.2.1) is nonnegative.

Lemma 2.2.1 *Suppose* (2.1.3) *is fulfilled and $R > 0$. Furthermore, let $w \in C^1([-R,R])$ satisfy $w(\pm R) = 0$ and* (2.2.2). *Then w is nonnegative and there is $\vartheta \in [0,R]$ such that $w = w_\vartheta$ holds, where we define*

$$w_\vartheta(x) := c_0\left[(R-\vartheta)^\alpha - (|x|-\vartheta)_+^\alpha\right] \qquad \textit{for } x \in [-R,R].$$

Proof. Since $\alpha > 1$ holds, $w_\vartheta \in C^1([-R,R])$ is satisfied for any $\vartheta \in [0,R]$. Let $w \in C^1([-R,R])$ be a solution of (2.2.2) with $w(\pm R) = 0$. Then (2.2.2) implies that

$$\Phi := |w_x|^{p-2}w_x$$

fulfills $\Phi \in W^{1,\infty}((-R,R))$ with $\Phi_x = -|w_x|^q$ a.e. in $(-R,R)$. Hence, Φ is a nonincreasing function on $[-R,R]$ due to $\Phi_x \le 0$ a.e. in $(-R,R)$. Thus, w_x is a nonincreasing function on $[-R,R]$ due to $p > 1$. As $w(\pm R) = 0$, we conclude that w is nonnegative in $[-R,R]$. Moreover, this implies $w_x(-R) \ge 0$ and $w_x(R) \le 0$.
If $w_x(-R) = 0$ is fulfilled, we conclude $w = w_R \equiv 0$, since w_x is nonincreasing and $w(\pm R) = 0$ holds. Similarly, $w = w_R \equiv 0$ is satisfied in case of $w_x(R) = 0$.
In case of $w_x(-R) > 0$ and $w_x(R) < 0$, there are unique $\vartheta_1, \vartheta_2 \in (-R,R)$ with $\vartheta_1 \le \vartheta_2$ such that $w_x > 0$ in $[-R,\vartheta_1)$, $w_x = 0$ in $[\vartheta_1,\vartheta_2]$ and $w_x < 0$ in $(\vartheta_2,R]$ is satisfied as w_x is nonincreasing. Hence, Φ is positive in $[-R,\vartheta_1)$ and therefore $\Phi_x = -w_x^q = -\Phi^{\frac{q}{p-1}}$ holds a.e. in $(-R,\vartheta_1)$. Thus, an integration yields

$$\frac{p-1}{p-1-q}\Phi^{\frac{p-1-q}{p-1}}(x) + x = \beta_1 \quad \text{for } x \in (-R,\vartheta_1)$$

with some constant $\beta_1 \in \mathbb{R}$. Now the limit $x \nearrow \vartheta_1$ implies $\beta_1 = \vartheta_1$ due to $w_x(\vartheta_1) = 0$ and $q < p-1$. Hence, we have

$$w_x(x) = \left(\frac{p-1-q}{p-1}(\vartheta_1 - x)\right)^{\frac{1}{p-1-q}} \quad \text{for } x \in (-R, \vartheta_1).$$

Using $w(-R) = 0$ and (2.2.3), a further integration implies

$$w(x) = c_0\left[(\vartheta_1 + R)^\alpha - (\vartheta_1 - x)^\alpha\right] \quad \text{for } x \in [-R, \vartheta_1]. \tag{2.2.4}$$

Similarly, we conclude

$$-\frac{p-1}{p-1-q}(-\Phi)^{\frac{p-1-q}{p-1}}(x) + x = \vartheta_2 \quad \text{for } x \in (\vartheta_2, R),$$

$$w_x(x) = -\left(\frac{p-1-q}{p-1}(x - \vartheta_2)\right)^{\frac{1}{p-1-q}} \quad \text{for } x \in (\vartheta_2, R)$$

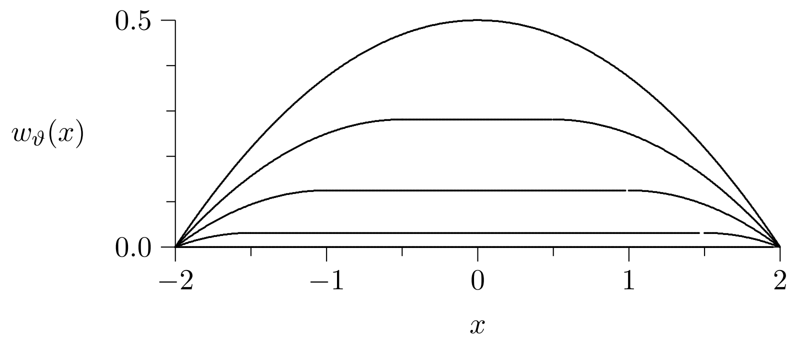
and finally, due to $w(R) = 0$,

$$w(x) = c_0\left[(R - \vartheta_2)^\alpha - (x - \vartheta_2)^\alpha\right] \quad \text{for } x \in [\vartheta_2, R]. \tag{2.2.5}$$

Furthermore, we get $w(\vartheta_1) = w(\vartheta_2)$ because $w_x \equiv 0$ in $[\vartheta_1, \vartheta_2]$ is satisfied. Thus, $(R + \vartheta_1)^\alpha = (R - \vartheta_2)^\alpha$ holds due to (2.2.4) and (2.2.5), and we conclude $\vartheta_1 = -\vartheta_2$. Hence, $\vartheta_2 \in [0, R]$ has to be fulfilled since $\vartheta_1 \leq \vartheta_2$, and finally (2.2.4) and (2.2.5) imply $w = w_{\vartheta_2}$ in $[-R, R]$. ■

As an example, we show the plot of some stationary solutions w_ϑ for a particular choice of R, p and q. In particular, w_ϑ is flat in $[-\vartheta, \vartheta]$ for $\vartheta \in (0, R)$ and $w_{\vartheta_1} \geq w_{\vartheta_2}$ holds for $0 \leq \vartheta_1 \leq \vartheta_2 \leq R$.

Figure 2.1: steady states w_ϑ for $\vartheta \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ in case of $R = 2$, $p = 5$ and $q = 3$



2.3 Approximate parabolic problems

In this section, we consider solutions of some suitably chosen regularized problems. Specifically, as smooth functions u satisfy

$$(|u_x|^{p-2}u_x)_x = (p-1)|u_x|^{p-2}u_{xx},$$

for $\varepsilon > 0$ let u_ε denote the classical solution of

$$\begin{cases} (u_\varepsilon)_t = (p-1)(|(u_\varepsilon)_x|^2+\varepsilon^2)^{\frac{p-2}{2}}(u_\varepsilon)_{xx} + (|(u_\varepsilon)_x|^2+\varepsilon^2)^{\frac{q}{2}}, & x\in\Omega,\ t\in(0,\infty), \\ u_\varepsilon|_{\partial\Omega} = 0, \\ u_\varepsilon|_{t=0} = u_{0\varepsilon}. \end{cases} \tag{2.3.1}$$

These functions u_ε will approximate a weak solution of (2.1.1). First, we state the existence of classical solutions to (2.3.1).

Lemma 2.3.1 *Suppose* (2.1.3) *is fulfilled,* $\varepsilon \in (0,1)$ *and* $u_{0\varepsilon} \in C^\infty(\bar\Omega)$ *satisfies* $u_{0\varepsilon} = 0$ *on* $\partial\Omega$*. Then there is a unique solution* $u_\varepsilon \in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty))$ *of* (2.3.1)*. This solution* u_ε *fulfills*

$$\min_{x\in\bar\Omega} u_{0\varepsilon}(x) \le u_\varepsilon \le \max_{x\in\bar\Omega} u_{0\varepsilon}(x) + \varepsilon^q t \qquad \textit{in } \bar\Omega\times[0,\infty).$$

Moreover, there are constants $C_0, C_1 \in [1,\infty)$ *which are independent of* ε *such that*

$$|u_\varepsilon| \le C_0(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \qquad \textit{and} \qquad |(u_\varepsilon)_x| \le C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \ \ \textit{in } \Omega\times[0,\infty)$$

is satisfied.

Proof. By standard parabolic theory, there is a unique classical solution $u_\varepsilon \in C^0(\bar\Omega \times [0,T_\varepsilon))\cap C^{2,1}(\Omega\times(0,T_\varepsilon))$ of (2.3.1) with some maximal existence time $T_\varepsilon \in (0,\infty]$ (see e.g., Section VI.5 in [A2.12]). Furthermore, it is possible to choose $c \ge 0$ properly such that $v_1(x,t) := c+\varepsilon^q t$, $(x,t)\in\bar\Omega\times[0,\infty)$, is a supersolution and $v_2 \equiv -c$ is a subsolution to (2.3.1) in $\bar\Omega\times[0,\infty)$. Hence, the comparison principle implies $T_\varepsilon = \infty$ and

$$\min_{x\in\bar\Omega} u_{0\varepsilon}(x) \le u_\varepsilon \le \max_{x\in\bar\Omega} u_{0\varepsilon}(x) + \varepsilon^q t \qquad \text{in } \bar\Omega\times[0,\infty).$$

Similarly, we conclude

$$\min_{x\in\bar\Omega} u_\varepsilon(x,t_0) \le u_\varepsilon(x,t) \le \max_{x\in\bar\Omega} u_\varepsilon(x,t_0)+\varepsilon^q(t-t_0) \qquad \text{for } (x,t)\in\bar\Omega\times[t_0,\infty) \text{ and any } t_0\ge 0. \tag{2.3.2}$$

Next, in case of $q \ge p-2$ we define

$$v(x,t) := \cosh R - \cosh x, \quad (x,t)\in\bar\Omega\times[0,\infty).$$

Then, for any $a \ge 1$ the function $z(x,t) := a\,v(x,t)$ satisfies

$$z_t - (p-1)(z_x^2+\varepsilon^2)^{\frac{p-2}{2}} z_{xx} - (z_x^2+\varepsilon^2)^{\frac{q}{2}}$$

$$
\begin{aligned}
&= -a^{p-1}(p-1)\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}} v_{xx} - a^q\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{q}{2}}\\
&= a^{p-1}\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}}\Big(-(p-1)v_{xx} - a^{q-p+1}\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{q-p+2}{2}}\Big)\\
&\geq a^{p-1}\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}}\Big(-v_{xx} - a^{q-p+1}(v_x^2+1)^{\frac{1}{2}}\Big)\\
&= a^{p-1}\Big(v_x^2+\frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}}\big(1-a^{q-p+1}\big)\cosh x \geq 0 \quad \text{in } \Omega\times(0,\infty)
\end{aligned}
$$

since $p>2$, $p-2\leq q<p-1$, $\varepsilon\in(0,1)$ and $v_{xx}\leq 0$ is fulfilled.
Moreover, $v_{xx}\leq 0$ and $a>0$ imply

$$(-z)_t-(p-1)((-z)_x^2+\varepsilon^2)^{\frac{p-2}{2}}(-z)_{xx}-((-z)_x^2+\varepsilon^2)^{\frac{q}{2}}\leq -\varepsilon^q\leq 0 \quad \text{in } \Omega\times(0,\infty).$$

As $|u_{0\varepsilon}(x)|\leq \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}\operatorname{dist}(x,\partial\Omega)$ and $v(x)\geq c_1\operatorname{dist}(x,\partial\Omega)$ holds for $x\in\bar\Omega$ with some $c_1>0$, we choose $a:=\max\{1,\frac{\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}}{c_1}\}$ and obtain $-z(x,0)\leq u_{0\varepsilon}(x)\leq z(x,0)$ for $x\in\bar\Omega$ and $u_\varepsilon=\pm z=0$ on $\partial\Omega$. Thus, we have

$$-z\leq u_\varepsilon\leq z \quad \text{in } \bar\Omega\times[0,\infty) \quad \text{in case of } q\geq p-2 \tag{2.3.3}$$

by comparison.
Now, for $i\in\{1,2\}$, we fix $x_1:=-2R$, $x_2:=2R$ and let

$$v_i(x):=c_0[(3R)^\alpha-|x-x_i|^\alpha],\quad x\in I_i:=[x_i-3R,x_i+3R],$$

where c_0 and α are defined in (2.2.3). Similar to Lemma 2.2.1, $v_i\in C^1(\bar I_i)\cap C^2(\bar I_i\setminus\{x_i\})$ satisfies the differential equation of (2.2.1) in $I_i\setminus\{x_i\}$ for $i\in\{1,2\}$. Next, in case of $q<p-2$ we set

$$z_i(x,t):=b\,v_i(x),\quad (x,t)\in\bar I_i\times[0,\infty),$$

where $b\geq 1$ and $i\in\{1,2\}$. Then, as $\Omega\subset I_i\setminus\{x_i\}$, we obtain for any $b\geq1$ and $i\in\{1,2\}$

$$
\begin{aligned}
&(z_i)_t-(p-1)((z_i)_x^2+\varepsilon^2)^{\frac{p-2}{2}}(z_i)_{xx}-((z_i)_x^2+\varepsilon^2)^{\frac{q}{2}}\\
&= -b^{p-1}(p-1)\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{p-2}{2}}(v_i)_{xx}-b^q\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{q}{2}}\\
&= b^{p-1}\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{p-2}{2}}\Big(-(p-1)(v_i)_{xx}-b^{q-p+1}\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{q-p+2}{2}}\Big)\\
&\geq b^{p-1}\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{p-2}{2}}\big(-(p-1)(v_i)_{xx}-b^{q-p+1}|(v_i)_x|^{q-p+2}\big)\\
&= b^{p-1}\Big((v_i)_x^2+\frac{\varepsilon^2}{b^2}\Big)^{\frac{p-2}{2}}\big(1-b^{q-p+1}\big)|(v_i)_x|^{q-p+2}\geq 0 \quad \text{in } \Omega\times(0,\infty),
\end{aligned}
$$

since $q<p-2$ is fulfilled. Due to $(v_i)_{xx}\leq 0$ in Ω, we conclude

$$(-z_i)_t-(p-1)((-z_i)_x^2+\varepsilon^2)^{\frac{p-2}{2}}(-z_i)_{xx}-((-z_i)_x^2+\varepsilon^2)^{\frac{q}{2}}\leq-\varepsilon^q\leq 0 \quad \text{in } \Omega\times(0,\infty)$$

for any $b > 0$ and $i \in \{1, 2\}$. Furthermore, there is $\tilde{c} > 0$ such that $v_i(x) \geq \tilde{c}\,\mathrm{dist}(x, \partial I_i)$ holds for $x \in \bar{I}_i$ and $i \in \{1, 2\}$. This implies $v_1(x) \geq \tilde{c}\,|x - R| \geq \tilde{c}\,\mathrm{dist}(x, \partial\Omega)$ and $v_2(x) \geq \tilde{c}\,|x + R| \geq \tilde{c}\,\mathrm{dist}(x, \partial\Omega)$ for $x \in \bar{\Omega}$. Since $|u_{0\varepsilon}(x)| \leq \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}\,\mathrm{dist}(x, \partial\Omega)$ is satisfied for $x \in \bar{\Omega}$, we choose $b := \max\{1, \frac{\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}}{\tilde{c}}\}$ and obtain $-z_i(x, 0) \leq u_{0\varepsilon}(x) \leq z_i(x, 0)$ for $x \in \bar{\Omega}$ as well as $-z_i \leq 0 = u_\varepsilon \leq z_i$ on $\partial\Omega$ for $i \in \{1, 2\}$. Thus, we obtain $-z_i \leq u_\varepsilon \leq z_i$ in $\bar{\Omega} \times [0, \infty)$ for $i \in \{1, 2\}$ by comparison. This implies

$$-(\min\{z_1, z_2\}) \leq u_\varepsilon \leq \min\{z_1, z_2\} \quad \text{in } \bar{\Omega} \times [0, \infty) \quad \text{in case of } q < p - 2. \tag{2.3.4}$$

Hence, whenever $1 < q < p - 1$, there are constants $C_0 \geq 1$ and $C \geq 1$, which are independent of $\varepsilon \in (0, 1)$, such that

$$|u_\varepsilon| \leq C_0(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \quad \text{in } \Omega \times [0, \infty)$$

and

$$|(u_\varepsilon)_x| \leq C\,(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \quad \text{on } \partial\Omega \times [0, \infty) \tag{2.3.5}$$

is fulfilled, due to $u_\varepsilon = z = \min\{z_1, z_2\} = 0$ on $\partial\Omega \times [0, \infty)$, (2.3.3) and (2.3.4). Consequently, parabolic regularity theory shows that $(u_\varepsilon)_x$ and $(u_\varepsilon)_{xx}$ are bounded in $\bar{\Omega} \times (\tau, T)$ for $0 < \tau < T < \infty$ and $(u_\varepsilon)_x \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ is satisfied (see [A2.12]). Furthermore, $y := (u_\varepsilon)_x$ fulfills

$$y_t = (p-1)(y^2+\varepsilon^2)^{\frac{p-2}{2}} y_{xx} + (p-1)(p-2)(y^2+\varepsilon^2)^{\frac{p-4}{2}} y(y_x)^2 + q(y^2+\varepsilon^2)^{\frac{q-2}{2}} y y_x \quad \text{in } \Omega \times (0, \infty).$$

Since (2.3.5) implies $|y| < C_1\,(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})$ on the parabolic boundary of $\Omega \times (0, \infty)$ with some $C_1 > 1$ independent of ε, we finally conclude

$$-C_1\,(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \leq (u_\varepsilon)_x \leq C_1\,(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \quad \text{in } \bar{\Omega} \times [0, \infty)$$

by comparison. ■

Moreover, u_ε is also Hölder continuous with respect to the time t. To prove this, we use an idea which is similar to the one used in Lemma 5 of [A2.10].

Lemma 2.3.2 *Suppose* (2.1.3) *is fulfilled,* $\varepsilon \in (0, 1)$ *and* u_ε *is the solution of* (2.3.1). *Then*

$$\begin{aligned} |u_\varepsilon(x_0, t_1) - u_\varepsilon(x_0, t_2)| \quad &\leq \quad \left(2C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^q\, |t_1 - t_2| \\ &\quad + p\left(2C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^{p-1} |t_1 - t_2|^{\frac{1}{2}} \end{aligned}$$

holds for any $x_0 \in \bar{\Omega}$ *and* $t_1, t_2 \in [0, \infty)$.

Proof. Since the claim is obvious in case of $t_1 = t_2$, we assume $t_1 \neq t_2$. By Lemma 2.3.1 we have $|u_\varepsilon(x, t)| \leq C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\,\mathrm{dist}(x, \partial\Omega)$ for $(x, t) \in \bar{\Omega} \times [0, \infty)$. Hence, in case of $\mathrm{dist}(x_0, \partial\Omega) \leq |t_1 - t_2|^{\frac{1}{2}}$ this implies

$$|u_\varepsilon(x_0, t_1) - u_\varepsilon(x_0, t_2)| \quad \leq \quad 2\,C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\,\mathrm{dist}(x_0, \partial\Omega)$$

$$\leq \quad p\left(2\,C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^{p-1}|t_1-t_2|^{\frac{1}{2}} \qquad (2.3.6)$$

due to $p>2$.
In case of $\operatorname{dist}(x_0,\partial\Omega) > |t_1-t_2|^{\frac{1}{2}}$ we choose $r := |t_1-t_2|^{\frac{1}{2}} > 0$. Thus, we get by Lemma 2.3.1 (due to $\varepsilon\in(0,1)$, $p>2$ and $C_1\geq 1$)

$$\begin{aligned}
&|u_\varepsilon(x_0,t_1)-u_\varepsilon(x_0,t_2)| \\
&= \frac{1}{2r}\left|\int_{x_0-r}^{x_0+r}(u_\varepsilon(x_0,t_1)-u_\varepsilon(x_0,t_2))\mathrm{d}x\right| \\
&= \frac{1}{2r}\left|\int_{x_0-r}^{x_0+r}(u_\varepsilon(x,t)-u_\varepsilon(x_0,t))\mathrm{d}x\Big|_{t=t_1}^{t_2} - \int_{t_1}^{t_2}\int_{x_0-r}^{x_0+r}(u_\varepsilon)_t(x,t)\mathrm{d}x\mathrm{d}t\right| \\
&\leq \frac{1}{2r}\cdot 2\,C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\int_{x_0-r}^{x_0+r}|x-x_0|\mathrm{d}x \\
&\quad +\frac{1}{2r}\left|\int_{t_1}^{t_2}\int_{x_0-r}^{x_0+r}\left[(p-1)((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{p-2}{2}}(u_\varepsilon)_{xx}+((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{q}{2}}\right](x,t)\mathrm{d}x\mathrm{d}t\right| \\
&\leq C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\,r+\frac{1}{2r}\left|\int_{t_1}^{t_2}\int_{x_0-r}^{x_0+r}((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{q}{2}}(x,t)\mathrm{d}x\mathrm{d}t\right| \\
&\quad +\frac{1}{2r}\left|\int_{t_1}^{t_2}\left[\int_0^{(u_\varepsilon)_x(x,t)}(p-1)(z^2+\varepsilon^2)^{\frac{p-2}{2}}\mathrm{d}z\right]_{x=x_0-r}^{x_0+r}\mathrm{d}t\right| \\
&\leq C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\,|t_1-t_2|^{\frac{1}{2}}+\left((C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}))^2+1\right)^{\frac{q}{2}}|t_1-t_2| \\
&\quad +\frac{|t_1-t_2|}{2r}\cdot 2C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})(p-1)\left((C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}))^2+1\right)^{\frac{p-2}{2}} \\
&\leq \left(2C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^q|t_1-t_2|+p\left(2C_1(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^{p-1}|t_1-t_2|^{\frac{1}{2}}.
\end{aligned}$$

This estimate and (2.3.6) imply the claim. ■

Furthermore, there is a Lyapunov functional for problem (2.3.1). This functional implies important estimates, which are used to prove the convergence of solutions of (2.1.1) to steady states.
For $\varepsilon>0$, we define

$$\varphi_\varepsilon(z) := \int_0^z\int_0^s(p-1)(\sigma^2+\varepsilon^2)^{\frac{p-q-2}{2}}\mathrm{d}\sigma\mathrm{d}s \qquad \text{for } z\in\mathbb{R}.$$

The idea of the following proof is similar to the one used in [A2.13] for $p = 2$ and is based on a technique developed by Zelenyak in [A2.25] (see also [A2.2, A2.19] for an application of this method to related problems).

Lemma 2.3.3 *Let* (2.1.3) *be satisfied,* $\varepsilon \in (0,1)$ *and* u_ε *denote the solution of* (2.3.1)*. Then for any* $t > 0$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\varphi_\varepsilon((u_\varepsilon)_x(x,t)) - u_\varepsilon(x,t)\right)\mathrm{d}x + \int_\Omega \left[((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\right](x,t)\mathrm{d}x = 0.$$

Proof. We fix $t > 0$. As $u_\varepsilon \in C^\infty(\Omega \times (0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ holds (see e.g. Theorem III.12.1 and Chapter VI in [A2.12]), we obtain (in some places omitting the argument (x,t))

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\varphi_\varepsilon((u_\varepsilon)_x(x,t)) - u_\varepsilon(x,t)\right)\mathrm{d}x \\
&\quad= \int_\Omega \left(\varphi_\varepsilon'((u_\varepsilon)_x)(u_\varepsilon)_{tx} - (u_\varepsilon)_t\right)\mathrm{d}x \\
&\quad= \varphi_\varepsilon'((u_\varepsilon)_x)(u_\varepsilon)_t\Big|_{x=-R}^{x=R} - \int_\Omega [\varphi_\varepsilon''((u_\varepsilon)_x)(u_\varepsilon)_{xx}+1](u_\varepsilon)_t\mathrm{d}x \\
&\quad= -\int_\Omega (u_\varepsilon)_t((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}\left[(p-1)((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{p-2}{2}}(u_\varepsilon)_{xx} + ((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{q}{2}}\right]\mathrm{d}x \\
&\quad= -\int_\Omega \left[((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\right](x,t)\mathrm{d}x
\end{aligned}$$

due to (2.3.1). ■

Now we immediately obtain an estimate for the derivative $(u_\varepsilon)_t$, which will be very useful.

Corollary 2.3.4 *Suppose* (2.1.3) *is satisfied,* $\varepsilon \in (0,1)$ *and* u_ε *denotes the solution of* (2.3.1)*. Then we have*

$$\int_0^\infty\int_\Omega ((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\mathrm{d}x\mathrm{d}t \le C_2\left(1+\|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}^2\right)^{\frac{p-q}{2}},$$

where C_2 *is a positive constant which is independent of* ε*.*

Proof. We fix $T > 0$. Since φ_ε is even and convex, we get in case of $p-2 \le q < p-1$

$$0 \le \varphi_\varepsilon(z) \le \int_0^z\int_0^s (p-1)|\sigma|^{p-q-2}\mathrm{d}\sigma\mathrm{d}s = \frac{(p-1)}{(p-q-1)(p-q)}|z|^{p-q} \quad \text{for } z \in \mathbb{R}.$$

In case of $1 < q < p - 2$, we have

$$0 \le \varphi_\varepsilon(z) \le |z|^2(p-1)(|z|^2+1)^{\frac{p-q-2}{2}} \le (p-1)(z^2+1)^{\frac{p-q}{2}} \qquad \text{for } z \in \mathbb{R}.$$

Altogether, there is a positive constant c, which only depends on p and q, such that

$$0 \le \varphi_\varepsilon(z) \le c(z^2+1)^{\frac{p-q}{2}} \qquad \text{for } z \in \mathbb{R}$$

is satisfied whenever $1 < q < p - 1$. Next, $(u_\varepsilon)_x$ is Hölder continuous in $\Omega \times [0, T]$ due to Theorem VI.2.3 and Remark VI.2.1 in [A2.12] and Lemma 2.3.1. Hence, we obtain by Lemmas 2.3.3 and 2.3.1

$$\begin{aligned}
&\int_0^T \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{2}} (u_\varepsilon)_t^2 \mathrm{d}x\mathrm{d}t \\
&= -\int_\Omega (\varphi_\varepsilon((u_\varepsilon)_x(x,T)) - u_\varepsilon(x,T))\,\mathrm{d}x + \int_\Omega (\varphi_\varepsilon((u_{0\varepsilon})_x(x)) - u_{0\varepsilon}(x))\,\mathrm{d}x \\
&\le \int_\Omega |u_\varepsilon(x,T)|\mathrm{d}x + \int_\Omega |u_{0\varepsilon}(x)|\mathrm{d}x + \int_\Omega \varphi_\varepsilon((u_{0\varepsilon})_x(x))\mathrm{d}x \\
&\le 2C_0|\Omega|\,(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) + c\,|\Omega| \left(\|(u_{0\varepsilon})_x\|^2_{L^\infty(\Omega)} + 1\right)^{\frac{p-q}{2}} \\
&\le C_2 \left(1 + \|(u_{0\varepsilon})_x\|^2_{L^\infty(\Omega)}\right)^{\frac{p-q}{2}}
\end{aligned}$$

with some positive constant C_2 which is independent of ε. As $T > 0$ was arbitrary, this implies the claim. ■

Furthermore, the preceding Corollary implies another estimate, which will be used to control terms involving $(u_\varepsilon)_{xx}$.

Corollary 2.3.5 *Let* (2.1.3) *be fulfilled,* $\varepsilon \in (0, 1)$ *and let* u_ε *denote the solution of* (2.3.1). *Then for* $0 \le t_1 < t_2 < \infty$ *we get*

$$\begin{aligned}
\int_{t_1}^{t_2} \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2p-q-4}{2}} (u_\varepsilon)_{xx}^2 \mathrm{d}x\mathrm{d}t \;\le\; & \frac{2C_2}{(p-1)^2}\left(1 + \|(u_{0\varepsilon})_x\|^2_{L^\infty(\Omega)}\right)^{\frac{p-q}{2}} \\
&+ \frac{2|\Omega|}{(p-1)^2}\,(t_2 - t_1)\left(2C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)})\right)^q .
\end{aligned}$$

Proof. For $0 \le t_1 < t_2 < \infty$ we conclude by (2.3.1), Lemma 2.3.1 and Corollary 2.3.4

$$\int_{t_1}^{t_2} \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2p-q-4}{2}} (u_\varepsilon)_{xx}^2 \mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{(p-1)^2} \int_{t_1}^{t_2} \int_\Omega \left(((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{4}} (u_\varepsilon)_t - ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{q}{4}} \right)^2 \mathrm{d}x\mathrm{d}t \\
&\leq \frac{2}{(p-1)^2} \int_{t_1}^{t_2} \int_\Omega \left(((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{2}} (u_\varepsilon)_t^2 + ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{q}{2}} \right) \mathrm{d}x\mathrm{d}t \\
&\leq \frac{2}{(p-1)^2} \int_0^\infty \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{2}} (u_\varepsilon)_t^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{2}{(p-1)^2} |\Omega| \, (t_2 - t_1) \left((C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}))^2 + 1 \right)^{\frac{q}{2}} \\
&\leq \frac{2}{(p-1)^2} C_2 \left(1 + \|(u_{0\varepsilon})_x\|^2_{L^\infty(\Omega)} \right)^{\frac{p-q}{2}} \\
&\quad + \frac{2}{(p-1)^2} |\Omega| \, (t_2 - t_1) \left(2C_1(1 + \|(u_{0\varepsilon})_x\|_{L^\infty(\Omega)}) \right)^q .
\end{aligned}
$$

Hence, the claim is proved. ■

2.4 Existence of a weak solution and convergence to steady states

We first give a definition of weak solutions to (2.1.1).

Definition 2.4.1 *A function $u \in C^0(\bar{\Omega} \times [0,\infty)) \cap L^p_{loc}([0,\infty); W^{1,p}(\Omega))$ is called a weak solution of (2.1.1), if $u(\cdot, 0) = u_0$, $u|_{\partial\Omega} = 0$ and for any $\xi \in C_0^\infty(\Omega)$ and $0 \leq s < t < \infty$*

$$
\int_\Omega (u(x,t) - u(x,s))\xi(x)\mathrm{d}x = \int_s^t \int_\Omega \Big(- (|u_x|^{p-2}u_x)(x,\tau)\, \xi_x(x) + |u_x|^q(x,\tau)\, \xi(x) \Big) \mathrm{d}x\mathrm{d}\tau \tag{2.4.1}
$$

is fulfilled.

As we approximate a weak solution of (2.1.1) by solutions of the regularized problems (2.3.1), we first state that we are able to choose the initial data $u_{0\varepsilon}$ in a suitable way.

Lemma 2.4.2 *Let (2.1.2) be fulfilled. Then there is a family $(u_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^\infty(\bar{\Omega})$ such that $u_{0\varepsilon} = 0$ on $\partial\Omega$,*

$$
\|u_{0\varepsilon} - u_0\|_{L^\infty(\Omega)} \leq \varepsilon \quad \text{and} \quad \|(u_{0\varepsilon})_x - (u_0)_x\|_{L^\infty(\Omega)} \leq \varepsilon
$$

holds for any $\varepsilon \in (0,1)$. Moreover, in case of $u_0 \geq 0$, $u_{0\varepsilon}$ is chosen such that additionally $u_{0\varepsilon} \geq 0$ is satisfied for any $\varepsilon \in (0,1)$.

Proof. We fix $\varepsilon \in (0,1)$. Due to (2.1.2) we have $(u_0)_x \in C^0(\bar\Omega)$ as well as $\int_\Omega (u_0)_x \mathrm{d}x = 0$. Thus, we are able to choose $v_\varepsilon \in C^\infty(\bar\Omega)$ such that $\int_\Omega (v_\varepsilon)_x \mathrm{d}x = 0$, and $\|(v_\varepsilon)_x - (u_0)_x\|_{L^\infty(\Omega)} \le \min\{\frac{\varepsilon}{2R}, \varepsilon\}$ holds. Furthermore, as $u_0 \ge 0$ implies $u_0(x) = \int_{-R}^x (u_0)_x(y)\mathrm{d}y \ge 0$ for $x \in \Omega$, in case of $u_0 \ge 0$ we choose v_ε such that additionally $\int_{-R}^x (v_\varepsilon)_x(y)\mathrm{d}y \ge 0$ for $x \in \Omega$ is fulfilled. Defining $u_{0\varepsilon}(x) := \int_{-R}^x (v_\varepsilon)_x(y)\mathrm{d}y$ for $x \in \bar\Omega$, $u_{0\varepsilon} \in C^\infty(\bar\Omega)$ fulfills $u_{0\varepsilon} = 0$ on $\partial\Omega$ as well as $\|u_{0\varepsilon} - u_0\|_{L^\infty(\Omega)} \le \varepsilon$ and $\|(u_{0\varepsilon})_x - (u_0)_x\|_{L^\infty(\Omega)} \le \varepsilon$. We additionally have $u_{0\varepsilon} \ge 0$ in case of $u_0 \ge 0$. ■

Now the results of the preceding section imply the existence of a weak solution to (2.1.1).

Theorem 2.4.3 *Suppose* (2.1.2) *and* (2.1.3) *are fulfilled,* $(u_{0\varepsilon})_{\varepsilon\in(0,1)} \subset C^\infty(\bar\Omega)$ *satisfy the conditions of Lemma 2.4.2 and* $(u_\varepsilon)_{\varepsilon\in(0,1)}$ *denote the corresponding solutions of* (2.3.1). *Then there is a sequence* $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ *with* $\varepsilon_k \searrow 0$ *as* $k \to \infty$ *and a global weak solution* u *of* (2.1.1) *satisfying* $u_{\varepsilon_k} \to u$ *in* $C^0_{loc}(\bar\Omega \times [0,\infty))$ *as* $k \to \infty$. *Furthermore,* u *fulfills*

$$\min_{x\in\bar\Omega} u_0(x) \le u \le \max_{x\in\bar\Omega} u_0(x) \ \text{ in } \bar\Omega \times [0,\infty)$$

and

$$\|u_x(t)\|_{L^\infty(\Omega)} \le C_1(2 + \|(u_0)_x\|_{L^\infty(\Omega)}) \text{ for any } t \in [0,\infty),$$

where C_1 *is defined in Lemma 2.3.1. Moreover,* $\|u(t)\|_{L^\infty(\Omega)}$ *is a nonincreasing function of* $t \ge 0$.

Proof. We fix $\varepsilon \in (0,1)$ and choose $u_{0\varepsilon}$ like in Lemma 2.4.2. Furthermore, let u_ε denote the classical solution of (2.3.1) evolving from $u_{0\varepsilon}$ which we obtain in Lemma 2.3.1. Due to Lemmas 2.3.1 and 2.4.2 we conclude

$$|u_\varepsilon| \le C_0(2+\|(u_0)_x\|_{L^\infty(\Omega)}) \quad \text{and} \quad |(u_\varepsilon)_x| \le C_1(2+\|(u_0)_x\|_{L^\infty(\Omega)}) \ \text{ in } \Omega\times[0,\infty) \quad (2.4.2)$$

for any $\varepsilon \in (0,1)$. We define

$$M_1 := C_1(2 + \|(u_0)_x\|_{L^\infty(\Omega)}) \quad (2.4.3)$$

to simplify the notation.
Next we fix $T \in (0,\infty)$. Then, by (2.4.2) and Lemma 2.3.2 the set $\mathcal{M} := \{u_\varepsilon \mid \varepsilon \in (0,1)\}$ is bounded in $C^0(\bar\Omega \times [0,T])$ and equicontinuous in $\bar\Omega \times [0,T]$. Hence, the Arzelà-Ascoli theorem implies that $\mathcal{M}$ is relatively compact in $C^0(\bar\Omega \times [0,T])$. Moreover, $\mathcal{M}$ is also bounded in $L^p((0,T); W_0^{1,p}(\Omega))$ by (2.4.2). Thus, we obtain a function $u \in C^0(\bar\Omega\times[0,T]) \cap L^p((0,T); W_0^{1,p}(\Omega))$ and a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k \searrow 0$ and

$$u_{\varepsilon_k} \to u \text{ uniformly in } \bar\Omega \times [0,T] \text{ and weakly in } L^p((0,T); W_0^{1,p}(\Omega)) \text{ as } k \to \infty. \quad (2.4.4)$$

Since $u_\varepsilon(t)$ is Lipschitz continuous in $\bar\Omega$ for any $t \in [0,T]$ with Lipschitz constant M_1 by (2.4.2) and (2.4.3), $u(t)$ is again Lipschitz continuous in $\bar\Omega$ for any $t \in [0,T]$ with Lipschitz constant M_1 by (2.4.4). This implies $u(t) \in W^{1,\infty}(\Omega)$ with

$$\|u_x(t)\|_{L^\infty(\Omega)} \le M_1 \quad \text{for any } t \in [0,T].$$

Next, we define

$$\varphi(z) := |z|^{p-1}z \quad \text{for } z \in \mathbb{R}. \tag{2.4.5}$$

Then, (2.4.2) and (2.4.3) imply

$$|\varphi((u_\varepsilon)_x)| \le M_1^p \quad \text{in } \Omega \times [0,T]. \tag{2.4.6}$$

Moreover, by (2.4.2), Corollaries 2.3.4 and 2.3.5 we obtain constants $c_1 = c_1(T, \|(u_0)_x\|_{L^\infty(\Omega)})$ and $c_2 = c_2(\|(u_0)_x\|_{L^\infty(\Omega)})$, which are independent of ε, such that

$$\int_0^T \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2p-q-4}{2}} (u_\varepsilon)_{xx}^2 \mathrm{d}x\mathrm{d}t \le c_1 \quad \text{and} \quad \int_0^\infty \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{2}} (u_\varepsilon)_t^2 \mathrm{d}x\mathrm{d}t \le c_2 \tag{2.4.7}$$

holds. This implies

$$\begin{aligned}
\int_0^T \int_\Omega [\varphi((u_\varepsilon)_x)]_x^2 \mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega [p\,|(u_\varepsilon)_x|^{p-1}(u_\varepsilon)_{xx}]^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_0^T \int_\Omega [p((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{p-1}{2}}(u_\varepsilon)_{xx}]^2 \mathrm{d}x\mathrm{d}t \\
&= p^2 \int_0^T \int_\Omega ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{q+2}{2}} ((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2p-q-4}{2}} (u_\varepsilon)_{xx}^2 \mathrm{d}x\mathrm{d}t \\
&\le p^2 \left(M_1^2 + 1\right)^{\frac{q+2}{2}} \cdot c_1.
\end{aligned} \tag{2.4.8}$$

Furthermore, due to $\Omega \subset \mathbb{R}$, there is $C > 0$ such that $\|\rho\|_{L^\infty(\Omega)} \le C\|\rho\|_{W_0^{1,2}(\Omega)}$ for all $\rho \in W_0^{1,2}(\Omega)$ is fulfilled. Hence, by (2.4.7), (2.4.2) and (2.4.3) we conclude

$$\begin{aligned}
&\|[\varphi((u_\varepsilon)_x)]_t\|_{L^1((0,T);(W_0^{1,2}(\Omega))^*)} \\
&= \int_0^T \left(\sup_{\rho \in C_0^\infty(\Omega), \|\rho\|_{W_0^{1,2}(\Omega)} \le 1} \int_\Omega p|(u_\varepsilon)_x|^{p-1}(u_\varepsilon)_{xt}\rho(x)\mathrm{d}x \right) \mathrm{d}t \\
&= \int_0^T \left(\sup_{\rho \in C_0^\infty(\Omega), \|\rho\|_{W_0^{1,2}(\Omega)} \le 1} p \int_\Omega -\left[|(u_\varepsilon)_x|^{p-1}(u_\varepsilon)_t\rho_x(x) \right.\right. \\
&\qquad \left.\left. +(p-1)|(u_\varepsilon)_x|^{p-3}(u_\varepsilon)_x(u_\varepsilon)_{xx}(u_\varepsilon)_t\rho(x)\right] \mathrm{d}x \right) \mathrm{d}t \\
&\le p \int_0^T \left(\sup_{\rho \in C_0^\infty(\Omega), \|\rho\|_{W_0^{1,2}(\Omega)} \le 1} \int_\Omega \left[((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2p+q-2}{4}} ((u_\varepsilon)_x^2 + \varepsilon^2)^{-\frac{q}{4}} |(u_\varepsilon)_t||\rho_x(x)| \right.\right.
\end{aligned}$$

$$+(p-1)\|\rho\|_{L^\infty(\Omega)}((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{2p+q-4}{4}}|(u_\varepsilon)_{xx}|((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{4}}|(u_\varepsilon)_t|\Big]\,\mathrm{d}x\Bigg)\mathrm{d}t$$

$$\begin{aligned}
\leq\ & p\int_0^T \sup_{\rho\in C_0^\infty(\Omega),\|\rho\|_{W_0^{1,2}(\Omega)}\leq 1}\Bigg((M_1^2+1)^{\frac{2p+q-2}{4}}\|\rho_x\|_{L^2(\Omega)}\Big(\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\mathrm{d}x\Big)^{\frac12} \\
& +(p-1)C\|\rho\|_{W_0^{1,2}(\Omega)}(M_1^2+1)^{\frac{2q}{4}}\cdot \\
& \cdot\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{2p-q-4}{4}}|(u_\varepsilon)_{xx}|((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{4}}|(u_\varepsilon)_t|\mathrm{d}x\Bigg)\mathrm{d}t \\
\leq\ & p(M_1^2+1)^{\frac{2p+q-2}{4}}\int_0^T\Bigg(\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\mathrm{d}x\Bigg)^{\frac12}\mathrm{d}t \\
& +p(p-1)C(M_1^2+1)^{\frac{q}{2}}\int_0^T\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{2p-q-4}{4}}|(u_\varepsilon)_{xx}|((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{4}}|(u_\varepsilon)_t|\mathrm{d}x\mathrm{d}t \\
\leq\ & p(M_1^2+1)^{\frac{2p+q-2}{4}}T^{\frac12}\Bigg(\int_0^T\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\mathrm{d}x\mathrm{d}t\Bigg)^{\frac12}+p(p-1)C(M_1^2+1)^{\frac{q}{2}}\cdot \\
& \cdot\Bigg(\int_0^T\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{2p-q-4}{2}}(u_\varepsilon)_{xx}^2\mathrm{d}x\mathrm{d}t\Bigg)^{\frac12}\cdot\Bigg(\int_0^T\int_\Omega((u_\varepsilon)_x^2+\varepsilon^2)^{-\frac{q}{2}}(u_\varepsilon)_t^2\mathrm{d}x\mathrm{d}t\Bigg)^{\frac12} \\
\leq\ & p(M_1^2+1)^{\frac{2p+q-2}{4}}T^{\frac12}c_2^{\frac12}+p(p-1)C(M_1^2+1)^{\frac{q}{2}}c_1^{\frac12}c_2^{\frac12}.
\end{aligned}\tag{2.4.9}$$

Altogether, (2.4.6), (2.4.8) and (2.4.9) imply

$$\|\varphi((u_\varepsilon)_x)\|_{L^2((0,T);W^{1,2}(\Omega))}\leq\tilde{C}\text{ and }\|[\varphi((u_\varepsilon)_x)]_t\|_{L^1((0,T);(W_0^{1,2}(\Omega))^*)}\leq\tilde{C}$$

with some $\tilde{C}>0$ which is independent of $\varepsilon\in(0,1)$. Thus, the set $\mathcal{O}:=\{\varphi((u_\varepsilon)_x)\,|\,\varepsilon\in(0,1)\}$ is relatively compact in $L^2((0,T);L^2(\Omega))$ by the Aubin-Lions lemma (see Theorem 2.3 in [A2.23]). As $\mathcal{O}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$, there is a subsequence of $(\varepsilon_k)_{k\in\mathbb{N}}$ (which is not relabeled) and a function $v\in L^2((0,T);W^{1,2}(\Omega))$ such that $\varphi((u_{\varepsilon_k})_x)$ converges to v as $k\to\infty$ strongly in $L^2((0,T);L^2(\Omega))$, weakly in $L^2((0,T);W^{1,2}(\Omega))$ and a.e. in $\Omega\times[0,T]$. Due to $p>1$, we therefore have $(u_{\varepsilon_k})_x(x,t)\to\operatorname{sign}(v(x,t))|v(x,t)|^{\frac1p}$ a.e. in $\Omega\times(0,T)$. Since $u_{\varepsilon_k}\to u$ weakly in $L^p((0,T);W_0^{1,p}(\Omega))$, this implies $u_x=\operatorname{sign}(v)|v|^{\frac1p}$ a.e. in $\Omega\times(0,T)$. Thus, we conclude

$$(u_{\varepsilon_k})_x(x,t)\to u_x(x,t)\quad\text{a.e. in }\Omega\times(0,T)\tag{2.4.10}$$

and $v=\varphi(u_x)$.

Moreover, we fix $\xi\in C_0^\infty(\Omega)$ and $0\leq s<t\leq T$. Defining

$$\zeta_\varepsilon(z):=\int_0^z(p-1)(s^2+\varepsilon^2)^{\frac{p-2}{2}}\mathrm{d}s\qquad\text{for }z\in\mathbb{R}\text{ and }\varepsilon\in[0,1),\tag{2.4.11}$$

(2.3.1) implies

$$\begin{aligned}
&\int_\Omega (u_{\varepsilon_k}(x,t) - u_{\varepsilon_k}(x,s))\xi(x)\mathrm{d}x \\
= &\int_s^t \int_\Omega \Big(-\zeta_{\varepsilon_k}((u_{\varepsilon_k})_x(x,\tau))\,\xi_x(x) + ((u_{\varepsilon_k})_x^2 + \varepsilon_k^2)^{\frac{q}{2}}(x,\tau)\,\xi(x)\Big)\mathrm{d}x\mathrm{d}\tau. \quad (2.4.12)
\end{aligned}$$

For any sequence $(z_k)_{k\in\mathbb{N}} \subset [-M_1, M_1]$ such that $z_k \to z$ as $k \to \infty$ we have

$$\begin{aligned}
|\zeta_{\varepsilon_k}(z_k) - |z|^{p-2}z| &\leq |\zeta_{\varepsilon_k}(z_k) - \zeta_{\varepsilon_k}(z)| + |\zeta_{\varepsilon_k}(z) - \zeta_0(z)| \\
&\leq (p-1)(M_1^2+1)^{\frac{p-2}{2}}|z_k - z| + |\zeta_{\varepsilon_k}(z) - \zeta_0(z)|
\end{aligned}$$

and therefore $\zeta_{\varepsilon_k}(z_k) \to |z|^{p-2}z$ as $k \to \infty$ by the dominated convergence theorem. Hence, as $k \to \infty$ in (2.4.12), we obtain that (2.4.1) is satisfied due to (2.4.2), (2.4.3), (2.4.4) and (2.4.10). Taking $T \in (0,\infty)$ arbitrary, upon a repeated extraction process we finally obtain a global weak solution u of (2.1.1).
Furthermore, (2.4.4) and (2.3.2) imply

$$\min_{x\in\bar\Omega} u(x,t_0) \leq u(x,t) \leq \max_{x\in\bar\Omega} u(x,t_0) \quad \text{for } (x,t) \in \bar\Omega \times [t_0,\infty) \text{ and any } t_0 \geq 0.$$

In particular, $\|u(t)\|_{L^\infty(\Omega)}$ is a nonincreasing function of $t \geq 0$. Altogether, the theorem is proved due to (2.4.2) and (2.4.10). ■

We remark that uniqueness of solutions to (2.1.1) is likely to be obtained within the theory of viscosity solutions, but we do not need it here.
Now we are ready to prove the main result concerning the large time behavior of u. We show that u converges to one of the weak solutions w_ϑ of (2.2.1) which were obtained in Lemma 2.2.1. In particular, this limit function is nonnegative independent of the sign of the initial data u_0. In the proof we use an idea which is similar to that used in [A2.24]. We particularly make use of the Lyapunov functional which is established in Lemma 2.3.3.

Theorem 2.4.4 *Suppose* (2.1.2) *and* (2.1.3) *are satisfied and* u *denotes the weak solution of* (2.1.1) *which is obtained in Theorem 2.4.3. Then there is a unique* $\vartheta \in [0, R]$ *such that* $\|w_\vartheta\|_{L^\infty(\Omega)} = \lim_{t\to\infty}\|u(t)\|_{L^\infty(\Omega)}$ *and*

$$\|u(t) - w_\vartheta\|_{L^\infty(\Omega)} \to 0 \quad \textit{as } t \to \infty$$

is fulfilled.

Proof. We fix a sequence $(\tilde t_m)_{m\in\mathbb{N}} \subset (1,\infty)$ such that $\tilde t_m \to \infty$ as $m \to \infty$. Then, due to the Sobolev embedding theorem with $p > 2$ and the one-dimensional case, the set $\mathcal{M} := \{u(\tilde t_m)\,|\, m \in \mathbb{N}\}$ is bounded in $C^{\frac{1}{2}}(\bar\Omega) \cap W_0^{1,p}(\Omega)$ by Theorem 2.4.3. Thus, by the Arzelà-Ascoli theorem we obtain a function $w \in C^0(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$ and a subsequence $(t_l)_{l\in\mathbb{N}}$ of $(\tilde{t}_m)_{m\in\mathbb{N}}$ such that

$$u(t_l) \to w \quad \text{strongly in } C^0(\bar{\Omega}) \text{ and weakly in } W_0^{1,p}(\Omega) \text{ as } l \to \infty. \tag{2.4.13}$$

For $\varepsilon \in (0,1)$ let u_ε denote the solution of (2.3.1) which is constructed in the proof of Theorem 2.4.3. Moreover, we define

$$\tilde{u}_l(x,\tau) := u(x, t_l + \tau) \quad \text{for } (x,\tau) \in \bar{\Omega} \times [-1,1]. \tag{2.4.14}$$

As Theorem 2.4.3, (2.4.2), (2.4.3) and Corollary 2.3.4 imply

$$\int_0^\infty \int_\Omega u_t^2 \mathrm{d}x\mathrm{d}t \le C_2(M_1^2+1)^{\frac{p}{2}} < \infty, \tag{2.4.15}$$

we conclude for $l \in \mathbb{N}$

$$\begin{aligned} &\int_{-1}^1 \int_\Omega |\tilde{u}_l(x,\tau) - w(x)|^2 \mathrm{d}x\mathrm{d}\tau \\ \le\ & 2\int_{-1}^1 \int_\Omega |u(x,t_l+\tau) - u(x,t_l)|^2 \mathrm{d}x\mathrm{d}\tau + 2\int_{-1}^1 \int_\Omega |u(x,t_l) - w(x)|^2 \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

Now we obtain due to $u, u_t \in L^2(\Omega \times (0, t_l+1))$

$$\begin{aligned} &\int_{-1}^1 \int_\Omega |\tilde{u}_l(x,\tau) - w(x)|^2 \mathrm{d}x\mathrm{d}\tau \\ &\quad\le\ 2\int_{-1}^1 \int_\Omega \left| \int_{t_l}^{t_l+\tau} u_t(x,s)\mathrm{d}s \right|^2 \mathrm{d}x\mathrm{d}\tau + 4|\Omega|\, \|u(t_l) - w\|_{L^\infty(\Omega)}^2 \\ &\quad\le\ 2\int_{-1}^1 \int_\Omega |\tau| \left| \int_{t_l}^{t_l+\tau} |u_t(x,s)|^2 \mathrm{d}s \right| \mathrm{d}x\mathrm{d}\tau + 4|\Omega|\, \|u(t_l) - w\|_{L^\infty(\Omega)}^2 \\ &\quad\le\ 2\int_{t_l-1}^\infty \int_\Omega |u_t(x,s)|^2 \mathrm{d}x\mathrm{d}s + 4|\Omega|\, \|u(t_l) - w\|_{L^\infty(\Omega)}^2. \end{aligned}$$

Hence, we have $\tilde{u}_l \to W$ in $L^2(\Omega \times (-1,1))$ as $l \to \infty$ by (2.4.13) and (2.4.15), where $W(x,\tau) := w(x)$ for $(x,\tau) \in \Omega \times (-1,1)$. Thus, there is a subsequence of $(t_l)_{l\in\mathbb{N}}$ (which we do not rename) such that

$$\tilde{u}_l \to W \quad \text{in } L^2(\Omega \times (-1,1)) \text{ and a.e. in } \Omega \times (-1,1) \text{ as } l \to \infty. \tag{2.4.16}$$

Furthermore, we choose φ like in (2.4.5). Then we obtain in a way completely similar to (2.4.6), (2.4.8) and (2.4.9) that

$$\|\varphi((u_\varepsilon)_x)\|_{L^2((t_l-1,t_l+1);W^{1,2}(\Omega))} \le \tilde{C} \text{ and } \|[\varphi((u_\varepsilon)_x)]_t\|_{L^1((t_l-1,t_l+1);(W_0^{1,2}(\Omega))^*)} \le \tilde{C}$$

holds for any $\varepsilon \in (0,1)$ and $l \in \mathbb{N}$ with a positive constant $\tilde{C}$ which is independent of ε and l (since $\tilde{C}$ only depends on the length of the interval (t_l-1, t_l+1)). This implies

$$\|\varphi(((\widetilde{u_\varepsilon})_l)_x)\|_{L^2((-1,1);W^{1,2}(\Omega))} \le \tilde{C} \text{ and } \|[\varphi(((\widetilde{u_\varepsilon})_l)_x)]_t\|_{L^1((-1,1);(W_0^{1,2}(\Omega))^*)} \le \tilde{C}$$

for every $\varepsilon \in (0,1)$ and $l \in \mathbb{N}$. Therefore, the Aubin-Lions lemma (see Theorem 2.3 in [A2.23]) implies that the set $\mathcal{M} := \{\varphi(((\widetilde{u_\varepsilon})_l)_x) \,|\, \varepsilon \in (0,1), l \in \mathbb{N}\}$ is relatively compact in $L^2((-1,1);L^2(\Omega))$. Moreover, (2.4.4), (2.4.10) and the choice of the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in the proof of Theorem 2.4.3 yield $\mathcal{N} := \{\varphi((\tilde{u}_l)_x) \,|\, l \in \mathbb{N}\} \subset \overline{\mathcal{M}}$. Hence, $\mathcal{N}$ is again relatively compact in $L^2((-1,1);L^2(\Omega))$. Thus, there is a subsequence of $(t_l)_{l\in\mathbb{N}}$ (which is not relabeled) and a function $v \in L^2((-1,1);W^{1,2}(\Omega))$ such that $\varphi((\tilde{u}_l)_x)$ converges to v as $l \to \infty$ strongly in $L^2((-1,1);L^2(\Omega))$, weakly in $L^2((-1,1);W^{1,2}(\Omega))$ and a.e. in $\Omega \times (-1,1)$. Due to $p > 1$, we therefore have $(\tilde{u}_l)_x(x,t) \to \operatorname{sign}(v(x,t))|v(x,t)|^{\frac{1}{p}}$ a.e. in $\Omega \times (-1,1)$. As $\{\tilde{u}_l \,|\, l \in \mathbb{N}\}$ is bounded in $L^p((-1,1);W_0^{1,p}(\Omega))$ by Theorem 2.4.3, again there is a subsequence of $(t_l)_{l\in\mathbb{N}}$ (which we again do not rename) such that $\tilde{u}_l \to w$ weakly in $L^p((-1,1);W_0^{1,p}(\Omega))$ by (2.4.16). This implies $w_x = \operatorname{sign}(v)|v|^{\frac{1}{p}}$ a.e. in $\Omega \times (-1,1)$ and hence

$$(\tilde{u}_l)_x(x,\tau) \to w_x(x) \quad \text{a.e. in } \Omega \times (-1,1) \tag{2.4.17}$$

and $v = \varphi(w_x)$ is satisfied.

Next, we fix $\xi \in C_0^\infty(\Omega)$ and $\rho \in C_0^\infty((-1,1))$ such that $\int_{-1}^1 \rho(\tau)\mathrm{d}\tau = 1$ and recall the definition of ζ_ε in (2.4.11). Thus, by (2.3.1) we obtain for any $k, l \in \mathbb{N}$

$$\begin{aligned}
&-\int_{t_l-1}^{t_l+1}\int_\Omega u_{\varepsilon_k}(x,t)\xi(x)\rho'(t-t_l)\mathrm{d}x\mathrm{d}t \\
&= \int_{t_l-1}^{t_l+1}\int_\Omega \Big(-\zeta_{\varepsilon_k}((u_{\varepsilon_k})_x(x,t))\,\xi_x(x) + ((u_{\varepsilon_k})_x^2 + \varepsilon_k^2)^{\frac{q}{2}}(x,t)\,\xi(x)\Big)\rho(t-t_l)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By (2.4.2), (2.4.3), (2.4.4) and (2.4.10) we obtain as $k \to \infty$ (in a completely similar way to the proof that (2.4.12) implies (2.4.1))

$$\begin{aligned}
&-\int_{t_l-1}^{t_l+1}\int_\Omega u(x,t)\xi(x)\rho'(t-t_l)\mathrm{d}x\mathrm{d}t \\
&= \int_{t_l-1}^{t_l+1}\int_\Omega \Big(-(|u_x|^{p-2}u_x)(x,t)\,\xi_x(x) + |u_x|^q(x,t)\,\xi(x)\Big)\rho(t-t_l)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This equation is equivalent to

$$\begin{aligned}0 &= \int_{-1}^{1}\int_{\Omega} \tilde{u}_l(x,\tau)\xi(x)\rho'(\tau)\mathrm{d}x\mathrm{d}\tau \\ &+ \int_{-1}^{1}\int_{\Omega} \Big(- (|(\tilde{u}_l)_x|^{p-2}(\tilde{u}_l)_x)(x,\tau)\,\xi_x(x) + |(\tilde{u}_l)_x|^q(x,\tau)\,\xi(x)\Big)\rho(\tau)\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

In the limit $l \to \infty$, (2.4.16), (2.4.17), Theorem 2.4.3 and the dominated convergence theorem imply

$$\begin{aligned}0 &= \int_{-1}^{1}\int_{\Omega} w(x)\xi(x)\rho'(\tau)\mathrm{d}x\mathrm{d}\tau \\ &+ \int_{-1}^{1}\int_{\Omega} \Big(- (|w_x|^{p-2}w_x)(x)\,\xi_x(x) + |w_x|^q(x)\,\xi(x)\Big)\rho(\tau)\mathrm{d}x\mathrm{d}\tau \\ &= \left(\int_{-1}^{1} \rho'(\tau)\mathrm{d}\tau\right)\left(\int_{\Omega} w(x)\xi(x)\mathrm{d}x\right) \\ &+ \left(\int_{-1}^{1} \rho(\tau)\mathrm{d}\tau\right)\left(\int_{\Omega} \Big(- (|w_x|^{p-2}w_x)(x)\,\xi_x(x) + |w_x|^q(x)\,\xi(x)\Big)\mathrm{d}x\right) \\ &= \int_{\Omega} \Big(- (|w_x|^{p-2}w_x)(x)\,\xi_x(x) + |w_x|^q(x)\,\xi(x)\Big)\mathrm{d}x\end{aligned}$$

due to the choice of ρ. As $\xi \in C_0^\infty(\Omega)$ was arbitrary, w satisfies (2.2.2). Moreover, w fulfills $w|_{\partial\Omega} = 0$ due to (2.4.13) and Theorem 2.4.3. Furthermore, (2.2.2) implies that $\Phi := |w_x|^{p-2}w_x$ is weakly differentiable with $\Phi_x = -|w_x|^q$ a.e. in Ω. As $q < p-1$ and $w_x \in L^p(\Omega)$ due to (2.4.13), we conclude $\Phi \in W^{1,\frac{p}{p-1}}(\Omega)$ and thus $\Phi \in C^0(\bar{\Omega})$. Hence, we obtain $w_x \in C^0(\bar{\Omega})$ since $w_x = \operatorname{sign}(\Phi)|\Phi|^{\frac{1}{p-1}}$. This implies $w \in C^1(\bar{\Omega})$ and thus $w = w_\vartheta$ for some $\vartheta \in [0,R]$ by Lemma 2.2.1. In particular, w is nonnegative. Thus, from (2.4.13) and Theorem 2.4.3, we conclude

$$c_0(R-\vartheta)^\alpha = \|w_\vartheta\|_{L^\infty(\Omega)} = \lim_{l\to\infty} \|u(t_l)\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)},$$

and therefore $\vartheta \in [0,R]$ is uniquely determined. Altogether, by (2.4.13) we conclude

$$\|u(t_l) - w_\vartheta\|_{L^\infty(\Omega)} \to 0 \quad \text{as } l \to \infty.$$

This implies the claim, since $(t_l)_{l\in\mathbb{N}}$ is a subsequence of $(\tilde{t}_m)_{m\in\mathbb{N}}$, which was an arbitrarily chosen sequence satisfying $\tilde{t}_m \to \infty$ as $m \to \infty$, and since ϑ is independent of the sequence

$(\tilde{t}_m)_{m\in\mathbb{N}}$. ∎

The following proposition shows that, if u_0 is nonnegative such that $u_0 \not\equiv 0$, $u(t)$ can only converge to a nontrivial steady state.

Proposition 2.4.5 *Let* (2.1.2) *and* (2.1.3) *be satisfied and let* u *denote the weak solution of* (2.1.1) *which is obtained in Theorem 2.4.3. Then, if* u_0 *is nonnegative with* $u_0 \not\equiv 0$, *we have* $\lim\limits_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)} > 0$.

Proof. Suppose u_0 is nonnegative such that $u_0 \not\equiv 0$. As $u_0 \in C^0(\bar{\Omega})$, there exist $x_0 \in \Omega$, $r > 0$ and $\delta > 0$ such that $u_0 \geq 2\delta$ in $I := (x_0 - r, x_0 + r)$ holds. Moreover, for $\varepsilon \in (0,1)$, let u_ε denote the solution of (2.3.1) which is constructed in the proof of Theorem 2.4.3. As $\|u_{0\varepsilon} - u_0\|_{L^\infty(\Omega)} \leq \varepsilon$ is satisfied by Lemma 2.4.2, we obtain $u_{0\varepsilon} \geq \delta$ in $\bar{I}$ for any $\varepsilon \in (0,\delta)$. Furthermore, in case of $q > p-2$ we define

$$\tilde{w}(x) := c_0[r^\alpha - |x - x_0|^\alpha] \qquad \text{for } x \in \bar{I},$$

where α and c_0 are defined in (2.2.3). Hence, as $q > p-2$ implies $\alpha > 2$, we have $\tilde{w} \in C^2(\bar{I})$. Moreover, similar to Lemma 2.2.1 we have

$$(p-1)|\tilde{w}_x|^{p-2}\tilde{w}_{xx} + |\tilde{w}_x|^q = 0 \quad \text{and} \quad (p-1)\tilde{w}_{xx} + |\tilde{w}_x|^{q-p+2} = 0 \quad \text{in } I.$$

Then, we choose $a \in (0,1)$ such that $ac_0 r^\alpha \leq \delta$ and set $z(x,t) := a\,\tilde{w}(x)$ for $(x,t) \in \bar{I} \times [0,\infty)$. Thus, in case of $q > p-2$ we conclude

$$\begin{aligned}
& z_t - (p-1)(z_x^2 + \varepsilon^2)^{\frac{p-2}{2}} z_{xx} - (z_x^2 + \varepsilon^2)^{\frac{q}{2}} \\
= \;& -a^{p-1}(p-1)\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}} \tilde{w}_{xx} - a^q\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{q}{2}} \\
= \;& a^{p-1}\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}} \left(-(p-1)\tilde{w}_{xx} - a^{q-p+1}\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{q-p+2}{2}}\right) \\
\leq \;& a^{p-1}\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}} \left(-(p-1)\tilde{w}_{xx} - a^{q-p+1}|\tilde{w}_x|^{q-p+2}\right) \\
\leq \;& a^{p-1}\Big(\tilde{w}_x^2 + \frac{\varepsilon^2}{a^2}\Big)^{\frac{p-2}{2}} \left(1 - a^{q-p+1}\right)|\tilde{w}_x|^{q-p+2} \leq 0 \quad \text{in } I \times (0,\infty)
\end{aligned}$$

as $p - 2 < q < p - 1$.
In case of $q \leq p-2$ we set $v(x) := r^2 - (x - x_0)^2$, $x \in \bar{I}$, and choose $b \in (0,1)$ such that $br^2 \leq \delta$ and $2(p-1) \leq b^{q-p+1}(4r^2+1)^{\frac{q-p+2}{2}}$ is fulfilled. Defining $z(x,t) := b\,v(x)$ for $(x,t) \in \bar{I} \times [0,\infty)$, we obtain

$$\begin{aligned}
& z_t - (p-1)(z_x^2 + \varepsilon^2)^{\frac{p-2}{2}} z_{xx} - (z_x^2 + \varepsilon^2)^{\frac{q}{2}} \\
= \;& b(z_x^2 + \varepsilon^2)^{\frac{p-2}{2}} \left(-(p-1)v_{xx} - b^{q-p+1}\Big(v_x^2 + \frac{\varepsilon^2}{b^2}\Big)^{\frac{q-p+2}{2}}\right) \\
\leq \;& b(z_x^2 + \varepsilon^2)^{\frac{p-2}{2}} \left(2(p-1) - b^{q-p+1}(4r^2+1)^{\frac{q-p+2}{2}}\right) \leq 0 \quad \text{in } I \times (0,\infty)
\end{aligned}$$

for any $\varepsilon \in (0, b)$ in case of $q \leq p-2$ due to $|v_x| \leq 2r$ in I.
Moreover, for any $\varepsilon \in (0, \delta)$, we obtain $z \leq u_\varepsilon$ on the parabolic boundary of $I \times [0, \infty)$ in both cases, as u_ε is nonnegative by Lemma 2.3.1 and Lemma 2.4.2 as well as $z(x, 0) \leq \delta \leq u_{0\varepsilon}(x)$ for $x \in \bar{I}$. Thus, for $1 < q < p-1$, the comparison principle implies $z \leq u_\varepsilon$ in $\bar{I} \times [0, \infty)$ for any $\varepsilon \in (0, \min\{\delta, b\})$. Therefore, we conclude $z \leq u$ in $\bar{I} \times [0, \infty)$ due to Theorem 2.4.3. As this implies $\|u(t)\|_{L^\infty(\Omega)} \geq \tilde{\delta} > 0$ for all $t \geq 0$, we obtain $\lim_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)} \geq \tilde{\delta} > 0$ by Theorem 2.4.3. ■

Finally, we obtain further information about the limit function w_ϑ from Theorem 2.4.4, if the initial data u_0 of the solution u only have one sign.

Corollary 2.4.6 *Suppose* (2.1.2) *and* (2.1.3) *are satisfied and u denotes the weak solution of* (2.1.1) *which is obtained in Theorem 2.4.3. Moreover, let $\vartheta \in [0, R]$ be chosen such that $\|u(t) - w_\vartheta\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ is fulfilled. Then, in case of $u_0 \geq 0$ with $u_0 \not\equiv 0$ we have $w_\vartheta \not\equiv 0$, whereas $w_\vartheta \equiv 0$ holds in case of $u_0 \leq 0$.*

Proof. In case of $u_0 \geq 0$ with $u_0 \not\equiv 0$, $\|w_\vartheta\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)} > 0$ holds by Proposition 2.4.5 and Theorem 2.4.4. This implies $w_\vartheta \not\equiv 0$ in this case.
In case of $u_0 \leq 0$, Theorem 2.4.3 and Theorem 2.4.4 yield $w_\vartheta \leq 0$. Hence, we obtain $w_\vartheta \equiv 0$ in this case as w_ϑ is nonnegative. ■

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Article 3:
Convergence to steady states for radially symmetric solutions to a quasilinear degenerate diffusive Hamilton-Jacobi equation

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Abstract

Convergence to a single steady state is shown for non-negative and radially symmetric solutions to a diffusive Hamilton-Jacobi equation with homogeneous Dirichlet boundary conditions, the diffusion being the p-Laplacian operator, $p \geq 2$, and the source term a power of the norm of the gradient of u. As a first step, the radially symmetric and non-increasing stationary solutions are characterized.

Key words: convergence to steady state, degenerate parabolic equation, viscosity solutions, gradient source term

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3.1 Introduction

We investigate the large time behaviour of non-negative and radially symmetric solutions to the initial-boundary value problem

$$\begin{cases} \partial_t u &= \Delta_p u + |\nabla u|^q, & x \in B, & t \in (0,\infty), \\ u &= 0, & x \in \partial B, & t \in (0,\infty), \\ u(x,0) &= u_0(x), & x \in B, & \end{cases} \tag{3.1.1}$$

where $B := \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and the p-Laplacian operator is defined by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u).$$

We further assume the initial condition

$$u_0 \in W_0^{1,\infty}(B) \text{ is radially symmetric and non-negative and } u_0 \not\equiv 0, \tag{3.1.2}$$

while the parameters p and q satisfy

$$p \geq 2 \quad \text{and} \quad 0 < q < p-1. \tag{3.1.3}$$

The partial differential equation in (3.1.1) is a second-order parabolic equation featuring a diffusion term (possibly quasilinear and degenerate if $p > 2$) and a source term $|\nabla u|^q$ counteracting the effect of diffusion and depending solely on the gradient of the solution. The competition between the diffusion and the source term is already revealed by the structure of steady states to (3.1.1). Indeed, while it follows from Theorem 1 in [A3.4] that zero is the only steady state in $\mathcal{C}(\bar{B})$ when $p \geq 2$ and $q \geq p-1$, several steady states may exist when $p \geq 2$ and $q \in (0, p-1)$ [A3.6, A3.14, A3.20]. Another typical feature of the competition between diffusion and source is the possibility of finite time blow-up in a suitable norm, and this phenomenon has been shown to occur for (3.1.1) when $p = 2$ and $q > 2$, see [A3.16] and the references therein. More precisely, it is established in [A3.18] that, when $p = 2$ and $q > 2$, there are classical solutions to (3.1.1) for which the L^∞-norm of the gradient blows up in finite time, the L^∞-norm of the solution remaining bounded. These solutions may actually be extended to all positive times in a unique way within the framework of viscosity solutions [A3.5, A3.21], the boundary condition being also satisfied in the viscosity sense. According to the latter, the homogeneous Dirichlet boundary condition might not always be fulfilled for all times, a property which is likely to be connected with the finite time blow-up of the gradient.
Coming back to the case where p and q fulfil (3.1.3) and several steady states may exist, a complete classification of steady states seems to be out of reach when B is replaced by an arbitrary open set of $\mathbb{R}^N$. Nevertheless, there are at least two situations in which the set of stationary solutions can be described, namely, when $N = 1$ and $B = (-1, 1)$ [A3.14, A3.20] and when $N \geq 2$ under the additional requirement that the steady states are radially symmetric and non-increasing, the latter being the first result of this paper. More precisely, we show that (3.1.1) has a one-parameter family of stationary solutions and that each stationary solution is characterized by the value of its maximum.

Theorem 3.1.1 *Assume* (3.1.3). *Let* $w \in W^{1,\infty}(B)$ *be a radially symmetric and non-increasing viscosity solution to* $-\Delta_p w - |\nabla w|^q = 0$ *in* B *satisfying* $w = 0$ *on* ∂B. *Then there is* $\vartheta \in [0,1]$ *such that* $w = w_\vartheta$, *where*

$$w_\vartheta(x) := c_0 \int_{\max\{|x|,\vartheta\}}^{1} \left(\rho - \vartheta^\beta \rho^{-(\beta-1)}\right)^{1/(p-1-q)} \mathrm{d}\rho, \qquad x \in \bar{B}, \tag{3.1.4}$$

for $\vartheta \in [0,1]$ *with*

$$\beta := 1 + \frac{(N-1)(p-1-q)}{p-1} > 1 \quad \textit{and} \quad c_0 := \left(\frac{p-1-q}{(p-1)\beta}\right)^{1/(p-1-q)} > 0. \tag{3.1.5}$$

In particular, we have $w_0(x) = (c_0/\alpha)\,(1-|x|^\alpha)$ *for* $x \in \bar{B}$, *where* $\alpha := (p-q)/(p-1-q) > 1$.

An interesting feature of the stationary solution w_ϑ to (3.1.1) for $\vartheta > 0$ is that it is flat on the ball $B_\vartheta(0) := \{x \in \mathbb{R}^N \ : \ |x| < \vartheta\}$, a property connected to the failure of the comparison principle for (3.1.1) for the range (3.1.3) of the parameters p and q .

Remark 3.1.2 *As already mentioned, for any* $M \in [0, c_0/\alpha]$ *there is one and only one* $\vartheta \in [0,1]$ *such that* $\|w_\vartheta\|_{L^\infty(B)} = M$ *as* $\|w_\vartheta\|_{L^\infty(B)}$ *is a decreasing function of* $\vartheta \in [0,1]$. *This property plays an important role in the forthcoming analysis of the large time behaviour of solutions to* (3.1.1).

Having a precise description of the set of steady states of (3.1.1) at our disposal, it is natural to investigate whether they attract the dynamics of (3.1.1) for large times. In other words, given a solution to (3.1.1), does it converge to a steady state as $t \to \infty$? A positive answer to this question is given in [A3.14, A3.20] when $N = 1$, $B = (-1,1)$, and p and q fulfil (3.1.3). The one-dimensional framework is fully exploited there as it allows the construction of a Liapunov functional by the technique developed in [A3.22]. Such a nice tool does not seem to be available here and we instead use the theory of viscosity solutions [A3.10] and more precisely the relaxed half-limits method introduced in [A3.7]. This approach has already been used in [A3.8, A3.15, A3.17] to investigate the large time behaviour of solutions to Hamilton-Jacobi equations and can be roughly summarized as follows: given a non-negative and radially symmetric solution u to (3.1.1) which is bounded in $W^{1,\infty}(B)$, the half-relaxed limits

$$u_*(x) := \liminf_{(s,\varepsilon)\to(t,0)} u(x,\varepsilon^{-1}s) \quad \text{and} \quad u^*(x) := \limsup_{(s,\varepsilon)\to(t,0)} u(x,\varepsilon^{-1}s), \quad x \in \bar{B},$$

are well-defined, do not depend on $t > 0$, and are Lipschitz continuous viscosity superso-lution and subsolution to

$$-\Delta_p z - |\nabla z|^q = 0 \quad \text{in} \quad B, \quad z = 0 \quad \text{on} \quad \partial B,$$

respectively, by Lemma 6.1 in [A3.10]. Clearly, $u_* \le u^*$ on $\bar{B}$ but we cannot apply the comparison principle at this stage to conclude that $u_* \ge u^*$ on $\bar{B}$. However, additional information are available in this particular case, namely that u_* and u^* are both non-negative, radially symmetric, non-increasing, and have the same maximal value. Extensive use of these properties allows us to prove that $u_* \ge u^*$, from which we readily conclude that $u_* = u^*$ is a Lipschitz continuous radially symmetric and non-increasing stationary solution to (3.1.1). Consequently, $u_* = u^* = w_\vartheta$ for some $\vartheta \in [0,1]$ by Theorem 3.1.1 and the assumption $u_0 \not\equiv 0$ prevents $\vartheta = 1$. The convergence result we obtain actually reads as follows.

Theorem 3.1.3 *Assume* (3.1.2) *and* (3.1.3) *and let u denote the (radially symmetric) viscosity solution to* (3.1.1). *Then there is a unique $\vartheta \in [0,1)$ such that*

$$\lim_{t\to\infty} \|u(t) - w_\vartheta\|_{\mathcal{C}(\bar{B})} = 0.$$

Notice that Theorem 3.1.3 applies in particular in the semilinear case $p = 2$ with $q \in (0,1)$ according to (3.1.3). In that case, an interesting phenomenon takes place in one space dimension [A3.14]: if $\vartheta \in (0,1)$ (a property which is certainly true when $\|u_0\|_{L^\infty(B)} < \|w_0\|_{L^\infty(B)}$ in view of Lemma 3.3.2) and the initial data is not too steep in a neighbourhood of zero, then the corresponding solution to (3.1.1) becomes instantaneously flat in a time-dependent neighbourhood of zero, its gradient thus undergoing finite time extinction near zero. Whether this phenomenon still occurs in higher space dimension and also for $p > 2$ is an interesting open question. It is however likely that it requires $q \in (0,1)$.
Still in the semilinear case $p = 2$, several results on the large time behaviour of solutions to (3.1.1) are also available when $q \ge 1$ and B is replaced by an arbitrary open set Ω of $\mathbb{R}^N$ [A3.1, A3.9, A3.19, A3.21], including the convergence to zero in $\mathcal{C}(\bar{\Omega})$ of global classical solutions.

The analysis in this paper being restricted to radially symmetric solutions, we define $r := |x|$ and switch between the notation $u = u(x,t)$ and $u = u(r,t)$, whenever this is convenient.
For further use, we introduce the following notations:

$$F(s,X) := -|s|^{p-2}\mathrm{trace}(X) - (p-2)|s|^{p-4}\langle Xs, s\rangle - |s|^q \quad \text{for } (s,X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}, \quad (3.1.6)$$

its radially symmetric counterpart

$$f(r,\mu,\zeta) := -(p-1)|\mu|^{p-2}\zeta - \frac{N-1}{r}|\mu|^{p-2}\mu - |\mu|^q \quad \text{for } (r,\mu,\zeta) \in (0,1)\times\mathbb{R}\times\mathbb{R}, \quad (3.1.7)$$

and the radially symmetric p-Laplacian operator

$$f_0(r,\mu,\zeta) := -(p-1)|\mu|^{p-2}\zeta - \frac{N-1}{r}|\mu|^{p-2}\mu \quad \text{for } (r,\mu,\zeta) \in (0,1)\times\mathbb{R}\times\mathbb{R}. \quad (3.1.8)$$

3.2 Radially symmetric and non-increasing stationary solutions

In this section, we prove Theorem 3.1.1, that is, if w is a radially symmetric, non-increasing, and Lipschitz continuous viscosity solution to the stationary equation

$$\begin{cases} -\Delta_p w - |\nabla w|^q = 0 & \text{in } B, \\ w = 0 & \text{on } \partial B, \end{cases} \tag{3.2.1}$$

then $w = w_\vartheta$ for some $\vartheta \in [0,1]$. To this end, we first observe that, as a function of $r = |x|$, w is a viscosity solution to $f(r, \partial_r w, \partial_r^2 w) = 0$ in $(0,1)$ with $w(1) = 0$ (recall that f is defined in (3.1.7)).

Next, as a preliminary step, let us first give a formal proof, assuming w to be in $\mathcal{C}^1(\bar{B})$ and solving (3.2.1) pointwise. In particular, we will derive an identity (see (3.2.3) below) which turns out to be valid for viscosity solutions as we shall see later on.

As w is radially symmetric and in $\mathcal{C}^1(\bar{B})$, we have $\partial_r w(0) = 0$. In addition, by (3.2.1),

$$\varphi(r) := r^{N-1}(|\partial_r w|^{p-2}\partial_r w)(r), \quad r \in [0,1],$$

fulfils $\varphi \in W^{1,\infty}((0,1))$ with $\partial_r \varphi(r) = -r^{N-1}|\partial_r w(r)|^q \le 0$ a.e. in $(0,1)$. Thus, φ is a non-increasing function in $[0,1]$. As, moreover, w is non-increasing with $w(1) = 0$, we have $\partial_r w(1) \le 0$.

Now, either $\partial_r w(1) = 0$ and thus $\varphi(1) = 0$. Since φ is non-increasing with $\varphi(0) = 0$, we conclude that $\varphi \equiv 0$. This implies $w = w_1 \equiv 0$.

Or $\partial_r w(1) < 0$, and the continuity and monotonicity of φ warrant that there is a unique $\vartheta \in [0,1)$ such that $\varphi = 0$ in $[0,\vartheta]$ and $\varphi < 0$ in $(\vartheta,1]$. Hence,

$$\partial_r \varphi(r) = -r^{[(N-1)(p-1-q)]/(p-1)}|\varphi(r)|^{q/(p-1)} = -r^{\beta-1}(-\varphi(r))^{q/(p-1)} \quad \text{in} \quad (\vartheta,1).$$

After integration we obtain

$$-\frac{p-1}{p-1-q}(-\varphi(r))^{(p-1-q)/(p-1)} + \frac{1}{\beta}r^\beta = \gamma \quad \text{for } r \in (\vartheta,1)$$

with some constant $\gamma \in \mathbb{R}$. Introducing

$$\chi(z) := \frac{p-1}{p-1-q}|z|^{p-2-q}z \qquad \text{for } z \in \mathbb{R}, \tag{3.2.2}$$

we end up with

$$r^{\beta-1}\chi(\partial_r w(r)) + \frac{1}{\beta}r^\beta = \gamma \quad \text{for } r \in (\vartheta,1) \tag{3.2.3}$$

as $\partial_r w < 0$ in $(\vartheta,1)$. Letting $r \searrow \vartheta$ implies $\gamma = \vartheta^\beta/\beta$ owing to $\partial_r w(\vartheta) = 0$ and $0 < q < p-1$.

Furthermore, due to $\partial_r w < 0$ in $(\vartheta,1)$, we have

$$-\frac{p-1}{p-1-q}\left(r^{(N-1)/(p-1)}(-\partial_r w(r))\right)^{p-1-q} = \frac{1}{\beta}\left(\vartheta^\beta - r^\beta\right) \quad \text{for } r \in (\vartheta,1).$$

Hence, we conclude

$$\partial_r w(r) = -\left(\frac{p-1-q}{(p-1)\beta}\left(r - \vartheta^\beta r^{-(\beta-1)}\right)\right)^{1/(p-1-q)} \qquad \text{for } r \in (\vartheta, 1).$$

Using $w(1) = 0$ and the definition of c_0, a further integration implies

$$w(r) = c_0 \int_r^1 \left(\rho - \vartheta^\beta \rho^{-(\beta-1)}\right)^{1/(p-1-q)} \mathrm{d}\rho = w_\vartheta(r) \qquad \text{for } r \in [\vartheta, 1].$$

Furthermore, we get $w(r) = w(\vartheta)$ for any $r \in [0, \vartheta]$ since $\partial_r w \equiv 0$ in $[0, \vartheta]$ and we conclude that $w = w_\vartheta$.

We now turn to the proof of Theorem 3.1.1 and first establish some preliminary results. We recall that, by the Rademacher theorem, a Lipschitz continuous function $v \in W^{1,\infty}((0,1))$ is differentiable a.e. and the measure of the differentiability set

$$D(v) := \{r_0 \in (0,1) \,:\, \partial_r v(r_0) \text{ exists }\}$$

is thus equal to one.

Lemma 3.2.1 *Let $v \in W^{1,\infty}((0,1))$ be a non-negative and non-increasing viscosity supersolution to*

$$f_0(r, \partial_r z, \partial_r^2 z) = 0 \quad \text{in } (0,1), \tag{3.2.4}$$

the Hamiltonian f_0 being defined in (3.1.8). *Then, if $r_1 \in D(v)$ and $r_2 \in D(v)$ are such that $r_1 < r_2$, we have*

$$r_2^{(N-1)/(p-1)} \partial_r v(r_2) \le r_1^{(N-1)/(p-1)} \partial_r v(r_1).$$

Proof. Take $0 < r_1 < r_2 < 1$ with $r_1, r_2 \in D(v)$ and assume for contradiction that

$$\xi_1 := r_1^{(N-1)/(p-1)} \partial_r v(r_1) < r_2^{(N-1)/(p-1)} \partial_r v(r_2) =: \xi_2.$$

As v is non-increasing we have $\xi_2 \le 0$. Now take $\xi_1 < \eta_1 < \eta_2 < \xi_2 \le 0$ and define Φ by

$$r^{(N-1)/(p-1)} \partial_r \Phi(r) = \eta_1 + (\eta_2 - \eta_1)\frac{r - r_1}{r_2 - r_1}, \qquad r \in [r_1, r_2],$$

along with $\Phi(r_1) = 0$.
On the one hand, $v - \Phi$ is continuous in $[r_1, r_2]$ and thus attains its minimum at a point $r_0 \in [r_1, r_2]$. On the other hand, we have

$$\partial_r (v - \Phi)(r_1) = \frac{\xi_1 - \eta_1}{r_1^{(N-1)/(p-1)}} < 0 \quad \text{and} \quad \partial_r (v - \Phi)(r_2) = \frac{\xi_2 - \eta_2}{r_2^{(N-1)/(p-1)}} > 0,$$

so that we cannot have $r_0 = r_1$ or $r_0 = r_2$. Thus, $r_0 \in (r_1, r_2)$ and, since v is a viscosity supersolution to (3.2.4), we have

$$-\frac{1}{r_0^{N-1}}\partial_r \left(r^{N-1}|\partial_r\Phi|^{p-2}\partial_r\Phi\right)(r_0) \geq 0.$$

Since $r^{(N-1)/(p-1)}\partial_r\Phi(r) \leq \eta_2 < 0$ for $r \in [r_1, r_2]$ we obtain

$$\begin{aligned} -\left(r^{N-1}|\partial_r\Phi|^{p-2}\partial_r\Phi\right)(r) &= r^{N-1}|\partial_r\Phi(r)|^{p-1} = \left(-r^{(N-1)/(p-1)}\partial_r\Phi(r)\right)^{p-1} \\ &= \left|\eta_1 + (\eta_2 - \eta_1)\frac{r - r_1}{r_2 - r_1}\right|^{p-1}. \end{aligned}$$

Differentiating and taking $r = r_0$, we end up with

$$\begin{aligned} 0 &\leq -\partial_r \left(r^{N-1}|\partial_r\Phi|^{p-2}\partial_r\Phi\right)(r_0) \\ &= (p-1)\left|\eta_1 + (\eta_2 - \eta_1)\frac{r_0 - r_1}{r_2 - r_1}\right|^{p-3}\left(\eta_1 + (\eta_2 - \eta_1)\frac{r_0 - r_1}{r_2 - r_1}\right)\frac{\eta_2 - \eta_1}{r_2 - r_1} < 0, \end{aligned}$$

and a contradiction. ∎

In order to show that a viscosity solution to (3.2.1) satisfies (3.2.3), we next prove that the left-hand side of (3.2.3) is non-increasing for a supersolution to (3.2.1).

Lemma 3.2.2 *Let $w \in W^{1,\infty}((0,1))$ be a non-increasing viscosity supersolution to $f(r, \partial_r z, \partial_r^2 z) = 0$ in $(0,1)$ such that $\|w\|_{L^\infty((0,1))} > 0$ and $w(1) = 0$, and define $r_0 \in [0,1]$ by*

$$r_0 := \inf\left\{r \in (0,1] \,:\, w(r) < \|w\|_{L^\infty((0,1))}\right\}.$$

If $r_1 \in D(w)$ and $r_2 \in D(w)$ are such that $r_0 < r_1 < r_2$, then

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} \geq r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta},$$

the parameter β and the function χ being defined in (3.1.5) *and* (3.2.2)*, respectively.*

Proof. The properties of w imply $r_0 \in [0,1)$. As w is non-increasing and Lipschitz continuous, the definition of r_0 yields that there is a sequence $(\varrho_n)_{n\geq 1}$ such that $\varrho_n \in D(w)$, $\partial_r w(\varrho_n) < 0$ and $\varrho_n \searrow r_0$ as $n \to \infty$. Pick $r_1 \in D(w) \cap (r_0, 1)$. For n large enough, we have $r_1 > \varrho_n$. Since w is clearly also a supersolution to (3.2.4), we infer from Lemma 3.2.1 that

$$r_1^{(N-1)/(p-1)}\partial_r w(r_1) \leq \varrho_n^{(N-1)/(p-1)}\partial_r w(\varrho_n) < 0$$

for n large enough. Consequently,

$$r_1^{(N-1)/(p-1)}\partial_r w(r_1) < 0 \quad \text{for } r_1 \in D(w) \cap (r_0, 1). \tag{3.2.5}$$

Assume now for contradiction that there are $r_1, r_2 \in (r_0, 1) \cap D(w)$ such that $r_1 < r_2$ and

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} < r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta}.$$

As $\partial_r w(r_1) < 0$ by (3.2.5), we have $\chi(\partial_r w(r_1)) < 0$ and we can choose two real numbers η_1 and η_2 such that

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} < \eta_1 < \eta_2 < r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta}\ , \qquad \eta_1 < \frac{r_1^\beta}{\beta},$$

and

$$a := 1 - \frac{\beta(\eta_2 - \eta_1)}{r_2^\beta - r_1^\beta} \in (0,1).$$

Indeed we first choose $\eta_1 \in (r_1^{\beta-1}\chi(\partial_r w(r_1)) + (r_1^\beta/\beta), r_1^\beta/\beta)$ and then $\eta_2 > \eta_1$ close enough to η_1 in order to have $a \in (0,1)$. Setting now

$$A := \eta_1 - (1-a)\frac{r_1^\beta}{\beta} = \eta_2 - (1-a)\frac{r_2^\beta}{\beta},$$

let Φ denote the solution to

$$r^{\beta-1}\chi(\partial_r \Phi(r)) + a\frac{r^\beta}{\beta} = A, \qquad r \in [r_1, r_2], \tag{3.2.6}$$

such that $\Phi(r_1) = 0$. Observe that the choice of a and A ensures that

$$r_i^{\beta-1}\chi(\partial_r \Phi(r_i)) + \frac{r_i^\beta}{\beta} = \eta_i \qquad \text{for } i = 1, 2. \tag{3.2.7}$$

Due to

$$A - a\frac{r_1^\beta}{\beta} = \eta_1 - \frac{r_1^\beta}{\beta} < 0$$

we conclude by (3.2.6) that

$$\chi(\partial_r \Phi(r)) = r^{-(\beta-1)}\left(A - a\frac{r^\beta}{\beta}\right) \le r^{-(\beta-1)}\left(A - a\frac{r_1^\beta}{\beta}\right) < 0 \qquad \text{for } r \in [r_1, r_2].$$

This implies that $\partial_r \Phi(r) < 0$ for $r \in [r_1, r_2]$, so that $\Phi \in \mathcal{C}^2([r_1, r_2])$ by (3.2.6). In addition,

$$(-\partial_r \Phi(r))^{p-1-q} = \frac{p-1-q}{p-1}\left(\frac{a}{\beta}r - Ar^{-(\beta-1)}\right), \quad r \in [r_1, r_2],$$

hence

$$\partial_r \Phi(r) = -\left[\frac{p-1-q}{p-1}\left(\frac{a}{\beta}r - Ar^{-(\beta-1)}\right)\right]^{1/(p-1-q)}, \qquad r \in [r_1, r_2].$$

Furthermore, due to (3.2.7) and the choice of η_1, we obtain

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} < \eta_1 = r_1^{\beta-1}\chi(\partial_r\Phi(r_1)) + \frac{r_1^\beta}{\beta}.$$

This implies $\chi(\partial_r w(r_1)) < \chi(\partial_r\Phi(r_1))$ and, since χ is increasing,

$$\partial_r w(r_1) < \partial_r\Phi(r_1).$$

Similarly, we conclude

$$\partial_r w(r_2) > \partial_r\Phi(r_2).$$

Now $w - \Phi$ is a continuous function in $[r_1, r_2]$ and thus attains its minimum at some $r_m \in [r_1, r_2]$. The above two inequalities prevent r_m to be equal to r_1 or r_2 and, since w is a viscosity supersolution to $f(r, \partial_r v, \partial_r^2 v) = 0$ in $(0, 1)$, we have

$$-\frac{1}{r_m^{N-1}}\partial_r\left(r^{N-1}|\partial_r\Phi(r)|^{p-2}\partial_r\Phi(r)\right)(r_m) - |\partial_r\Phi(r_m)|^q \geq 0.$$

But as $\partial_r\Phi < 0$, (3.2.6) implies

$$\begin{aligned}
&-\partial_r\left(r^{N-1}|\partial_r\Phi(r)|^{p-2}\partial_r\Phi(r)\right) = \partial_r\left(r^{N-1}|\partial_r\Phi(r)|^{p-1}\right)\\
= \;& \partial_r\left(\left|\frac{p-1-q}{p-1}r^{\beta-1}\chi(\partial_r\Phi(r))\right|^{(p-1)/(p-1-q)}\right)\\
= \;& -ar^{\beta-1}\left|\frac{p-1-q}{p-1}r^{\beta-1}\chi(\partial_r\Phi(r))\right|^{[(p-1)/(p-1-q)]-2}\left(\frac{p-1-q}{p-1}r^{\beta-1}\chi(\partial_r\Phi(r))\right)\\
= \;& ar^{(\beta-1)(p-1)/(p-1-q)}\left|\frac{p-1-q}{p-1}\chi(\partial_r\Phi(r))\right|^{[(p-1)/(p-1-q)]-1}\\
= \;& ar^{N-1}|\partial_r\Phi(r)|^q \qquad \text{for } r \in [r_1, r_2],
\end{aligned} \tag{3.2.8}$$

so that

$$-\frac{1}{r_m^{N-1}}\partial_r\left(r^{N-1}|\partial_r\Phi(r)|^{p-2}\partial_r\Phi(r)\right)(r_m) - |\partial_r\Phi(r_m)|^q = (a-1)|\partial_r\Phi(r_m)|^q < 0$$

since $a < 1$, and a contradiction. ■

In a similar way we now establish that the left-hand side of (3.2.3) is non-decreasing for viscosity subsolutions to (3.2.1).

Lemma 3.2.3 *Let $w \in W^{1,\infty}((0,1))$ be a non-increasing viscosity subsolution to $f(r, \partial_r z, \partial_r^2 z) = 0$ in $(0,1)$ such that $\|w\|_{L^\infty((0,1))} > 0$ and $w(1) = 0$, and define $r_0 \in [0,1]$ by*

$$r_0 := \inf\left\{r \in (0,1] \,:\, w(r) < \|w\|_{L^\infty((0,1))}\right\}.$$

If $r_1 \in D(w)$ and $r_2 \in D(w)$ are such that $r_0 < r_1 < r_2$, then

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} \leq r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta}.$$

Proof. The properties of w imply $r_0 \in [0,1)$. Assume for contradiction that there are $r_1, r_2 \in (r_0, 1) \cap D(w)$ such that $r_1 < r_2$ and

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} > r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta}.$$

We may then choose $\eta_1, \eta_2 \in \mathbb{R}$ such that

$$r_1^{\beta-1}\chi(\partial_r w(r_1)) + \frac{r_1^\beta}{\beta} > \eta_1 > \eta_2 > r_2^{\beta-1}\chi(\partial_r w(r_2)) + \frac{r_2^\beta}{\beta},$$

and define

$$a := 1 + \frac{\beta(\eta_1 - \eta_2)}{r_2^\beta - r_1^\beta} > 1 \qquad \text{and} \qquad A := \eta_1 + (a-1)\frac{r_1^\beta}{\beta} = \eta_2 + (a-1)\frac{r_2^\beta}{\beta}.$$

Let Φ denote the solution to

$$r^{\beta-1}\chi(\partial_r \Phi(r)) + a\frac{r^\beta}{\beta} = A, \qquad r \in [r_1, r_2], \tag{3.2.9}$$

such that $\Phi(r_1) = 0$. Thanks to the choice of a and A, we have

$$r_i^{\beta-1}\chi(\partial_r \Phi(r_i)) + \frac{r_i^\beta}{\beta} = \eta_i \qquad \text{for } i = 1, 2, \tag{3.2.10}$$

and the monotonicity of w implies that

$$A - a\frac{r_1^\beta}{\beta} = \eta_1 - \frac{r_1^\beta}{\beta} < r_1^{\beta-1}\chi(\partial_r w(r_1)) \le 0.$$

Consequently,

$$\chi(\partial_r \Phi(r)) = r^{-(\beta-1)}\left(A - a\frac{r^\beta}{\beta}\right) \le r^{-(\beta-1)}\left(A - a\frac{r_1^\beta}{\beta}\right) < 0 \qquad \text{for } r \in [r_1, r_2],$$

hence $\partial_r \Phi(r) < 0$ for $r \in [r_1, r_2]$. We then conclude from (3.2.9) that $\Phi \in \mathcal{C}^2([r_1, r_2])$. Furthermore, due to (3.2.10), the choice of η_1 and η_2, and the monotonicity of χ, we obtain

$$\partial_r w(r_1) > \partial_r \Phi(r_1) \qquad \text{and} \qquad \partial_r w(r_2) < \partial_r \Phi(r_2).$$

Now $w - \Phi$ is a continuous function in $[r_1, r_2]$ and thus attains its maximum at some point $r_m \in [r_1, r_2]$. The above two inequalities prevent r_m to be equal to r_1 or r_2 and, since w is a viscosity subsolution to $f(r, \partial_r v, \partial_r^2 v) = 0$ in $(0,1)$, we have

$$-\frac{1}{r_m^{N-1}}\partial_r\left(r^{N-1}|\partial_r \Phi(r)|^{p-2}\partial_r \Phi(r)\right)(r_m) - |\partial_r \Phi(r_m)|^q \le 0.$$

But, owing to $\partial_r\Phi(r) < 0$, (3.2.9) and $a > 1$, we conclude similarly to (3.2.8) that

$$-\frac{1}{r_m^{N-1}}\partial_r\left(r^{N-1}|\partial_r\Phi(r)|^{p-2}\partial_r\Phi(r)\right)(r_m) - |\partial_r\Phi(r_m)|^q = (a-1)|\partial_r\Phi(r_m)|^q > 0$$

and end up with a contradiction. ■

We are now in a position to prove Theorem 3.1.1. The keystone of the proof is that, according to Lemmas 3.2.2 and 3.2.3, any non-increasing viscosity solution to $f(r, \partial_r v, \partial_r^2 v) = 0$ in $(0,1)$ satisfying $w(1) = 0$ has to fulfil (3.2.3).

Proof of Theorem 3.1.1.
Let $w \in W^{1,\infty}((0,1))$ be a non-increasing viscosity solution to $f(r, \partial_r v, \partial_r^2 v) = 0$ in $(0,1)$ satisfying $w(1) = 0$. Either $w \equiv 0 = w_1$ or $M := \|w\|_{L^\infty((0,1))} > 0$ and we define $r_0 \in [0,1)$ by

$$r_0 := \inf\{r \in (0,1] \,:\, w(r) < M\}.$$

Now, owing to Lemmas 3.2.2 and 3.2.3, there is a constant $\gamma \in \mathbb{R}$ such that

$$r^{\beta-1}\chi(\partial_r w(r)) + \frac{r^\beta}{\beta} = \gamma \tag{3.2.11}$$

for any $r \in (r_0, 1) \cap D(w)$ and thus a.e. in $(r_0, 1)$. Combining the monotonicity of w and χ with (3.2.11), we, moreover, deduce that

$$\gamma \leq \frac{r_0^\beta}{\beta} \tag{3.2.12}$$

and

$$\partial_r w(r) = -\left[\frac{p-1-q}{p-1}\left(\frac{r}{\beta} - \gamma r^{-(\beta-1)}\right)\right]^{1/(p-1-q)} \qquad \text{for a.e. } r \in (r_0, 1).$$

Integrating and using the boundary condition $w(1) = 0$, we obtain

$$w(r) = \int_r^1 \left[\frac{p-1-q}{(p-1)\beta}\left(\rho - \gamma\beta\rho^{-(\beta-1)}\right)\right]^{1/(p-1-q)} \mathrm{d}\rho \qquad \text{for any } r \in [r_0, 1].$$

Recalling $w(r) \equiv M$ for $r \in [0, r_0]$ and the definition of c_0, we conclude

$$w(r) = c_0 \int_{\max\{r,r_0\}}^1 \left(\rho - \gamma\beta\rho^{-(\beta-1)}\right)^{1/(p-1-q)} \mathrm{d}\rho, \qquad r \in [0,1]. \tag{3.2.13}$$

It remains to show that $\gamma = r_0^\beta/\beta$ in order to obtain that $w = w_{r_0}$.
Consider first the case $r_0 = 0$. Since $\beta > 1$, the Lipschitz continuity of w yields $\gamma = 0 =$

r_0^β/β by letting $r \searrow 0$ in (3.2.11).
Next, if $r_0 \in (0,1)$, we assume for contradiction that $\gamma < r_0^\beta/\beta$. Then we fix $\vartheta \in [0, r_0)$ such that $\gamma < \vartheta^\beta/\beta$ and choose $\Lambda > 1$ such that

$$\Lambda^{p-1-q} < 1 + \vartheta^\beta - \gamma\beta.$$

This choice of Λ implies that the function

$$g(r) := \left(1 - \gamma\beta r^{-\beta}\right) - \Lambda^{p-1-q}\left(1 - \vartheta^\beta r^{-\beta}\right), \quad r \in (r_0, 1),$$

satisfies

$$g'(r) = \beta^2 r^{-\beta-1}\left(\gamma - \Lambda^{p-1-q}\frac{\vartheta^\beta}{\beta}\right) \le \beta^2 r^{-\beta-1}\left(\gamma - \frac{\vartheta^\beta}{\beta}\right) < 0, \quad r \in (r_0, 1),$$

and thus

$$g(r) \ge g(1) \ge 1 - \gamma\beta - \Lambda^{p-1-q} + \vartheta^\beta > 0, \quad r \in [r_0, 1].$$

Consequently,

$$\left(1 - \gamma\beta r^{-\beta}\right) > \Lambda^{p-1-q}\left(1 - \vartheta^\beta r^{-\beta}\right), \quad r \in [r_0, 1],$$

and it follows from (3.2.13) that

$$\begin{aligned} \partial_r w(r) &= -c_0 r^{1/(p-1-q)}\left(1 - \gamma\beta r^{-\beta}\right)^{1/(p-1-q)} \\ &< -c_0 r^{1/(p-1-q)}\Lambda\left(1 - \vartheta^\beta r^{-\beta}\right)^{1/(p-1-q)} = \Lambda\partial_r w_\vartheta(r), \quad r \in (r_0, 1). \end{aligned}$$

In particular, $w(r) - \Lambda w_\vartheta(r) \le w(r_0) - \Lambda w_\vartheta(r_0)$ for $r \in [r_0, 1]$. Furthermore,

$$w(r) - \Lambda w_\vartheta(r) = w(r_0) - \Lambda w_\vartheta(r) \le w(r_0) - \Lambda w_\vartheta(r_0), \quad r \in [0, r_0],$$

thanks to the monotonicity of w_ϑ, and the function $w - \Lambda w_\vartheta$ has a global maximum at r_0. Since $w_\vartheta \in \mathcal{C}^2((\vartheta, 1))$, $\vartheta < r_0$, and w is a viscosity subsolution to $f(r, \partial_r v, \partial_r^2 v) = 0$ in $(0,1)$, we conclude that

$$f(r_0, \partial_r(\Lambda w_\vartheta)(r_0), \partial_r^2(\Lambda w_\vartheta)(r_0)) \le 0.$$

However, as $\Lambda > 1$ and $\vartheta < r_0$, we clearly have

$$f(r_0, \partial_r(\Lambda w_\vartheta)(r_0), \partial_r^2(\Lambda w_\vartheta)(r_0)) = \left(\Lambda^{p-1} - \Lambda^q\right)|\partial_r w_\vartheta(r_0)|^q > 0,$$

and the contradiction. Therefore, $\gamma = r_0^\beta/\beta$ and $w = w_{r_0}$, which completes the proof. ∎

3.3 Some properties of solutions to (3.1.1)

We now focus on time-dependent solutions to (3.1.1) and establish some qualitative properties of non-negative and radially symmetric viscosity solutions to (3.1.1) which are needed to analyse their large time behaviour.

Proposition 3.3.1 *Assume that u_0, p, and q fulfil* (3.1.2) *and* (3.1.3). *There is a unique non-negative viscosity solution $u \in \mathcal{C}(\bar{B} \times [0,\infty))$ to* (3.1.1) *such that $u(x,t) = 0$ for $x \in \partial B$ and $x \longmapsto u(x,t)$ is radially symmetric and belongs to $W^{1,\infty}(B)$ for all $t \ge 0$. In addition, there is a constant $A_0 > 0$ depending only on p, q, and u_0 such that*

$$0 \le u(x,t) \le A_0 \quad \textit{and} \quad -A_0 \le \nabla u(x,t) \cdot \frac{x}{|x|} \le W(t)\,, \quad (x,t) \in \bar{B} \times [0,\infty), \tag{3.3.1}$$

with

$$\begin{aligned} W(t) &:= \left(\left(2\|\nabla u_0\|_{L^\infty(B)} \right)^{2-p} + (p-2)(N-1)t \right)^{-1/(p-2)} \quad \textit{if} \quad p > 2\,, \\ W(t) &:= 2\|\nabla u_0\|_{L^\infty(B)}\ e^{-(N-1)t} \quad \textit{if} \quad p = 2\,. \end{aligned}$$

Since $W(t) \longrightarrow 0$ as $t \to \infty$, the upper bound on ∇u in (3.3.1) ensures that, as a function of r, $u(t)$ becomes more and more monotone as time increases. Its proof relies on the comparison principle applied to the equation satisfied by $\partial_r u$ and thus explicitly makes use of the assumed radial symmetry of the solution. It would be interesting to figure out whether a similar property is enjoyed by solutions to (3.1.1) starting from arbitrary initial data.

Proof of Proposition 3.3.1. We first derive the expected properties on suitable approximations to (3.1.1) which we introduce now. For $\varepsilon \in (0,1)$, let $a_\varepsilon \in \mathcal{C}^\infty([0,\infty))$ and $b_\varepsilon \in \mathcal{C}^\infty([0,\infty))$ be two functions such that:

- a_ε is bounded and increasing and $a_\varepsilon(\xi) := (\varepsilon^2 + \xi)^{(p-2)/2}$ for $\xi \in [0, \varepsilon^{-1}]$,
- b_ε is increasing, Lipschitz continuous, and $b_\varepsilon(\xi) := (\varepsilon^2 + \xi)^{q/2} - \varepsilon^q$ for $\xi \in [0, \varepsilon^{-1}]$.

In addition, owing to the properties (3.1.2) of u_0, there exists a sequence $(u_{0\varepsilon})_{\varepsilon \in (0,1)}$ of non-negative and radially symmetric functions in $\mathcal{C}^\infty(\bar{B})$ such that

$$\|u_{0\varepsilon}\|_{L^\infty(B)} \le \|u_0\|_{L^\infty(B)} + \varepsilon\ , \quad \|\nabla u_{0\varepsilon}\|_{L^\infty(B)} \le 2\ \|\nabla u_0\|_{L^\infty(B)}, \tag{3.3.2}$$

and

$$\lim_{\varepsilon \to 0} \|u_{0\varepsilon} - u_0\|_{\mathcal{C}(\bar{B})} = 0.$$

Fix $\varepsilon \in (0,1)$. According to the properties of a_ε, b_ε and $u_{0\varepsilon}$, it follows from [A3.13] that the initial-boundary value problem

$$\left\{ \begin{array}{lcllll} \partial_t u_\varepsilon & = & \mathrm{div}(a_\varepsilon(|\nabla u_\varepsilon|^2)\nabla u_\varepsilon) + b_\varepsilon(|\nabla u_\varepsilon|^2), & x \in B, & t \in (0,\infty), \\ u_\varepsilon & = & 0, & x \in \partial B, & t \in (0,\infty), \\ u_\varepsilon|_{t=0} & = & u_{0\varepsilon}, & x \in B, & \end{array} \right. \tag{3.3.3}$$

has a unique non-negative classical solution u_ε. In addition, $x \longmapsto u_\varepsilon(t,x)$ is radially symmetric for every $t \geq 0$ and the comparison principle entails that

$$0 \leq u_\varepsilon(x,t) \leq \|u_{0\varepsilon}\|_{L^\infty(B)} \leq \|u_0\|_{L^\infty(B)} + \varepsilon, \qquad (x,t) \in \bar{B} \times [0,\infty). \tag{3.3.4}$$

We next derive some estimates on the gradient of u_ε and begin with the normal trace $\partial_r u_\varepsilon(1,t)$. Let $\mathcal{L}_\varepsilon$ be the parabolic operator

$$\mathcal{L}_\varepsilon z := \partial_t z - \frac{1}{r^{N-1}} \partial_r \left(r^{N-1} a_\varepsilon \left(|\partial_r z|^2 \right) \partial_r z \right) - b_\varepsilon \left(|\partial_r z|^2 \right), \quad (r,t) \in (0,1) \times (0,\infty),$$

and fix

$$A_0 \in (\sqrt{3}\varepsilon, \varepsilon^{-1/2}) \quad \text{such that} \quad A_0 \geq 2^{1/(p-1-q)} + 2\left(1 + \|u_0\|_{L^\infty(B)} + \|\nabla u_0\|_{L^\infty(B)}\right). \tag{3.3.5}$$

Then, thanks to the properties of a_ε, b_ε and (3.3.5), the function ψ defined by $\psi(r) := A_0(1-r)$ for $r \in [0,1]$ satisfies

$$\begin{aligned}
\mathcal{L}_\varepsilon \psi(r) &= \frac{1}{r^{N-1}} \partial_r \left(r^{N-1} a_\varepsilon \left(A_0^2 \right) A_0 \right) - b_\varepsilon \left(A_0^2 \right) = \frac{N-1}{r}\, a_\varepsilon \left(A_0^2 \right) A_0 - b_\varepsilon \left(A_0^2 \right) \\
&\geq \left(\varepsilon^2 + A_0^2 \right)^{(p-2)/2} A_0 - \left(\varepsilon^2 + A_0^2 \right)^{q/2} + \varepsilon^q \\
&\geq \left(\varepsilon^2 + A_0^2 \right)^{(p-2)/2} \left(\sqrt{\varepsilon^2 + A_0^2} - \varepsilon \right) - \left(\varepsilon^2 + A_0^2 \right)^{q/2} \\
&\geq \left(\varepsilon^2 + A_0^2 \right)^{(p-1)/2} \left(1 - \frac{\varepsilon}{\sqrt{\varepsilon^2 + A_0^2}} \right) - \left(\varepsilon^2 + A_0^2 \right)^{q/2} \\
&\geq \frac{1}{2} \left(\varepsilon^2 + A_0^2 \right)^{(p-1)/2} - \left(\varepsilon^2 + A_0^2 \right)^{q/2} \geq 0, \qquad r \in (0,1].
\end{aligned}$$

Furthermore, (3.3.2), (3.3.4) and (3.3.5) entail that

$$u_\varepsilon \left(\frac{1}{2}, t \right) \leq 1 + \|u_0\|_{L^\infty(B)} \leq \frac{A_0}{2} = \psi \left(\frac{1}{2} \right), \quad t \geq 0,$$

and

$$u_{0\varepsilon}(r) = -\int_r^1 \partial_r u_{0\varepsilon}(\varrho)\, d\varrho \leq 2\|\nabla u_0\|_{L^\infty(B)}(1-r) \leq \psi(r), \quad r \in \left(\frac{1}{2}, 1 \right).$$

Since $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in $(1/2,1) \times (0,\infty)$, the comparison principle ensures that $u_\varepsilon(r,t) \leq A_0(1-r)$ for $(r,t) \in (1/2,1) \times (0,\infty)$. Since $u_\varepsilon(1,t) = 0$, this implies in particular that $0 \leq -\partial_r u_\varepsilon(1,t) \leq A_0$ for $t \geq 0$. Recalling that $u_\varepsilon(t)$ is radially symmetric and smooth, we thus have

$$-A_0 \leq \partial_r u_\varepsilon(1,t) \leq 0 = \partial_r u_\varepsilon(0,t), \quad t \geq 0. \tag{3.3.6}$$

We next estimate the gradient of u_ε in B. For that purpose, we introduce the parabolic operator

$$\mathcal{M}_\varepsilon z \;:=\; \partial_t z - \partial_r \left[\left(a_\varepsilon \left(z^2 \right) + 2a_\varepsilon' \left(z^2 \right) z^2 \right) \partial_r z \right]$$

$$- \left[\frac{N-1}{r} \left(a_\varepsilon \left(z^2 \right) + 2a_\varepsilon' \left(z^2 \right) z^2 \right) + 2b_\varepsilon' \left(z^2 \right) z^2 \right] \partial_r z + \frac{N-1}{r^2} a_\varepsilon \left(z^2 \right) z$$

for $(r,t) \in (0,1) \times (0,\infty)$ and readily deduce from (3.3.3) that

$$\mathcal{M}_\varepsilon \partial_r u_\varepsilon = 0 \quad \text{in } (0,1) \times (0,\infty). \tag{3.3.7}$$

Observe next that $\partial_r u_\varepsilon(r,0) \geq -2\|\nabla u_0\|_{L^\infty(B)} \geq -A_0$ by (3.3.2) and (3.3.5) and

$$\mathcal{M}_\varepsilon(-A_0) = -\frac{N-1}{r^2} a_\varepsilon \left(A_0^2 \right) A_0 \leq 0 \,,$$

which, together with (3.3.6), (3.3.7) and the comparison principle implies that

$$-A_0 \leq \partial_r u_\varepsilon(r,t), \qquad (r,t) \in [0,1] \times [0,\infty). \tag{3.3.8}$$

Finally, let $W_\varepsilon \in \mathcal{C}^1([0,\infty))$ be the solution to the ordinary differential equation

$$\frac{dW_\varepsilon}{dt} + (N-1)\, a_\varepsilon \left(W_\varepsilon^2 \right) W_\varepsilon = 0, \qquad W_\varepsilon(0) = 2\|\nabla u_0\|_{L^\infty(B)}. \tag{3.3.9}$$

Then W_ε is positive and decreasing, $W_\varepsilon(0) \geq \partial_r u_\varepsilon(r,0)$ for $r \in (0,1)$ by (3.3.2), and $\mathcal{M}_\varepsilon W_\varepsilon \geq 0$ in $(0,1) \times (0,\infty)$ by (3.3.9). Recalling (3.3.7), we deduce from the comparison principle that

$$\partial_r u_\varepsilon(r,t) \leq W_\varepsilon(t), \qquad (r,t) \in [0,1] \times [0,\infty). \tag{3.3.10}$$

Finally, we argue as in Lemma 5 of [A3.12] to deduce from (3.3.3), (3.3.4), (3.3.8) and (3.3.10) that there is a constant C depending on $\|\nabla u_0\|_{L^\infty(B)}$, p, q and N, such that

$$|u_\varepsilon(x,t_1) - u_\varepsilon(x,t_2)| \leq C(|t_1 - t_2| + |t_1 - t_2|^{1/2}) \tag{3.3.11}$$

for any $x \in \bar{B}$, $t_1, t_2 \in [0,\infty)$ and $\varepsilon \in (0,1)$. Indeed, consider $t_1 \neq t_2$ and set $\tau := |t_1 - t_2|^{1/2} > 0$ and $L := \max\{A_0, 2\|\nabla u_0\|_{L^\infty(B)}\}$. Since (3.3.8), (3.3.10) and the Dirichlet boundary conditions imply that $|u_\varepsilon(x,t)| \leq L\, \mathrm{dist}(x,\partial B)$ for $(x,t) \in \bar{B} \times [0,\infty)$, we have

$$|u_\varepsilon(x_0,t_1) - u_\varepsilon(x_0,t_2)| \leq 2L\, \mathrm{dist}(x_0,\partial B) \leq 2L\tau \quad \text{if} \quad \mathrm{dist}(x_0,\partial B) \leq \tau. \tag{3.3.12}$$

If $\mathrm{dist}(x_0,\partial B) > \tau$ and $\varepsilon \in (0,1/L)$, we infer from (3.3.3), the properties of $(a_\varepsilon, b_\varepsilon)$, and $|\nabla u_\varepsilon| \leq L$ in $B \times [0,\infty)$ that

$$\begin{aligned} &|u_\varepsilon(x_0,t_1) - u_\varepsilon(x_0,t_2)| \\ &\quad = \frac{1}{|B|\tau^N} \left| \int_{\{|x-x_0|<\tau\}} (u_\varepsilon(x_0,t_1) - u_\varepsilon(x_0,t_2)) \mathrm{d}x \right| \\ &\quad = \frac{1}{|B|\tau^N} \left| \int_{\{|x-x_0|<\tau\}} (u_\varepsilon(x,t) - u_\varepsilon(x_0,t)) \mathrm{d}x \Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \int_{\{|x-x_0|<\tau\}} \partial_t u_\varepsilon(x,t) \mathrm{d}x\mathrm{d}t \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2L}{|B|\tau^N} \int_{\{|x-x_0|<\tau\}} |x-x_0| \mathrm{d}x \\
&\quad + \frac{1}{|B|\tau^N} \left| \int_{t_1}^{t_2} \int_{\{|x-x_0|<\tau\}} \left[\mathrm{div}(a_\varepsilon(|\nabla u_\varepsilon|^2)\nabla u_\varepsilon) + b_\varepsilon(|\nabla u_\varepsilon|^2) \right](x,t) \mathrm{d}x\mathrm{d}t \right| \\
&\leq \frac{2LN}{N+1}\tau + \frac{1}{|B|\tau^N} \left| \int_{t_1}^{t_2} \int_{\{|x-x_0|<\tau\}} (\varepsilon^2 + |\nabla u_\varepsilon|^2)^{q/2}(x,t) \mathrm{d}x\mathrm{d}t \right| \\
&\quad + \frac{1}{|B|\tau^N} \left| \int_{t_1}^{t_2} \int_{\{|x-x_0|=\tau\}} \left[a_\varepsilon(|\nabla u_\varepsilon|^2)|\nabla u_\varepsilon| \right](y,t) \mathrm{d}S\mathrm{d}t \right| \\
&\leq \frac{2LN}{N+1}\tau + (1+L^2)^{q/2}|t_1-t_2| + \frac{N}{\tau}(1+L^2)^{(p-2)/2}L|t_1-t_2| \\
&\leq \frac{2LN}{N+1}\tau + (1+L^2)^{q/2}\,\tau^2 + N(1+L^2)^{p/2}\,\tau.
\end{aligned}
$$

Combining (3.3.12) and the above estimate gives the claim (3.3.11).

We can now pass to the limit as $\varepsilon \to 0$. Owing to (3.3.4), (3.3.8), (3.3.10) and (3.3.11), $(u_\varepsilon)_\varepsilon$ is bounded in, say, $C^{0,1/2}(B \times (0,\infty))$ because the uniform Lipschitz continuity in r implies a uniform $C^{0,1/2}$-bound in r; thus $(u_\varepsilon)_\varepsilon$ is relatively compact in $\mathcal{C}(\bar{B} \times [0,T])$ for all $T > 0$. It follows from the stability theorem in Section 6 of [A3.10] and the comparison principle for (3.1.1) (Theorem 2.1 in [A3.11]) that $(u_\varepsilon)_\varepsilon$ converges uniformly towards the unique viscosity solution u to (3.1.1) on compact subsets of $\bar{B} \times [0,\infty)$. The properties of u and the bounds listed in Proposition 3.3.1 then readily follow from this convergence, the properties of u_ε, (3.3.4), (3.3.8) and (3.3.10), the function W being the solution to the ordinary differential equation

$$\frac{dW}{dt} + (N-1)\,|W|^{p-2}W = 0, \qquad W(0) = 2\|\nabla u_0\|_{L^\infty(B)}.$$

In fact, $W(t) = \left(W(0)^{2-p} + (p-2)(N-1)t\right)^{-1/(p-2)}$ if $p > 2$ and $W(t) := W(0)e^{-(N-1)t}$ if $p = 2$ for $t \geq 0$. ■

By (3.3.1), the trajectory $\{u(t) : t \geq 0\}$ of the solution u to (3.1.1) is bounded in $L^\infty(B)$. More precise information are gathered in the next lemma.

Lemma 3.3.2 *Assume that u_0, p and q fulfil* (3.1.2) *and* (3.1.3). *Let u be the viscosity solution to* (3.1.1) *described in Proposition 3.3.1. Then $t \longmapsto \|u(t)\|_{L^\infty(B)}$ is a non-increasing function and*

$$M_\infty := \lim_{t\to\infty} \|u(t)\|_{L^\infty(B)} > 0. \tag{3.3.13}$$

Proof. Any positive constant being obviously a supersolution to (3.1.1), the time monotonicity of the $L^\infty(B)$-norm of u readily follows from the comparison principle. Next, since $u_0 \not\equiv 0$ by (3.1.2), there is $x_0 \in B$, $\varrho > 0$, and $m > 0$ such that

$$B_\varrho(x_0) := \{x \in \mathbb{R}^N \ : \ |x - x_0| < \varrho\} \subset B \quad \text{and} \quad u_0(x) \geq m \quad \text{for} \quad x \in B_\varrho(x_0).$$

Introducing $v_\lambda(x) := \lambda^{(p-q)/(p-1-q)} w_0(|x - x_0|/\lambda)$ for $x \in B_\lambda(x_0)$ and $\lambda \in (0,1)$ (the function w_0 being defined in Theorem 3.1.1), a simple computation shows that v_λ is a solution to $-\Delta_p v_\lambda - |\nabla v_\lambda|^q = 0$ in $B_\lambda(x_0)$ with $v_\lambda(x) = 0 \leq u(x,t)$ for $(x,t) \in \partial B_\lambda(x_0) \times (0,\infty)$. Furthermore, if $\lambda = \lambda_m := \min\{1 - |x_0|, (m\alpha/c_0)^{(p-1-q)/(p-q)}\}$, we have $v_{\lambda_m}(x) \leq m \leq u_0(x)$ for $x \in B_{\lambda_m}(x_0)$. The comparison principle (Theorem 2.1 in [A3.11]) then warrants that $u(x,t) \geq v_{\lambda_m}(x)$ for $(x,t) \in B_{\lambda_m}(x_0) \times (0,\infty)$. In particular, $\|u(t)\|_{L^\infty(B)} \geq \|v_{\lambda_m}\|_{L^\infty(B_{\lambda_m}(x_0))}$ for all $t \geq 0$, whence $M_\infty \geq \|v_{\lambda_m}\|_{L^\infty(B_{\lambda_m}(x_0))} > 0$. ■

3.4 Convergence to steady states

We introduce the half-relaxed limits

$$u_*(x) := \liminf_{(s,\varepsilon)\to(t,0)} u(x, \varepsilon^{-1}s), \quad x \in \bar{B},$$

and

$$u^*(x) := \limsup_{(s,\varepsilon)\to(t,0)} u(x, \varepsilon^{-1}s), \quad x \in \bar{B},$$

which are well defined and do not depend on $t > 0$. Moreover, we infer from the stability theorem (see Lemma 6.1 in [A3.10]) that

$$u^* \text{ is a viscosity subsolution to } F(\nabla z, D^2 z) = 0 \text{ in } B, \tag{3.4.1}$$

$$u_* \text{ is a viscosity supersolution to } F(\nabla z, D^2 z) = 0 \text{ in } B. \tag{3.4.2}$$

Next we state some useful properties of the half-relaxed limits.

Lemma 3.4.1 *The half-relaxed limits u_* and u^* enjoy the following properties:*

$$u_* \in W^{1,\infty}(B), \quad u^* \in W^{1,\infty}(B), \tag{3.4.3}$$

$$0 \leq u_*(x) \leq u^*(x), \quad x \in \bar{B}, \tag{3.4.4}$$

$$u_* \textit{ and } u^* \textit{ are radially symmetric and non-increasing,} \tag{3.4.5}$$

$$u_*(0) = u^*(0) = M_\infty := \lim_{t\to\infty} \|u(t)\|_{L^\infty(B)} > 0, \tag{3.4.6}$$

$$u_*(x) = u^*(x) = 0 \qquad \textit{for } x \in \partial B. \tag{3.4.7}$$

Proof. By (3.3.1) there is $L := \max\{A_0, W(0)\} > 0$ such that

$$u(x, \varepsilon^{-1}s) \leq u(y, \varepsilon^{-1}s) + L|x - y| \qquad \text{for all } (x, y, \varepsilon^{-1}s) \in \bar{B} \times \bar{B} \times [0,\infty)\,, \tag{3.4.8}$$

from which we deduce that u_* and u^* are Lipschitz continuous in B by taking the lim sup or lim inf in ε and s. This proves (3.4.3), while (3.4.4) comes directly from the definition of u_* and u^* and the facts that u is non-negative, radially symmetric for any $t \geq 0$ and vanishes identically on $\partial B \times (0, \infty)$. The proof of (3.4.7) uses, in addition, the uniform Lipschitz and $C^{0,1/2}$-bounds we have for u in space and time, respectively.
In order to prove (3.4.5), we use Proposition 3.3.1: there is a decreasing function W such that $W(t) \to 0$ as $t \to \infty$ and

$$u(x,t) \leq u(y,t) + W(t)(|x| - |y|) \quad \text{for} \quad (x,y) \in \bar{B} \times \bar{B} \quad \text{such that} \quad |x| \geq |y|. \quad (3.4.9)$$

Using this inequality with $t = \varepsilon^{-1}s$ and taking the lim sup or lim inf in ε and s lead to either $u_*(x) \leq u_*(y)$ or $u^*(x) \leq u^*(y)$ for any $(x,y) \in \bar{B} \times \bar{B}$ such that $|x| \geq |y|$ because $W(t) \to 0$ as $t \to \infty$, hence to (3.4.5).
It remains to show (3.4.6). To this end, we recall that M_∞ is well-defined and positive by (3.3.13) and first claim that

$$\lim_{t\to\infty} u(0,t) = M_\infty. \quad (3.4.10)$$

Indeed, (3.4.9) implies

$$u(x,t) \leq u(0,t) + W(t)|x| \leq u(0,t) + W(t) \leq \|u(t)\|_{L^\infty(B)} + W(t), \qquad x \in B,$$

whence

$$\|u(t)\|_{L^\infty(B)} \leq u(0,t) + W(t) \leq \|u(t)\|_{L^\infty(B)} + W(t),$$

and (3.4.10) due to $W(t) \to 0$ as $t \to \infty$.
Moreover, by the definition of the half-relaxed limits, we have $u_*(0) = u^*(0) = M_\infty$ and

$$\|u_*\|_{L^\infty(B)} \leq \|u^*\|_{L^\infty(B)} \leq M_\infty.$$

This completes the proof of (3.4.6). ■

Now, owing to the monotonicity and radial symmetry of u_* and u^*, there are $r_* \in [0,1]$ and $r^* \in [0,1]$ such that

$$u_*(x) = M_\infty \text{ if } |x| \leq r_* \text{ and } u_*(x) < M_\infty \text{ if } |x| \in (r_*, 1], \quad (3.4.11)$$

$$u^*(x) = M_\infty \text{ if } |x| \leq r^* \text{ and } u^*(x) < M_\infty \text{ if } |x| \in (r^*, 1]. \quad (3.4.12)$$

Due to (3.4.4), (3.4.6) and (3.4.7), we have

$$0 \leq r_* \leq r^* < 1. \quad (3.4.13)$$

Next, we show that Λu_* is a strict supersolution to the stationary equation in a subset of B for $\Lambda > 1$.

Lemma 3.4.2 *Fix $\Lambda > 1$ and $\delta \in (0, 1 - r_*)$. Then there are $r_\delta \in (r_*, r_* + \delta)$ and $\varepsilon_{\delta,\Lambda} > 0$ such that Λu_* is a viscosity supersolution to $f(r, \partial_r z, \partial_r^2 z) = \varepsilon_{\delta,\Lambda}$ in $(r_\delta, 1)$. In addition, $\varepsilon_{\delta,\Lambda} \to 0$ as $\Lambda \searrow 1$.*

Proof. Fix $\delta \in (0, 1-r_*)$. Then, due to (3.4.3), (3.4.5) and (3.4.11), there is $r_\delta \in (r_*, r_*+\delta)$ such that u_* is differentiable at r_δ and $\partial_r u_*(r_\delta) < 0$. Since u_* is a viscosity supersolution to $f(r, \partial_r z, \partial_r^2 z) = 0$ in $(0,1)$, it is also a viscosity supersolution to $f_0(r, \partial_r z, \partial_r^2 z) = 0$ in $(0,1)$ and it follows from Lemma 3.2.1 that

$$\partial_r u_*(r) \leq r^{(N-1)/(p-1)} \partial_r u_*(r) \leq r_\delta^{(N-1)/(p-1)} \partial_r u_*(r_\delta) =: -m_\delta < 0$$

for a.e. $r \in (r_\delta, 1)$. Integrating and using the continuity of u_* we conclude that

$$u_*(r) \leq u_*(r_1) - m_\delta (r - r_1) \tag{3.4.14}$$

for all $r_1 \in [r_\delta, 1]$ and $r \in [r_1, 1]$.

Consider $\Lambda > 1$, $\Phi \in \mathcal{C}^2((r_\delta, 1))$ and assume that $\Lambda u_* - \Phi$ has a local minimum at some $r_0 \in (r_\delta, 1)$. Then $u_* - (\Phi/\Lambda)$ has a local minimum at r_0 and (3.4.2) implies

$$-\frac{1}{r_0^{N-1}} \partial_r \left(r^{N-1} \left| \partial_r \left(\frac{\Phi}{\Lambda} \right) \right|^{p-2} \partial_r \left(\frac{\Phi}{\Lambda} \right) \right) (r_0) - \left| \partial_r \left(\frac{\Phi}{\Lambda} \right) (r_0) \right|^q \geq 0,$$

$$-\frac{1}{r_0^{N-1}} \partial_r \left(r^{N-1} \left| \partial_r \Phi \right|^{p-2} \partial_r \Phi \right) (r_0) - \Lambda^{p-1-q} \; |\partial_r \Phi(r_0)|^q \geq 0.$$

Thus, we have

$$-\frac{1}{r_0^{N-1}} \partial_r \left(r^{N-1} |\partial_r \Phi|^{p-2} \partial_r \Phi \right) (r_0) - |\partial_r \Phi(r_0)|^q \geq \left(\Lambda^{p-1-q} - 1 \right) |\partial_r \Phi(r_0)|^q. \tag{3.4.15}$$

Now, since $\Lambda u_* - \Phi$ has a local minimum at r_0, we infer from (3.4.14) that, for $r \in [r_0, r_0+\eta]$ with $\eta > 0$ small enough,

$$u_*(r_0) \leq \frac{\Phi(r_0)}{\Lambda} + u_*(r) - \frac{\Phi(r)}{\Lambda} \leq \frac{\Phi(r_0)}{\Lambda} + u_*(r_0) - m_\delta(r - r_0) - \frac{\Phi(r)}{\Lambda}.$$

Hence,

$$\frac{\Phi(r)}{\Lambda} - \frac{\Phi(r_0)}{\Lambda} \leq -m_\delta (r - r_0)$$

and thus

$$\frac{1}{\Lambda} \partial_r \Phi(r_0) \leq -m_\delta < 0$$

which implies $|\partial_r \Phi(r_0)| \geq \Lambda m_\delta$. Consequently, (3.4.15) becomes

$$f(r_0, \partial_r \Phi(r_0), \partial_r^2 \Phi(r_0)) \geq \left(\Lambda^{p-1-q} - 1 \right) \Lambda^q m_\delta^q =: \varepsilon_{\delta,\Lambda} > 0,$$

which ends the proof. ■

We are now able to prove that the half-relaxed limits u_* and u^* coincide.

Lemma 3.4.3 *We have $u_* = u^*$ on $\bar{B}$.*

Proof. We fix $\Lambda > 1 > \lambda > 0$ such that $\lambda > r_*$ and

$$\delta := \frac{M_\infty}{\|\nabla u_*\|_{L^\infty(B)}} \left(1 - \lambda^{(p-q)/(p-1-q)}\right) \in (0, \lambda - r_*).$$

Defining now

$$U(r) := \Lambda u_*(r), \quad r \in [0,1], \quad \text{and} \quad V(r) := \lambda^{(p-q)/(p-1-q)} u^* \left(\frac{r}{\lambda}\right), \quad r \in [0,\lambda],$$

we obtain due to (3.4.13)

$$U(r) \geq u_*(r) = M_\infty \geq V(r) \quad \text{for } r \in [0, r_*]. \tag{3.4.16}$$

Furthermore, we infer from the Lipschitz continuity of u_* that, for $r \in (r_*, r_* + \delta]$,

$$\begin{aligned} U(r) &\geq u_*(r) \geq u_*(r_*) - \|\nabla u_*\|_{L^\infty(B)} |r - r_*| \\ &= M_\infty - \|\nabla u_*\|_{L^\infty(B)} |r - r_*| \geq M_\infty - \delta \|\nabla u_*\|_{L^\infty(B)} \\ &\geq \lambda^{(p-q)/(p-1-q)} M_\infty \geq V(r). \end{aligned}$$

Recalling (3.4.16), we have thus shown that

$$U(r) \geq V(r) \quad \text{for} \quad r \in [0, r_* + \delta]. \tag{3.4.17}$$

Next, we define $I_\lambda := (r_* + \delta, \lambda)$. On the one hand, V is a viscosity subsolution to $f(r, \partial_r z, \partial_r^2 z) = 0$ in I_λ. Indeed, take $\Phi \in \mathcal{C}^2(I_\lambda)$ and assume that $V - \Phi$ has a local maximum at $r_1 \in I_\lambda$. Then $u^* - \Psi$ has a local maximum at r_1/λ, where $\Psi(r) := \lambda^{-(p-q)/(p-1-q)} \Phi(\lambda r)$ for $r \in ((r_* + \delta)/\lambda, 1)$. Owing to (3.4.1), we obtain

$$f\left(\frac{r_1}{\lambda}, \partial_r \Psi\left(\frac{r_1}{\lambda}\right), \partial_r^2 \Psi\left(\frac{r_1}{\lambda}\right)\right) \leq 0.$$

Consequently,

$$\begin{aligned} 0 &\geq \lambda^{q/(p-1-q)} f\left(\frac{r_1}{\lambda}, \lambda^{-1/(p-1-q)} \partial_r \Phi(r_1), \lambda^{1-1/(p-1-q)} \partial_r^2 \Phi(r_1)\right) \\ &= -(p-1) |\partial_r \Phi(r_1)|^{p-2} \partial_r^2 \Phi(r_1) - \frac{N-1}{r_1} |\partial_r \Phi(r_1)|^{p-2} \partial_r \Phi(r_1) - |\partial_r \Phi(r_1)|^q \\ &= f(r_1, \partial_r \Phi(r_1), \partial_r^2 \Phi(r_1)) \end{aligned}$$

and V is a viscosity subsolution to $f(r, \partial_r z, \partial_r^2 z) = 0$ in I_λ. On the other hand, it follows from Lemma 3.4.2 that U is a viscosity supersolution to $f(r, \partial_r z, \partial_r^2 z) = \varepsilon_{\delta,\Lambda}$ in I_λ with some $\varepsilon_{\delta,\Lambda} > 0$. As furthermore $V(r) = 0 \leq U(r)$ for $r = \lambda$ and $U(r) \geq V(r)$ for $r = r_* + \delta$ due to (3.4.17), we conclude that

$$U(r) \geq V(r) \qquad \text{for } r \in [r_* + \delta, \lambda]$$

by Section 5C in [A3.10]. Using (3.4.17), we end up with

$$\Lambda u_*(r) \geq \lambda^{(p-q)/(p-1-q)} u^* \left(\frac{r}{\lambda}\right) \qquad \text{for } r \in [0, \lambda].$$

Letting now $\Lambda \searrow 1$ and $\lambda \nearrow 1$, we conclude $u_* \geq u^*$ in $[0,1]$ which, together with (3.4.4), implies $u^* = u_*$. ■

Finally, we prove Theorem 3.1.3.

Proof of Theorem 3.1.3.
Defining $u_\infty := u_* = u^*$ by Lemma 3.4.3, (3.4.1), (3.4.2) and Lemma 3.4.1 imply that u_∞ is a radially symmetric, non-increasing, and Lipschitz continuous viscosity solution to $F(\nabla z, D^2 z) = 0$ in B satisfying $u_\infty = 0$ on ∂B. Moreover, $\|u_\infty\|_{L^\infty(B)} = M_\infty > 0$ due to (3.4.6). Hence, owing to Theorem 3.1.1, there is a unique $\vartheta \in [0,1)$ such that $u_\infty = w_\vartheta$. In particular, the equality $u_* = u^*$ and the definition of u_* and u^* provide the uniform convergence of $u(t)$ towards $u^* = w_\vartheta$ in every compact subset of B as $t \to \infty$, see Lemme 4.1 in [A3.3], or Lemma V.1.9 in [A3.2]. Combining this local convergence with (3.4.3) and (3.4.7) gives

$$\lim_{t\to\infty} \|u(t) - w_\vartheta\|_{\mathcal{C}(\bar{B})} = 0$$

and the claim is proved. ■

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Article 4:
Convergence to separate variables solutions for a degenerate parabolic equation with gradient source

by Philippe Laurençot[4] and Christian Stinner

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Abstract

The large time behaviour of nonnegative solutions to a quasilinear degenerate diffusion equation with a source term depending solely on the gradient is investigated. After a suitable rescaling of time, convergence to a unique profile is shown for global solutions. The proof relies on the half-relaxed limits technique within the theory of viscosity solutions and on the construction of suitable supersolutions and barrier functions to obtain optimal temporal decay rates and boundary estimates. Blowup of weak solutions is also studied.

Key words: convergence, diffusive Hamilton-Jacobi equation, friendly giant, viscosity solution, half-relaxed limits, blowup

MSC 2010: 35B40, 35K92, 35D40, 35B44

4.1 Introduction

Qualitative properties of nonnegative solutions to

$$\partial_t u - \Delta_p u = |\nabla u|^q , \qquad (t,x) \in Q := (0,\infty) \times \Omega , \tag{4.1.1}$$

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$$u = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, \tag{4.1.2}$$
$$u(0) = u_0, \qquad x \in \Omega, \tag{4.1.3}$$

vary greatly according to the relative strength of the (possibly nonlinear and degenerate) diffusion $\Delta_p u := \text{ div } \left(|\nabla u|^{p-2}\, \nabla u\right)$ and the source term $|\nabla u|^q$ which is measured by the exponents $p \geq 2$ and $q > 0$. More precisely, if $q \in (0, p-1)$, the comparison principle fails to be valid for the corresponding stationary equation [A4.3] and the existence of non-zero steady states is expected. The latter is known to be true for $p = 2$ and $q \in (0,1)$ for a general bounded domain Ω [A4.5, A4.17] and for $p > 2$ and $q \in (0, p-1)$ if $\Omega = B(0,1)$ is the open unit ball of $\mathbb{R}^N$ [A4.6, A4.24]. A complete classification of nonnegative steady states seems nevertheless to be lacking in general, except in space dimension $N = 1$ [A4.17, A4.24] and when $\Omega = B(0,1)$ for radially symmetric solutions [A4.6]. In these two particular cases, there is a one-parameter family $(w_\vartheta)_{\vartheta\in[0,1]}$ of stationary solutions to (4.1.1)-(4.1.2) with the properties $w_0 = 0$ and $w_\vartheta < w_{\vartheta'}$ in Ω if $\vartheta < \vartheta'$. In addition, each nonnegative solution to (4.1.1)-(4.1.3) converges as $t \to \infty$ to one of these steady states [A4.6, A4.17, A4.24] and the available classification of the steady states plays an important role in the convergence proof. The classification of nonnegative steady states to (4.1.1)-(4.1.2) and the large time behaviour of nonnegative solutions to (4.1.1)-(4.1.3) thus remain unsolved problems when $q \in (0, p-1)$ and Ω is an arbitrary bounded domain of $\mathbb{R}^N$, $N \geq 2$.

The situation is more clear for $q \geq p-1$ as the comparison principle [A4.3] guarantees that zero is the only stationary solution to (4.1.1)-(4.1.2). Convergence to zero of nonnegative solutions to (4.1.1)-(4.1.3) is then expected in that case but the dynamics turn out to be more complicated as the gradient source term $|\nabla u|^q$ induces finite time blowup for some solutions. More precisely, when $p = 2$, global existence and convergence to zero for large times of solutions to (4.1.1)-(4.1.3) are shown in [A4.8, A4.23, A4.25] when either $q \in [1,2]$ or $q > 2$ and $\|u_0\|_{C^1}$ is sufficiently small. The smallness condition on u_0 for $p = 2$ and $q > 2$ cannot be removed as finite time gradient blowup occurs for "large" initial data in that case [A4.22]. The blowup of the gradient then takes place on the boundary of Ω [A4.23] and additional information on the blowup rate and location of the blowup points are provided in [A4.14, A4.18]. In addition, the continuation of solutions after the blowup time is studied in [A4.4] within the theory of viscosity solutions. Coming back to the convergence to zero of global solutions, still for $p = 2$, the temporal decay rate and the limiting profile are identified in [A4.8] when $q \in (1,2]$ and shown to be that of the linear heat equation.

To our knowledge, the slow diffusion case $p > 2$ has not been studied and the main purpose of this paper is to investigate what happens when $q \geq p-1$ and $p > 2$. Our results may be summarized as follows: let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ (at least C^2) and consider an initial condition u_0 having the following properties:

$$u_0 \in C_0(\bar{\Omega}) := \{z \in C(\bar{\Omega}) \ : \ z = 0 \quad \text{on} \quad \partial\Omega\}, \quad u_0 \geq 0, \quad u_0 \not\equiv 0. \tag{4.1.4}$$

Then

(a) if $q = p-1$, there is a unique global (viscosity) solution u to (4.1.1)-(4.1.3) and $t^{1/(p-2)}u(t)$ converges as $t \to \infty$ to a unique profile f which does not depend on u_0.

In addition, $u_\infty : (t,x) \longmapsto t^{-1/(p-2)}\ f(x)$ is the unique solution to (4.1.1)-(4.1.2) with an initial condition identically infinite in Ω, see Theorem 4.1.2 below. The availability of solutions having infinite initial value in Ω (also called *friendly giants*) and their stability are well-known for the porous medium equation $\partial_t z - \Delta z^m = 0$, $m > 1$, the p-Laplacian equation $\partial_t z - \Delta_p z = 0$, $p > 2$, and some related equations sharing a similar variational structure, see [A4.19, A4.21, A4.26] for instance, but also for the semilinear diffusive Hamilton-Jacobi equation with gradient absorption $\partial_t z - \Delta z + |\nabla z|^q = 0$, $q > 1$ [A4.11].

(b) if $q \in (p-1, p]$, there is a unique global (viscosity) solution u to (4.1.1)-(4.1.3) and $t^{1/(p-2)}u(t)$ converges as $t \to \infty$ to a unique profile f_0 which does not depend on u_0. In that case, $(t,x) \longmapsto t^{-1/(p-2)}\ f_0(x)$ is the unique solution to the p-Laplacian equation $\partial_t z - \Delta_p z = 0$ with homogeneous Dirichlet boundary conditions and an initial condition identically infinite in Ω, see Theorem 4.1.4 below. Therefore, the gradient source term $|\nabla u|^q$ does not show up in the large time dynamics.

(c) if $q > p$ and u_0 is sufficiently small, there is a unique global (viscosity) solution u to (4.1.1)-(4.1.3) and $t^{1/(p-2)}u(t)$ converges as $t \to \infty$ to f_0 as in the previous case, see Theorem 4.1.4 below.

(d) if $q > p$ and u_0 is sufficiently large, then there is no global Lipschitz continuous weak solution to (4.1.1)-(4.1.3), see Proposition 4.5.3 below. Let us point out that, since the notion of solution used for this result differs from that used for the previous cases, it only provides an indication that the smallness condition is needed in case **(c)**.

Before stating precisely the main results, we point out that (4.1.1) is a quasilinear degenerate parabolic equation which is unlikely to have classical solutions. It turns out that a suitable framework for the well-posedness of the initial-boundary value problem (4.1.1)-(4.1.3) is the theory of viscosity solutions (see, e.g., [A4.1, A4.2, A4.10]) and we first define the notion of solutions to be used throughout this paper.

Definition 4.1.1 *Consider $u_0 \in C_0(\bar\Omega)$ satisfying (4.1.4). A function $u \in C([0,\infty) \times \bar\Omega)$ is a solution to (4.1.1)-(4.1.3) if u is a viscosity solution to (4.1.1) in Q and satisfies*

$$u(t,x) = 0\,, \quad (t,x) \in (0,\infty) \times \partial\Omega\,, \quad \textit{and} \quad u(0,x) = u_0(x)\,, \quad x \in \bar\Omega\,.$$

We begin with the case $p > 2$ and $q = p-1$.

Theorem 4.1.2 *Assume that $p > 2$, $q = p-1$, and consider $u_0 \in C_0(\bar\Omega)$ satisfying (4.1.4). Then, there is a unique solution u to (4.1.1)-(4.1.3) in the sense of Definition 4.1.1 and*

$$\lim_{t\to\infty} \left\| t^{1/(p-2)}\ u(t) - f \right\|_\infty = 0\,, \tag{4.1.5}$$

where $f \in C_0(\bar\Omega)$ is the unique positive solution to

$$-\Delta_p f - |\nabla f|^{p-1} - \frac{f}{p-2} = 0 \quad \textit{in} \quad \Omega, \quad f > 0 \quad \textit{in} \quad \Omega\,, \quad f = 0 \quad \textit{on} \quad \partial\Omega\,. \tag{4.1.6}$$

Furthermore, if $u_0 \in W^{1,\infty}(\Omega)$*, then* $\nabla u(t) \in L^\infty(\Omega)$ *for all* $t \geq 0$ *and*

$$\ell[u_0] := \sup_{t\geq 0} \{\|\nabla u(t)\|_\infty\} < \infty\,. \tag{4.1.7}$$

Let us emphasize here that Theorem 4.1.2 not only gives a description of the large time behaviour of u, but also provides the existence and uniqueness of the positive solution f to (4.1.6). To investigate the large time behaviour of u, no Liapunov functional seems to be available and we instead use the half-relaxed limits technique [A4.7, A4.10]. To this end, several steps are needed, including a comparison principle for (4.1.6) which is established in Section 4.2 and upper bounds which guarantee on the one hand that the solutions to (4.1.1)-(4.1.3) decay at the expected temporal decay rate and on the other hand that there is no loss of boundary conditions as discussed for instance in [A4.4]. The latter is achieved by the construction of suitable barrier functions. Also of importance is the positivity of the half-relaxed limits which allows us to apply the comparison lemma from Section 4.2.

Remark 4.1.3 *Consider* $G_0 \in W^{1,\infty}(\Omega)$ *satisfying* (4.1.4) *and denote the corresponding solution to* (4.1.1)-(4.1.3) *with* $q = p-1$ *by* G*. Owing to the homogeneity of the equation in that case, the function* g *defined by* $g(t,x) = G(t/\ell[G_0]^{p-2}, x)/\ell[G_0]$ *for* $(t,x) \in [0,\infty)\times\bar\Omega$ *is a solution to* (4.1.1)-(4.1.2) *with* $q = p-1$ *and initial condition* $G_0/\ell[G_0]$ *and satisfies* $\|\nabla g(t)\|_\infty \leq 1$ *for all* $t \geq 0$*. As we shall see below in Section 4.5,* g *turns out to be a supersolution of* (4.1.1)-(4.1.2) *for any* $q > p-1$ *and plays an important role in the construction of global solutions to* (4.1.1)-(4.1.2) *for* $q > p$*, see Theorem 4.1.4.*

We next turn to the case $q > p-1$ and establish the following result.

Theorem 4.1.4 *Assume that* $p > 2$*,* $q > p-1$*, and consider* $u_0 \in C_0(\bar\Omega)$ *satisfying* (4.1.4)*. If* $q > p$*, assume further that there is* $G_0 \in W^{1,\infty}(\Omega)$ *satisfying* (4.1.4) *such that*

$$u_0(x) \leq \frac{G_0(x)}{\ell[G_0]}\,, \qquad x \in \bar\Omega\,, \tag{4.1.8}$$

where $\ell[G_0]$ *is defined in* (4.1.7)*. Then, there is a unique solution* u *to* (4.1.1)-(4.1.3) *in the sense of Definition 4.1.1 and*

$$\lim_{t\to\infty} \left\| t^{1/(p-2)}\ u(t) - f_0 \right\|_\infty = 0\,, \tag{4.1.9}$$

where $f_0 \in C_0(\bar\Omega)$ *is the unique positive solution to*

$$-\Delta_p f_0 - \frac{f_0}{p-2} = 0 \quad in \quad \Omega, \quad f_0 > 0 \quad in \quad \Omega\,, \quad f_0 = 0 \quad on \quad \partial\Omega\,. \tag{4.1.10}$$

For $q \in [p-1,p]$, the well-posedness of (4.1.1)-(4.1.3) easily follows from [A4.12] as already noticed in [A4.4] for $p = 2$. For $q > p$ and an initial condition u_0 satisfying (4.1.8), it is a consequence of the Perron method and the comparison principle [A4.10]. As for the large time behaviour, the existence and uniqueness of f_0 is shown in [A4.19] and the main contribution of Theorem 4.1.4 is the convergence (4.1.9). The convergence proof follows

the same lines as that of Theorem 4.1.2 but a new difficulty has to be overcome in the case $q = p$ for the boundary estimates. We also show that, when $q \in (p-1, p]$, powers of the positive solution f to (4.1.6) with an exponent in $(0, 1]$ allow us to construct separate variables supersolutions to (4.1.1)-(4.1.2).
Finally, the announced blowup result is proved in Section 4.5.3 by a classical argument [A4.15, A4.20].

For further use, we introduce some notations: for $\xi \in \mathbb{R}^N$ and $X \in \mathcal{S}(N)$, $\mathcal{S}(N)$ being the space of $N \times N$ real-valued symmetric matrices, we define the functions F_0 and F by

$$\begin{aligned} F_0(\xi, X) &:= -|\xi|^{p-2}\ tr(X) - (p-2)\ |\xi|^{p-4}\ \langle X\xi, \xi\rangle\,, \qquad (4.1.11)\\ F(\xi, X) &:= F_0(\xi, X) - |\xi|^q\,. \qquad (4.1.12)\end{aligned}$$

4.2 A comparison lemma

An important tool for the uniqueness of the positive solution to (4.1.6) and the identification of the half-relaxed limits is the following comparison lemma between positive supersolutions and nonnegative subsolutions to the elliptic equation in (4.1.6).

Lemma 4.2.1 *Let $w \in USC(\bar\Omega)$ and $W \in LSC(\bar\Omega)$ be respectively a bounded upper semicontinuous (usc) viscosity subsolution and a bounded lower semicontinuous (lsc) viscosity supersolution to*

$$-\Delta_p \zeta - |\nabla \zeta|^{p-1} - \frac{\zeta}{p-2} = 0 \quad in \quad \Omega\,, \qquad (4.2.1)$$

such that

$$w(x) = W(x) = 0 \quad for \quad x \in \partial\Omega\,, \qquad (4.2.2)$$

$$W(x) > 0 \quad for \quad x \in \Omega\,. \qquad (4.2.3)$$

Then

$$w \le W \quad in \quad \bar\Omega\,. \qquad (4.2.4)$$

We remark that under the additional assumption $w > 0$ the result of Lemma 4.2.1 would follow from [A4.9, Theorem 2.1] which applies to a more general class of elliptic equations. However, we use different arguments to prove Lemma 4.2.1.
Proof. For $n \ge N_0$ large enough, $\Omega_n := \{x \in \Omega\ :\ d(x, \partial\Omega) > 1/n\}$ is a non-empty open subset of Ω. Since $\overline{\Omega_n}$ is compact and W is lower semicontinuous, the function W has a minimum in $\overline{\Omega_n}$ and the positivity (4.2.3) of W in $\overline{\Omega_n}$ implies that

$$\mu_n := \min_{\overline{\Omega_n}} \{W\} > 0\,. \qquad (4.2.5)$$

Similarly, the compactness of $\bar\Omega \setminus \Omega_n$ and the upper semicontinuity and boundedness of w ensure that w has at least one point of maximum x_n in $\bar\Omega \setminus \Omega_n$ and we set

$$\eta_n := w(x_n) = \max_{\bar\Omega \setminus \Omega_n} \{w\} \ge 0\,, \qquad (4.2.6)$$

the maximum being nonnegative since $\partial\Omega \subset \bar{\Omega} \setminus \Omega_n$ and w vanishes on $\partial\Omega$ by (4.2.2). We claim that

$$\lim_{n\to\infty} \eta_n = 0\,. \tag{4.2.7}$$

Indeed, owing to the compactness of $\bar{\Omega}$ and the definition of Ω_n, there are $y \in \partial\Omega$ and a subsequence of $(x_n)_{n\ge N_0}$ (not relabeled) such that $x_n \to y$ as $n \to \infty$. Since $w(y) = 0$, we deduce from the upper semicontinuity of w that

$$\lim_{\varepsilon\to 0} \sup \{w(x) \ : \ x \in B(y,\varepsilon) \cap \bar{\Omega}\} \le 0\,.$$

Given $\varepsilon > 0$ small enough, there is n_ε large enough such that $x_n \in B(y,\varepsilon) \cap \bar{\Omega}$ for $n \ge n_\varepsilon$ from which we deduce that

$$0 \le \eta_n = w(x_n) \le \sup\{w(x) \ : \ x \in B(y,\varepsilon) \cap \bar{\Omega}\}\,, \quad n \ge n_\varepsilon\,.$$

Therefore,

$$0 \le \limsup_{n\to\infty} \eta_n \le \sup\{w(x) \ : \ x \in B(y,\varepsilon) \cap \bar{\Omega}\}\,,$$

and letting $\varepsilon \to 0$ allows us to conclude that zero is a cluster point of $(\eta_n)_{n\ge N_0}$ as $n \to \infty$. The claim (4.2.7) then follows from the monotonicity of $(\eta_n)_{n\ge N_0}$.
Now, fix $s \in (0,\infty)$. For $\delta > 0$ and $n \ge N_0$, we define

$$\begin{aligned} z_n(t,x) &:= (t+s)^{-1/(p-2)}\, w(x) - s^{-1/(p-2)}\, \eta_n\,, \quad (t,x) \in [0,\infty) \times \bar{\Omega}\,, \\ Z_\delta(t,x) &:= (t+\delta)^{-1/(p-2)}\, W(x)\,, \quad (t,x) \in [0,\infty) \times \bar{\Omega}\,. \end{aligned}$$

It then follows from the assumptions on w and W that z_n and Z_δ are respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to

$$\partial_t \zeta - \Delta_p \zeta - |\nabla\zeta|^{p-1} = 0 \qquad \text{in} \quad (0,\infty) \times \Omega\,,$$

and satisfy

$$Z_\delta(t,x) = 0 \ge -s^{-1/(p-2)}\, \eta_n = z_n(t,x)\,, \quad (t,x) \in (0,\infty) \times \partial\Omega\,.$$

Moreover, if

$$0 < \delta < \left[\frac{\mu_n}{1 + \|w\|_\infty}\right]^{p-2} s\,, \tag{4.2.8}$$

it follows from (4.2.5) and (4.2.8) that, for $x \in \Omega_n$,

$$Z_\delta(0,x) = \delta^{-1/(p-2)}\, W(x) \ge \delta^{-1/(p-2)}\, \mu_n \ge s^{-1/(p-2)}\, \|w\|_\infty \ge z_n(0,x)\,,$$

and from (4.2.6) that, for $x \in \bar{\Omega} \setminus \Omega_n$,

$$Z_\delta(0,x) \ge 0 \ge s^{-1/(p-2)}\, (w(x) - \eta_n) = z_n(0,x)\,.$$

We are then in a position to apply the comparison principle [A4.10, Theorem 8.2] to conclude that

$$z_n(t,x) \le Z_\delta(t,x)\,, \qquad (t,x) \in [0,\infty) \times \bar{\Omega}\,, \tag{4.2.9}$$

for any $\delta > 0$ and $n \geq N_0$ satisfying (4.2.8). According to (4.2.8), the parameter δ can be taken to be arbitrarily small in (4.2.9) from which we deduce that

$$(t+s)^{-1/(p-2)}\, w(x) - s^{-1/(p-2)}\, \eta_n \leq t^{-1/(p-2)}\, W(x)\,, \qquad (t,x) \in (0,\infty)\times\bar{\Omega}\,,$$

for $n \geq N_0$. We next pass to the limit as $n \to \infty$ with the help of (4.2.7) to conclude that

$$(t+s)^{-1/(p-2)}\, w(x) \leq t^{-1/(p-2)}\, W(x)\,, \qquad (t,x) \in (0,\infty)\times\bar{\Omega}\,.$$

We finally let $s \to 0$ and take $t=1$ in the above inequality to obtain (4.2.4). ■

A straightforward consequence of Lemma 4.2.1 is the uniqueness of the friendly giant.

Corollary 4.2.2 *There is at most one positive viscosity solution to* (4.1.6)*.*

Arguing in a similar way, we have a similar result for the p-Laplacian:

Lemma 4.2.3 *Let $w \in USC(\bar{\Omega})$ and $W \in LSC(\bar{\Omega})$ be respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to*

$$-\Delta_p \zeta - \frac{\zeta}{p-2} = 0 \quad in \quad \Omega\,, \tag{4.2.10}$$

satisfying (4.2.2) *and* (4.2.3)*. Then $w \leq W$ in $\bar{\Omega}$.*

4.3 Well-posedness: $q \in [p-1,p]$

Proposition 4.3.1 *Assume that $q \in [p-1,p]$ and consider $u_0 \in C_0(\bar{\Omega})$ satisfying* (4.1.4)*. Then, there is a unique solution u to* (4.1.1)-(4.1.3) *in the sense of Definition 4.1.1.*

Proof. Since the comparison principle holds true for (4.1.1)-(4.1.3) by [A4.10, Theorem 8.2], Proposition 4.3.1 follows at once from [A4.12, Corollary 6.2] provided that $\Sigma_-^p = \Sigma_+^p = (0,\infty)\times\partial\Omega$, where the sets Σ_-^p and Σ_+^p are defined as follows: denoting the distance $d(x,\partial\Omega)$ from $x \in \bar{\Omega}$ to $\partial\Omega$ by $d(x)$, d is a smooth function in a neighbourhood of $\partial\Omega$ in $\bar{\Omega}$ and $(t,x) \in (0,\infty)\times\partial\Omega$ belongs to Σ_-^p if either

$$\liminf_{(y,\alpha)\to(x,0)} \left[F\left(\frac{\nabla d(y) + o_\alpha(1)}{\alpha}, -\frac{\nabla d(y)\otimes\nabla d(y) + o_\alpha(1)}{\alpha^2} \right) + \frac{o_\alpha(1)}{\alpha} \right] > 0\,, \tag{4.3.1}$$

or

$$\liminf_{(y,\alpha)\to(x,0)} \left[F\left(\frac{\nabla d(y) + o_\alpha(1)}{\alpha}, \frac{D^2 d(y) + o_\alpha(1)}{\alpha} \right) + \frac{o_\alpha(1)}{\alpha} \right] > 0\,. \tag{4.3.2}$$

Similarly, $(t,x) \in (0,\infty)\times\partial\Omega$ belongs to Σ_+^p if either

$$\limsup_{(y,\alpha)\to(x,0)} \left[F\left(-\frac{\nabla d(y) + o_\alpha(1)}{\alpha}, \frac{\nabla d(y)\otimes\nabla d(y) + o_\alpha(1)}{\alpha^2} \right) + \frac{o_\alpha(1)}{\alpha} \right] < 0\,, \tag{4.3.3}$$

or

$$\limsup_{(y,\alpha)\to(x,0)} \left[F\left(-\frac{\nabla d(y)+o_\alpha(1)}{\alpha}, -\frac{D^2 d(y)+o_\alpha(1)}{\alpha} \right) + \frac{o_\alpha(1)}{\alpha} \right] < 0\,. \tag{4.3.4}$$

Now, consider $t>0$ and $x\in\partial\Omega$. For $y\in\bar\Omega$ sufficiently close to x and $\alpha\in(0,1)$, we have

$$\begin{aligned} & \alpha^p \left[F\left(\frac{\nabla d(y)+o_\alpha(1)}{\alpha}, -\frac{\nabla d(y)\otimes\nabla d(y)+o_\alpha(1)}{\alpha^2} \right) + \frac{o_\alpha(1)}{\alpha} \right] \\ = \ & |\nabla d(y)+o_\alpha(1)|^{p-2}\ (|\nabla d(y)|^2+o_\alpha(1)) + (p-2)\ |\nabla d(y)+o_\alpha(1)|^{p-4}\ (|\nabla d(y)|^4 \\ & +o_\alpha(1)) - \alpha^{p-q}\ |\nabla d(y)+o_\alpha(1)|^q + \alpha^{p-1}\ o_\alpha(1) \\ = \ & (p-1)\ |\nabla d(y)|^p - \alpha^{p-q}\ |\nabla d(y)|^q + o_\alpha(1)\,. \end{aligned}$$

Since $|\nabla d(x)|=1$ and $p\ge q$, the right-hand side of the above inequality converges as $(y,\alpha)\to(x,0)$ either to $p-1$ if $q<p$ or to $p-2$ if $q=p$, both limits being positive since $p>2$. Therefore, the condition (4.3.1) is satisfied so that (t,x) belongs to Σ_-^p. Similarly, for $y\in\bar\Omega$ sufficiently close to x and $\alpha\in(0,1)$, we have

$$\begin{aligned} & \alpha^p \left[F\left(\frac{-\nabla d(y)+o_\alpha(1)}{\alpha}, \frac{\nabla d(y)\otimes\nabla d(y)+o_\alpha(1)}{\alpha^2} \right) + \frac{o_\alpha(1)}{\alpha} \right] \\ = \ & -|\nabla d(y)+o_\alpha(1)|^{p-2}\ (|\nabla d(y)|^2+o_\alpha(1)) - (p-2)\ |\nabla d(y)+o_\alpha(1)|^{p-4}\ (|\nabla d(y)|^4 \\ & +o_\alpha(1)) - \alpha^{p-q}\ |\nabla d(y)+o_\alpha(1)|^q + \alpha^{p-1}\ o_\alpha(1) \\ = \ & -(p-1)\ |\nabla d(y)|^p - \alpha^{p-q}\ |\nabla d(y)|^q + o_\alpha(1)\,, \end{aligned}$$

from which we readily infer that the condition (4.3.3) is satisfied. Therefore, (t,x) belongs to Σ_+^p and we have thus shown that $\Sigma_-^P=\Sigma_+^p=(0,\infty)\times\partial\Omega$ as expected. ■

4.4 Large time behaviour: $q\in[p-1,p]$

As already mentioned in the Introduction, the proofs of Theorems 4.1.2 and 4.1.4 involve several steps: we first show in the next section (Section 4.4.1) that the temporal decay rate of $\|u(t)\|_\infty$ is indeed $t^{-1/(p-2)}$. To this end we construct suitable supersolutions which differ according to whether $q=p-1$ or $q>p-1$. In a second step (Section 4.4.2), we prove boundary estimates for large times which guarantee that no loss of boundary conditions occurs throughout the time evolution. Here again, we need to perform different proofs for $q\in[p-1,p)$ and $q=p$. The half-relaxed limits technique is then employed in Section 4.4.3 to show the expected convergence after introducing self-similar variables, and the existence of a positive solution f to (4.1.6) as well. The last result of this section states that, if u_0 is bounded from above by $B\ f^\beta$ for some $B>0$ and $\beta\in(0,1]$, a similar bound holds true for $u(t)$ for positive times but with a possibly lower exponent β (Section 4.4.4).

4.4.1 Upper bounds

Lemma 4.4.1 *Assume that $q = p-1$. There is $C_1 > 0$ depending only on p, q, Ω, and $\|u_0\|_\infty$ such that*

$$u(t,x) \le C_1\ (1+t)^{-1/(p-2)}\,, \qquad (t,x) \in (0,\infty)\times\bar{\Omega}\,. \tag{4.4.1}$$

Proof. Consider $x_0 \notin \bar{\Omega}$ and $R_0 > 0$ such that $\Omega \subset B(x_0, R_0)$. For $A > 0$, $R > R_0$, $t \ge 0$, and $x \in \mathbb{R}^N$, we put $r = |x - x_0|$,

$$S(t,x) := A\ (1+t)^{-1/(p-2)}\ \sigma(r)\,, \qquad \sigma(r) := \frac{p-1}{p}\ \left(e^{pR/(p-1)} - e^{pr/(p-1)}\right)\,,$$

and assume that

$$A \ge \max\left\{ \left(\frac{e^{pR/(p-1)}}{(p-1)(p-2)}\right)^{1/(p-2)} , \frac{\|u_0\|_\infty}{\sigma(R_0)} \right\}. \tag{4.4.2}$$

Since x_0 does not belong to $\bar{\Omega}$, the function S is C^∞-smooth in $[0,\infty)\times\bar{\Omega}$ and it follows from (4.4.2) that, for $(t,x) \in Q$,

$$\begin{aligned}
& (1+t)^{(p-1)/(p-2)}\ \left\{\partial_t S(t,x) + F(\nabla S(t,x), D^2 S(t,x))\right\} \\
= \ & -\frac{A}{p-2}\ \sigma(r) - A^{p-1}\ |\sigma'(r)|^{p-1} \\
& -(p-1)\ A^{p-1}\ |\sigma'(r)|^{p-2}\ \sigma''(r) - (N-1)\ A^{p-1}\ \frac{|\sigma'(r)|^{p-2}\sigma'(r)}{r} \\
= \ & A\ \left[A^{p-2}\ \left(p-1+\frac{N-1}{r}\right)\ e^{pr} - \frac{\sigma(r)}{(p-2)}\right] \\
\ge \ & A\ \left((p-1)\ A^{p-2} - \frac{e^{pR/(p-1)}}{(p-2)}\right) \ge 0\,.
\end{aligned}$$

Therefore, the condition (4.4.2) guarantees that S is a supersolution to (4.1.1) in Q. In addition, since $|x - x_0| < R_0 < R$ for $x \in \Omega$, we have

$$u(t,x) = 0 \le A\ (t+1)^{-1/(p-2)}\ \sigma(R_0) \le S(t,x)\,, \qquad (t,x) \in (0,\infty)\times\partial\Omega\,,$$

and

$$u_0(x) \le \|u_0\|_\infty \le A\ \sigma(R_0) \le S(0,x)\,, \qquad x \in \bar{\Omega}\,,$$

by (4.4.2). The comparison principle then implies that $u(t,x) \le S(t,x)$ for $(t,x) \in [0,\infty)\times\bar{\Omega}$, and Lemma 4.4.1 follows from this inequality. ∎

Lemma 4.4.2 *Assume that $q > p-1$. There is $C_1 > 0$ depending only on p, q, Ω, and $\|u_0\|_\infty$ such that*

$$u(t,x) \le C_1\ (1+t)^{-1/(p-2)}\,, \qquad (t,x) \in (0,\infty)\times\bar{\Omega}\,. \tag{4.4.3}$$

Proof. Consider $x_0 \notin \bar\Omega$ and $R_0 > 0$ such that $\Omega \subset B(x_0, R_0)$. For $A > 0$, $\delta > 0$, $R > R_0$, $t \geq 0$, and $x \in \mathbb{R}^N$, we put $r = |x - x_0|$,

$$S(t,x) := A\ (1+\delta t)^{-1/(p-2)}\ \varphi(r)\,, \qquad \varphi(r) := \frac{p-1}{p}\ \left(R^{p/(p-1)} - r^{p/(p-1)} \right)\,,$$

and assume that

$$A = \left(\frac{N}{2R_0^{q/(p-1)}} \right)^{1/(q-p+1)}\,,\quad R = \left(R_0^{p/(p-1)} + \frac{p\|u_0\|_\infty}{(p-1)A} \right)^{(p-1)/p}\,,\quad \delta = \frac{N(p-2)A^{p-2}}{2R^{p/(p-1)}}\,.$$

Since x_0 does not belong to $\bar\Omega$, the function S is C^∞-smooth in $[0,\infty) \times \bar\Omega$ and it follows from the properties $\Omega \subset B(x_0,R_0)$ and $q > p-1$ that, for $(t,x) \in Q$,

$$\begin{aligned} & (1+\delta t)^{(p-1)/(p-2)}\ \left\{ \partial_t S(t,x) + F(\nabla S(t,x), D^2 S(t,x)) \right\} \\ = \ & -\frac{A\delta}{p-2}\ \varphi(r) + N\ A^{p-1} - A^q\ (1+\delta t)^{-(q-p+1)/(p-2)}\ r^{q/(p-1)} \\ \geq \ & A^{p-1}\ \left[N - A^{q-p+1}\ R_0^{q/(p-1)} - \frac{\delta R^{p/(p-1)}}{(p-2)A^{p-2}} \right] \\ \geq \ & 0\,. \end{aligned}$$

Therefore, the function S is a supersolution to (4.1.1) in Q and the choice of A and R also guarantees that

$$u_0(x) \leq \|u_0\|_\infty \leq A\ \varphi(R_0) \leq S(0,x)\,, \qquad x \in \bar\Omega\,.$$

Finally,

$$u(t,x) = 0 \leq A\ (1+\delta t)^{-1/(p-2)}\ \varphi(R_0) \leq S(t,x)\,, \qquad (t,x) \in (0,\infty) \times \partial\Omega\,,$$

since $|x - x_0| < R_0 < R$ for $x \in \Omega$ and we infer from the comparison principle that $u(t,x) \leq S(t,x)$ for $(t,x) \in [0,\infty) \times \bar\Omega$. Lemma 4.4.2 then follows from this inequality. ∎

4.4.2 Lipschitz estimates

Lemma 4.4.3 *Assume that $q \in [p-1, p)$. Then there is $L_1 > 0$ depending only on p, q, Ω, and $\|u_0\|_\infty$ such that*

$$|u(t,x)| = |u(t,x) - u(t,x_0)| \leq \frac{L_1}{(1+t)^{1/(p-2)}}\ |x - x_0|\,, \quad (t,x,x_0) \in [1,\infty) \times \bar\Omega \times \partial\Omega\,.$$

Proof. Since the boundary $\partial\Omega$ of Ω is smooth, it satisfies the uniform exterior sphere condition by [A4.13, Section 14.6], that is, there is $R_\Omega > 0$ such that, for each $x_0 \in \partial\Omega$, there is $y_0 \in \mathbb{R}^N$ satisfying $|x_0 - y_0| = R_\Omega$ and $B(y_0, R_\Omega) \cap \Omega = \emptyset$.

We fix positive real numbers A, M, and δ such that

$$A := \max\left\{ M, \frac{eC_1}{e-1}, \left(\frac{4e^{p-1}}{p-2}\right)^{1/(p-2)} \right\}, \quad M := \frac{2^{1/(p-2)}\|u_0\|_\infty}{2^{1/(p-2)}-1}\,, \tag{4.4.4}$$

and

$$0 < \delta < \min\left\{ 1, \frac{(p-2)R_\Omega}{N-1}, \left(\frac{1}{2A^{q-p+1}}\right)^{1/(p-q)} \right\}, \quad \Omega_\delta := \{x \in \Omega \ : \ d(x,\partial\Omega) > \delta\} \neq \emptyset\,, \tag{4.4.5}$$

the constant C_1 being defined in Lemma 4.4.1 if $q = p-1$ and Lemma 4.4.2 if $q \in (p-1,p)$. We next consider $t_0 \geq 1$, $x_0 \in \partial\Omega$, and let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R_\Omega$ and $B(y_0, R_\Omega) \cap \Omega = \emptyset$. We define the open subset U_{δ,x_0} of $\mathbb{R}^N$ by

$$U_{\delta,x_0} := \{x \in \Omega \ : R_\Omega < |x-y_0| < R_\Omega + \delta\}\,,$$

and the function

$$S_{\delta,x_0}(t,x) := \frac{A}{(1+t)^{1/(p-2)}} \ \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta}\right) + \frac{M}{(1+t)^{1/(p-2)}} - \frac{M}{(1+t_0)^{1/(p-2)}}$$

for $(t,x) \in [0,t_0] \times \overline{U_{\delta,x_0}}$. Since $y_0 \notin \overline{U_{\delta,x_0}}$, the function S_{δ,x_0} is C^∞-smooth in $[0,t_0] \times \overline{U_{\delta,x_0}}$. For $(t,x) \in (0,t_0) \times U_{\delta,x_0}$, we set $r := |x-y_0| - R_\Omega \in (0,\delta)$ and compute

$$\begin{aligned}
& \frac{(1+t)^{(p-1)/(p-2)}}{A^{p-1}} \ \left(\partial_t S_{\delta,x_0} - \Delta_p S_{\delta,x_0} - |\nabla S_{\delta,x_0}|^q\right)(t,x) \\
= \quad & -\frac{1-e^{-r/\delta}}{(p-2)A^{p-2}} - \frac{(N-1)e^{-(p-1)r/\delta}}{(r+R_\Omega)\delta^{p-1}} + \frac{(p-1)e^{-(p-1)r/\delta}}{\delta^p} \\
& -\frac{e^{-qr/\delta}}{\delta^q} \ \frac{A^{q-p+1}}{(1+t)^{(q-p+1)/(p-2)}} - \frac{M}{(p-2)A^{p-1}} \\
\geq \quad & \frac{e^{-(p-1)r/\delta}}{\delta^p} \ \left[p-1 - \frac{N-1}{r+R_\Omega}\ \delta - \frac{A^{q-p+1}\ \delta^{p-q}}{e^{(q-p+1)r/\delta}} - \frac{\delta^p e^{(p-1)r/\delta}}{(p-2)A^{p-2}} - \frac{M\delta^p e^{(p-1)r/\delta}}{(p-2)A^{p-1}} \right] \\
\geq \quad & \frac{e^{-(p-1)r/\delta}}{\delta^p} \ \left[p-1 - \frac{N-1}{R_\Omega}\ \delta - A^{q-p+1}\ \delta^{p-q} - \frac{e^{p-1}}{(p-2)A^{p-2}} - \frac{Me^{p-1}}{(p-2)A^{p-1}} \right] \\
\geq \quad & \frac{e^{-(p-1)r/\delta}}{\delta^p} \ \left[1 - A^{q-p+1}\ \delta^{p-q} - \frac{2e^{p-1}}{(p-2)A^{p-2}} \right] \geq 0\,,
\end{aligned}$$

the last two inequalities being a consequence of the choice (4.4.4) and (4.4.5) of δ, A, and M. Therefore, S_{δ,x_0} is a supersolution to (4.1.1) in $(0,\infty) \times U_{\delta,x_0}$. Moreover, since $t_0 \geq 1$, we have

$$S_{\delta,x_0}(0,x) \geq M - \frac{M}{2^{1/(p-2)}} = \|u_0\|_\infty \geq u_0(x)\,, \quad x \in \overline{U_{\delta,x_0}}\,, \tag{4.4.6}$$

by (4.4.4). It also follows from (4.4.1) and (4.4.3) that $u(t,x) \leq C_1\ (1+t)^{-1/(p-2)}$ for $t \geq 0$ and $x \in \bar{\Omega}$. Then, if $(t,x) \in (0,t_0) \times \partial U_{\delta,x_0}$, either $x \in \partial\Omega$ and $u(t,x) = 0 \leq S_{\delta,x_0}(t,x)$.

Or $r = |x - y_0| - R_\Omega = \delta$ and it follows from (4.4.4) that

$$S_{\delta,x_0}(t,x) \ge \frac{A(1-e^{-1})}{(1+t)^{1/(p-2)}} \ge \frac{C_1}{(1+t)^{1/(p-2)}} \ge u(t,x)\,.$$

We then deduce from the comparison principle [A4.10, Theorem 8.2] that $u(t,x) \le S_{\delta,x_0}(t,x)$ for $t \in [0,t_0]$ and $x \in \overline{U_{\delta,x_0}}$. In particular, for $t = t_0$,

$$u(t_0,x) \le \frac{A}{(1+t_0)^{1/(p-2)}} \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta}\right), \quad x \in \overline{U_{\delta,x_0}}\,.$$

Consequently,

$$0 \le u(t_0,x) - u(t_0,x_0) = u(t_0,x) \le \frac{A}{(1+t_0)^{1/(p-2)}} \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta}\right), \quad x \in \overline{U_{\delta,x_0}}\,,$$

whence, since $|x_0 - y_0| - R_\Omega = 0$,

$$0 \le u(t_0,x) - u(t_0,x_0) \le \frac{A}{\delta(1+t_0)^{1/(p-2)}} |x - x_0|\,, \quad x \in \overline{U_{\delta,x_0}}\,. \tag{4.4.7}$$

Consider finally $x \in \Omega$ and $x_0 \in \partial\Omega$. If $|x - x_0| \ge \delta/2$, it follows from (4.4.1) and (4.4.3) that

$$|u(t_0,x) - u(t_0,x_0)| = u(t_0,x) \le \frac{2C_1}{\delta(1+t_0)^{1/(p-2)}} |x - x_0|\,.$$

If $|x - x_0| < \delta/2$, let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R_\Omega$ and $B(y_0,R_\Omega) \cap \Omega = \emptyset$. Since $x \in \Omega$, $|x - y_0| > R_\Omega$ and

$$|x - y_0| \le |x - x_0| + |x_0 - y_0| < R_\Omega + \delta\,.$$

Consequently, $x \in U_{\delta,x_0}$ and we infer from (4.4.7) that

$$|u(t_0,x) - u(t_0,x_0)| \le \frac{A}{\delta(1+t_0)^{1/(p-2)}} |x - x_0|\,.$$

We have thus established Lemma 4.4.3 with $L_1 := \max\{2C_1, A\}/\delta$ for $(t,x,x_0) \in [1,\infty) \times \Omega \times \partial\Omega$. The extension to $[1,\infty) \times \bar{\Omega} \times \partial\Omega$ then readily follows thanks to the continuity of u up to the boundary of Ω. ■

The previous proof does not apply to the case $q = p$ as the term $A^{q-p+1}\,\delta^{p-q}$ cannot be made arbitrarily small by a suitable choice of δ. Still, a similar result is valid for $q = p$ but first requires a change of variable.

Lemma 4.4.4 *Assume that $q = p$. Then there is $L_1 > 0$ depending only on p, Ω, and $\|u_0\|_\infty$ such that*

$$|u(t,x)| = |u(t,x) - u(t,x_0)| \le \frac{L_1}{(1+t)^{1/(p-2)}} |x - x_0|\,, \quad (t,x,x_0) \in [1,\infty) \times \bar{\Omega} \times \partial\Omega\,.$$

Proof. We define $h := e^{u/(p-1)} - 1$ and notice that

$$\frac{u}{p-1} \le h \le \frac{e^{u/(p-1)}}{p-1}\ u\,. \tag{4.4.8}$$

By (4.1.1)-(4.1.3) and [A4.2, Corollaire 2.1] (or [A4.1, Proposition 2.5]), h is a viscosity solution to

$$\begin{aligned} \partial_t \left[\left(\frac{1+h}{p-1} \right)^{p-1} \right] - \Delta_p h &= 0 \qquad \text{in} \quad Q\,, & (4.4.9)\\ h &= 0 \qquad \text{on} \quad (0,\infty) \times \partial\Omega\,, & (4.4.10)\\ h(0) &= e^{u_0/(p-1)} - 1 \qquad \text{in} \quad \Omega\,. & (4.4.11) \end{aligned}$$

We fix positive real numbers A, M, and δ such that

$$A := \max\left\{ 1, M, \frac{eC_1}{(p-1)(e-1)}\ e^{C_1/(p-1)} \right\}, \quad M := \frac{2^{1/(p-2)} e^{\|u_0\|_\infty/(p-1)}}{2^{1/(p-2)} - 1}\,, \tag{4.4.12}$$

and

$$0 < \delta < \min\left\{ 1, \frac{(p-2)R_\Omega}{N-1}, \left(\frac{p-2}{2e^{p-1}} \right)^{1/p} \ \left(\frac{3}{p-1} \right)^{-(p-2)/p} \right\}, \quad \Omega_\delta \ne \emptyset\,, \tag{4.4.13}$$

the constant C_1 and the set Ω_δ being defined in Lemma 4.4.2 and (4.4.5), respectively. We next consider $t_0 \ge 1$, $x_0 \in \partial\Omega$, and let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R_\Omega$ and $B(y_0, R_\Omega) \cap \Omega = \emptyset$, the definition of R_Ω and the existence of y_0 being stated at the beginning of the proof of Lemma 4.4.3. We still define the open subset U_{δ,x_0} of $\mathbb{R}^N$ by

$$U_{\delta,x_0} := \{ x \in \Omega\ : R_\Omega < |x - y_0| < R_\Omega + \delta \}\,,$$

and the function

$$S_{\delta,x_0}(t,x) := \frac{A}{(1+t)^{1/(p-2)}}\ \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta} \right) + \frac{M}{(1+t)^{1/(p-2)}} - \frac{M}{(1+t_0)^{1/(p-2)}}$$

for $(t,x) \in [0,t_0] \times \overline{U_{\delta,x_0}}$. Since $y_0 \notin \overline{U_{\delta,x_0}}$, the function S_{δ,x_0} is C^∞-smooth in $[0,t_0] \times \overline{U_{\delta,x_0}}$. For $(t,x) \in (0,t_0) \times U_{\delta,x_0}$, we set $r := |x - y_0| - R_\Omega \in (0,\delta)$ and compute

$$\begin{aligned} & \frac{(1+t)^{(p-1)/(p-2)}}{A^{p-1}} \left(\partial_t \left[\left(\frac{1+S_{\delta,x_0}}{p-1} \right)^{p-1} \right] - \Delta_p S_{\delta,x_0} \right)(t,x) \\ = & -\frac{(1-e^{-r/\delta})}{(p-2)(p-1)^{p-2}}\ \frac{(1+S_{\delta,x_0})^{p-2}}{A^{p-2}} - \frac{M}{(p-2)(p-1)^{p-2}} \frac{(1+S_{\delta,x_0})^{p-2}}{A^{p-1}} \\ & + \frac{(p-1)e^{-(p-1)r/\delta}}{\delta^p} - \frac{(N-1)e^{-(p-1)r/\delta}}{(r+R_\Omega)\delta^{p-1}} \\ \ge & \frac{e^{-(p-1)r/\delta}}{\delta^p} \left[p - 1 - \frac{N-1}{R_\Omega}\ \delta - \frac{\delta^p\ e^{(p-1)r/\delta}}{(p-2)(p-1)^{p-2}}\ \left(\frac{1+2A}{A} \right)^{p-2} \right. \end{aligned}$$

$$
\left. - \frac{M\delta^p e^{(p-1)r/\delta}(1+2A)^{p-2}}{(p-2)(p-1)^{p-2}A^{p-1}} \right]
$$
$$
\geq \frac{e^{-(p-1)r/\delta}}{\delta^p} \left[1 - \frac{2\delta^p\ e^{p-1}}{(p-2)} \left(\frac{3}{p-1} \right)^{p-2} \right] \geq 0\,,
$$

the last two inequalities being a consequence of the choice (4.4.12) and (4.4.13) of δ, A, and M. Therefore, S_{δ,x_0} is a supersolution to (4.4.9) in $(0,\infty)\times U_{\delta,x_0}$. Moreover, since $t_0 \geq 1$, we have

$$
S_{\delta,x_0}(0,x) \geq M - \frac{M}{2^{1/(p-2)}} = e^{\|u_0\|_\infty/(p-1)} \geq h(0,x)\,, \quad x \in \overline{U_{\delta,x_0}}\,,
$$

by (4.4.12). It next follows from (4.4.3) and (4.4.8) that

$$
h(t,x) \leq \frac{e^{u(t,x)/(p-1)}}{p-1}\ u(t,x) \leq \frac{C_1 e^{C_1/(p-1)}}{p-1}\ (1+t)^{-1/(p-2)}\,, \quad (t,x) \in [0,\infty)\times\bar{\Omega}\,. \quad (4.4.14)
$$

Then, if $(t,x) \in (0,t_0)\times\partial U_{\delta,x_0}$, either $x \in \partial\Omega$ and $h(t,x) = 0 \leq S_{\delta,x_0}(t,x)$. Or $r = |x-y_0| - R_\Omega = \delta$ and it follows from (4.4.12) and (4.4.14) that

$$
S_{\delta,x_0}(t,x) \geq \frac{A(1-e^{-1})}{(1+t)^{1/(p-2)}} \geq \frac{C_1 e^{C_1/(p-1)}}{(p-1)(1+t)^{1/(p-2)}} \geq h(t,x)\,.
$$

We then deduce from the comparison principle [A4.10, Theorem 8.2] that $h(t,x) \leq S_{\delta,x_0}(t,x)$ for $t \in [0,t_0]$ and $x \in \overline{U_{\delta,x_0}}$. In particular, owing to (4.4.8), for $t = t_0$,

$$
\frac{u(t_0,x)}{p-1} \leq h(t_0,x) \leq \frac{A}{(1+t_0)^{1/(p-2)}}\ \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta} \right)\,, \quad x \in \overline{U_{\delta,x_0}}\,,
$$

and we argue as in the proof of Lemma 4.4.3 to complete the proof. ∎

We next proceed as in [A4.16] to deduce the Lipschitz continuity of $u(t)$ from Lemmas 4.4.3 and 4.4.4 if u_0 is Lipschitz continuous.

Corollary 4.4.5 *Assume that $q \in [p-1,p]$ and $u_0 \in W^{1,\infty}(\Omega)$ satisfies* (4.1.4). *Then there is $L_2 > 0$ depending only on p, q, Ω, and $\|u_0\|_{W^{1,\infty}(\Omega)}$ such that*

$$
|u(t,x) - u(t,y)| \leq L_2\ |x-y|\,, \quad (t,x,y) \in [0,\infty)\times\bar{\Omega}\times\bar{\Omega}\,.
$$

Proof. In a first step we show that Lemma 4.4.3 and Lemma 4.4.4 are also valid for small times $t \in [0,1]$. To this end we first note that the Lipschitz continuity of u_0 implies the existence of $L_0 > 0$ such that

$$
|u_0(x) - u_0(y)| \leq L_0|x-y|, \qquad (x,y) \in \bar{\Omega}\times\bar{\Omega}. \qquad (4.4.15)
$$

We now assume $q \in [p-1, p)$ and proceed similarly to the proof of Lemma 4.4.3. Keeping the notations of Lemma 4.4.3, we choose $M = 0$ and two positive real numbers A and δ such that

$$A := \max\left\{ eL_0, \frac{eC_1}{e-1}, \left(\frac{4e^{p-1}}{p-2}\right)^{1/(p-2)} \right\} \tag{4.4.16}$$

and (4.4.5) is satisfied. We next fix $t_0 > 0$ and $x_0 \in \partial\Omega$ and define as in Lemma 4.4.3

$$S_{\delta,x_0}(t,x) := \frac{A}{(1+t)^{1/(p-2)}} \left(1 - e^{-(|x-y_0|-R_\Omega)/\delta}\right), \qquad (t,x) \in [0,t_0] \times \overline{U_{\delta,x_0}}.$$

Owing to (4.4.5) and (4.4.16), S_{δ,x_0} is a supersolution to (4.1.1) in $(0,\infty) \times U_{\delta,x_0}$ and $S_{\delta,x_0}(t,x) \geq u(t,x)$ for $(t,x) \in (0,t_0) \times \partial U_{\delta,x_0}$, the computations being the same as in Lemma 4.4.3. Also, in view of (4.4.15), (4.4.16), and the mean value theorem we obtain

$$\begin{aligned} S_{\delta,x_0}(0,x) &= A\left(1 - e^{-(|x-y_0|-R_\Omega)/\delta}\right) \geq A\ e^{-1}\frac{|x-y_0|-R_\Omega}{\delta} = \frac{A}{e\delta}\ \mathrm{dist}(x, \partial B(y_0, R_\Omega)) \\ &\geq \frac{A}{e\delta}\ \mathrm{dist}(x,\partial\Omega) \geq L_0\ \mathrm{dist}(x,\partial\Omega) \geq u_0(x), \qquad x \in \overline{U_{\delta,x_0}}, \end{aligned} \tag{4.4.17}$$

where we have used that $B(y_0, R_\Omega) \cap \Omega = \emptyset$ and u_0 vanishes identically on $\partial\Omega$. The comparison principle then ensures that $S_{\delta,x_0} \geq u$ in $[0,t_0] \times \overline{U_{\delta,x_0}}$ and we next argue as in the end of the proof of Lemma 4.4.3 to conclude that

$$0 \leq u(t,x) = u(t,x) - u(t,x_0) \leq \frac{L_1}{(1+t)^{1/(p-2)}}\ |x-x_0|, \quad (t,x,x_0) \in [0,\infty) \times \bar{\Omega} \times \partial\Omega, \tag{4.4.18}$$

with $L_1 := \max\{2C_1, A\}/\delta$. Using similar changes in the proof of Lemma 4.4.4, we can establish (4.4.18) also in the case $q = p$.

Setting next $L_2 := \max\{L_0, L_1\}$ and borrowing an idea from the proof of [A4.16, Theorem 5], we fix $h \in \mathbb{R}^N$ such that $\Omega \cap \Omega_h \neq \emptyset$, where $\Omega_h := \{x \in \mathbb{R}^N\ :\ x - h \in \Omega\}$. Then the functions $u_1(t,x) := u(t,x-h) - L_2|h|$ and $u_2(t,x) := u(t,x-h) + L_2|h|$ defined for $(t,x) \in [0,\infty) \times \overline{\Omega_h}$ are viscosity solutions to (4.1.1) in $(0,\infty) \times \Omega_h$ with Dirichlet boundary conditions $u_2(t,x) = -u_1(t,x) = L_2|h|$ for $(t,x) \in (0,\infty) \times \partial\Omega_h$. In view of (4.4.15) and (4.4.18), we have $u_1 \leq u \leq u_2$ on the parabolic boundary of $(0,\infty) \times (\Omega \cap \Omega_h)$, so that the comparison principle [A4.10, Theorem 8.2] guarantees that $u_1 \leq u \leq u_2$ in $[0,\infty) \times \overline{\Omega \cap \Omega_h}$. This completes the proof of the claim. ■

4.4.3 Convergence

Let U be the solution to the p-Laplacian equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \partial_t U - \Delta_p U &= 0, \qquad (t,x) \in Q, & (4.4.19) \\ U &= 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, & (4.4.20) \end{aligned}$$

$$U(0) = u_0\,, \qquad x \in \Omega\,. \tag{4.4.21}$$

Owing to the nonnegativity of $|\nabla u|^q$, the comparison principle [A4.10, Theorem 8.2] ensures that

$$0 \le U(t,x) \le u(t,x)\,, \qquad (t,x) \in [0,\infty) \times \bar{\Omega}\,. \tag{4.4.22}$$

We introduce the scaling variable $s = \ln t$ for $t > 0$ and the new unknown functions v and V defined by

$$u(t,x) = t^{-1/(p-2)}\, v(\ln t, x)\,, \qquad (t,x) \in (0,\infty) \times \bar{\Omega}\,, \tag{4.4.23}$$

$$U(t,x) = t^{-1/(p-2)}\, V(\ln t, x)\,, \qquad (t,x) \in (0,\infty) \times \bar{\Omega}\,. \tag{4.4.24}$$

Then v is a viscosity solution to

$$\partial_s v - \Delta_p v - e^{-(q-p+1)s/(p-2)}\, |\nabla v|^q - \frac{v}{p-2} = 0\,, \qquad (s,x) \in Q\,, \tag{4.4.25}$$

$$v = 0\,, \qquad (s,x) \in (0,\infty) \times \partial\Omega\,, \tag{4.4.26}$$

$$v(0) = u(1)\,, \qquad x \in \Omega\,. \tag{4.4.27}$$

In addition, owing to (4.4.1) (if $q = p-1$), (4.4.3) (if $q > p-1$), Lemma 4.4.3 (if $q \in [p-1,p)$), Lemma 4.4.4 (if $q = p$), and (4.4.22), we have

$$V(s,x) \le v(s,x) \le C_1\,, \qquad (s,x) \in [0,\infty) \times \bar{\Omega}\,, \tag{4.4.28}$$

$$|v(s,x) - v(s,y)| \le L_1\, |x-y|\,, \qquad (s,x,y) \in [0,\infty) \times \bar{\Omega} \times \partial\Omega\,. \tag{4.4.29}$$

We next define for $\varepsilon \in (0,1)$

$$w_\varepsilon(s,x) := v\left(\frac{s}{\varepsilon}, x\right)\,, \qquad (s,x) \in [0,\infty) \times \bar{\Omega}\,,$$

and the half-relaxed limits

$$w_*(x) := \liminf_{(\sigma,y,\varepsilon)\to(s,x,0)} w_\varepsilon(\sigma,y)\,, \qquad w^*(x) := \limsup_{(\sigma,y,\varepsilon)\to(s,x,0)} w_\varepsilon(\sigma,y)\,,$$

for $(s,x) \in (0,\infty) \times \bar{\Omega}$. Observe that w_* and w^* are well-defined according to (4.4.28) and indeed do not depend on $s > 0$. In addition, it readily follows from (4.4.26) and (4.4.29) that

$$w_*(x) = w^*(x) = 0\,, \quad x \in \partial\Omega\,. \tag{4.4.30}$$

Also, w_ε is a solution to

$$\varepsilon\, \partial_s w_\varepsilon - \Delta_p w_\varepsilon - e^{-((q-p+1)s)/((p-2)\varepsilon)}\, |\nabla w_\varepsilon|^q - \frac{w_\varepsilon}{p-2} = 0 \qquad \text{in} \quad Q\,, \tag{4.4.31}$$

$$w_\varepsilon = 0 \qquad \text{on} \quad (0,\infty) \times \partial\Omega\,, \tag{4.4.32}$$

$$w_\varepsilon(0) = u(1) \qquad \text{in} \quad \Omega\,. \tag{4.4.33}$$

At this point, we distinguish the two cases $q = p-1$ and $q \in (p-1,p]$:

Case 1: $q = p-1$. We use the stability of semicontinuous viscosity solutions [A4.10, Lemma 6.1] to deduce from (4.4.31) that

$$w_* \text{ is a supersolution to (4.2.1) in } \Omega\,, \tag{4.4.34}$$

$$w^* \text{ is a subsolution to (4.2.1) in } \Omega\,. \tag{4.4.35}$$

In addition, as $V(s) \to f_0$ in $L^\infty(\Omega)$ as $s \to \infty$ by [A4.19, Theorem 1.3], it also follows from (4.4.28) and the definition of w_* and w^* that

$$f_0(x) \le w_*(x) \le w^*(x) \le C_1\,, \qquad x \in \bar{\Omega}\,. \tag{4.4.36}$$

Since $f_0 > 0$ in Ω by [A4.19, Theorem 1.1], we deduce from (4.4.36) that w_* and w^* are positive and bounded in Ω and vanish on $\partial\Omega$ by (4.4.30). Owing to (4.4.34) and (4.4.35), we are then in a position to apply Lemma 4.2.1 to conclude that $w^* \le w_*$ in $\bar{\Omega}$. Recalling (4.4.36), we have thus shown that $w_* = w^*$ in $\bar{\Omega}$. Setting $f := w_* = w^*$, we infer from (4.4.30), (4.4.34), (4.4.35), and (4.4.36) that $f \in C_0(\bar{\Omega})$ is a positive viscosity solution to (4.2.1) so that it solves (4.1.6). We have thus proved the existence of a positive solution to (4.1.6), its uniqueness being granted by Corollary 4.2.2. A further consequence of the equality $w_* = w^*$ is that $\|w_\varepsilon(1) - f\|_\infty \to 0$ as $\varepsilon \to 0$ (see, e.g., [A4.2, Lemme 4.1] or [A4.1, Lemma 5.1.9]). In other words,

$$\lim_{s\to\infty} \|v(s) - f\|_\infty = 0\,, \tag{4.4.37}$$

which implies (4.1.5) once written in terms of u. Finally, Corollary 4.4.5 gives the last statement of Theorem 4.1.2.

Case 2: $q \in (p-1, p]$. We use once more the stability of semicontinuous viscosity solutions [A4.10, Lemma 6.1] to deduce from (4.4.31) that

$$w_* \text{ is a supersolution to (4.2.10) in } \Omega\,, \tag{4.4.38}$$

$$w^* \text{ is a subsolution to (4.2.10) in } \Omega\,. \tag{4.4.39}$$

In addition, as $V(s) \to f_0$ in $L^\infty(\Omega)$ as $s \to \infty$ by [A4.19, Theorem 1.3], it also follows from (4.4.28) and the definition of w_* and w^* that

$$f_0(x) \le w_*(x) \le w^*(x) \le C_1\,, \qquad x \in \bar{\Omega}\,. \tag{4.4.40}$$

Since $f_0 > 0$ in Ω by [A4.19, Theorem 1.1] and a solution to (4.2.10), we apply Lemma 4.2.3 to conclude that $w^* \le f_0$ in $\bar{\Omega}$. Recalling (4.4.40), we have proved that $w_* = w^* = f_0$ in $\bar{\Omega}$. We then complete the proof of Theorem 4.1.4 for $q \in (p-1, p]$ in the same way as that of Theorem 4.1.2.

4.4.4 Improved upper bounds

Interestingly, the positive solution f to (4.1.6) can be also used to construct supersolutions to (4.1.1)-(4.1.2) for $q \in (p-1, p]$.

Proposition 4.4.6 *Assume that $q \in (p-1,p]$ and there are $\beta \in (0,1]$ and $B > 0$ such that*

$$u_0(x) \le B\ f(x)^\beta\,, \qquad x \in \bar\Omega\,. \tag{4.4.41}$$

Then there is $\gamma \in (0,\beta]$ such that

$$u(t,x) \le \frac{\|f\|_\infty^{1-\gamma}}{\gamma\left(\|f\|_\infty^{p-2} + \gamma t\right)^{1/(p-2)}}\ f(x)^\gamma \le \frac{f(x)^\gamma}{\gamma \|f\|_\infty^\gamma}\,, \qquad (t,x) \in [0,\infty) \times \bar\Omega\,. \tag{4.4.42}$$

Proof. We fix $\gamma \in (0,1)$ such that

$$\gamma \le \min\left\{\frac{p-2}{p-1}, \beta, \frac{1}{B\|f\|_\infty^\beta}\right\}, \tag{4.4.43}$$

and, for $(t,x) \in [0,\infty) \times \bar\Omega$, we define

$$\Sigma(t,x) = \frac{A f(x)^\gamma}{\gamma(1+\delta t)^{1/(p-2)}} \quad \text{with} \quad A := \frac{1}{\|f\|_\infty^\gamma} \quad \text{and} \quad \delta = \frac{\gamma}{\|f\|_\infty^{p-2}}\,.$$

We claim that

$$\Sigma \text{ is a supersolution to (4.1.1) in } Q \text{ for } q \in [p-1,p]\,. \tag{4.4.44}$$

Indeed, let $\phi \in C^2(Q)$ and consider $(t_0,x_0) \in Q$ where $\Sigma - \phi$ has a local minimum. Since Σ is smooth with respect to the time variable, this property implies that

$$\partial_t \phi(t_0,x_0) = -\frac{\delta A}{\gamma(p-2)}\ \frac{f(x_0)^\gamma}{(1+\delta t_0)^{(p-1)/(p-2)}}\,, \tag{4.4.45}$$

and that $x \mapsto \Sigma(t_0,x) - \phi(t_0,x)$ has a local minimum at x_0. In other words, the function $x \mapsto f(x)^\gamma - \gamma\,(1+\delta t_0)^{1/(p-2)}\ \phi(t_0,x)/A$ has a local minimum at x_0 and we infer from (4.1.6), the positivity of f in Ω, and [A4.2, Corollaire 2.1] (or [A4.1, Proposition 2.5]) that $g := f^\gamma$ is a viscosity solution to

$$-\Delta_p g - \frac{(1-\gamma)(p-1)}{\gamma}\ \frac{|\nabla g|^p}{g} - |\nabla g|^{p-1} - \frac{\gamma^{p-1}}{p-2}\ g^{(1-(1-\gamma)(p-1))/\gamma} = 0 \quad \text{in} \quad \Omega\,.$$

Consequently,

$$\begin{aligned}
&-\frac{\gamma^{p-1}}{A^{p-1}}\ (1+\delta t_0)^{(p-1)/(p-2)}\ \Delta_p\phi(t_0,x_0)\\
&-\frac{(1-\gamma)(p-1)\gamma^{p-1}}{A^p}\ (1+\delta t_0)^{p/(p-2)}\frac{|\nabla\phi(t_0,x_0)|^p}{f(x_0)^\gamma}\\
&-\frac{\gamma^{p-1}}{A^{p-1}}\ (1+\delta t_0)^{(p-1)/(p-2)}\ |\nabla\phi(t_0,x_0)|^{p-1} - \frac{\gamma^{p-1}}{p-2}\ f(x_0)^{1-(1-\gamma)(p-1)} \ge 0\,,
\end{aligned}$$

from which we deduce, since $\gamma \in (0,1)$,

$$\begin{aligned} -\Delta_p\phi(t_0,x_0) \quad &\ge \quad \frac{(1-\gamma)(p-1)}{A}\ (1+\delta t_0)^{1/(p-2)} \frac{|\nabla\phi(t_0,x_0)|^p}{f(x_0)^\gamma} \\ &+ \quad |\nabla\phi(t_0,x_0)|^{p-1} + \frac{A^{p-1}}{p-2}\ \frac{f(x_0)^{1-(1-\gamma)(p-1)}}{(1+\delta t_0)^{(p-1)/(p-2)}}\,. \end{aligned} \tag{4.4.46}$$

By (4.4.45) and (4.4.46), we have

$$\begin{aligned} \partial_t\phi(t_0,x_0) - \Delta_p\phi(t_0,x_0) - |\nabla\phi(t_0,x_0)|^q \quad &\ge \quad \frac{|\nabla\phi(t_0,x_0)|^{p-1}}{f(x_0)^\gamma}\ R_1 \\ &+ \frac{A^{p-1} f(x_0)^{1-(1-\gamma)(p-1)}}{(1+\delta t_0)^{(p-1)/(p-2)}}\ \frac{R_2}{p-2}\,, \end{aligned} \tag{4.4.47}$$

with

$$\begin{aligned} R_1 \quad &:= \quad \frac{(1-\gamma)(p-1)}{A}\ (1+\delta t_0)^{1/(p-2)}|\nabla\phi(t_0,x_0)| + f(x_0)^\gamma - f(x_0)^\gamma\ |\nabla\phi(t_0,x_0)|^{q-p+1}\,, \\ R_2 \quad &:= \quad 1 - \frac{\delta}{\gamma A^{p-2}}\ f(x_0)^{(1-\gamma)(p-2)}\,. \end{aligned}$$

On the one hand, (4.4.43) guarantees that $(1-\gamma)(p-1) \ge 1$ which, together with Young's inequality and the assumption $q \in (p-1,p]$, leads us to

$$R_1 \ge \|f\|_\infty^\gamma\ |\nabla\phi(t_0,x_0)| + f(x_0)^\gamma - (q-p+1)\ f(x_0)^\gamma\ |\nabla\phi(t_0,x_0)| - (p-q)\ f(x_0)^\gamma \ge 0\,.$$

On the other hand, the choice of A and δ gives

$$R_2 = 1 - \left(\frac{f(x_0)}{\|f\|_\infty}\right)^{(1-\gamma)(p-2)} \ge 0\,.$$

Combining the previous two inequalities with (4.4.47) completes the proof of the claim (4.4.44).

Now, $u = \Sigma = 0$ on $(0,\infty)\times\partial\Omega$ while, since $\beta \ge \gamma$, we infer from (4.4.43) and the choice of A that, for $x \in \bar\Omega$,

$$u_0(x) \le B\ f(x)^\beta = \frac{Af(x)^\gamma}{\gamma}\ \frac{\gamma B f(x)^{\beta-\gamma}}{A} \le \Sigma(0,x)\ \frac{\gamma B\|f\|_\infty^{\beta-\gamma}}{A} \le \Sigma(0,x)\,.$$

We then deduce from the comparison principle [A4.10, Theorem 8.2] that $u(t,x) \le \Sigma(t,x)$ for $(t,x) \in [0,\infty)\times\bar\Omega$ and the proof of Proposition 4.4.6 is complete. ∎

4.5 Well-posedness and blowup: $q > p$

4.5.1 Well-posedness

We finally turn to the case $q > p$ and first show that a suitable solution to (4.1.1)-(4.1.2) with $q = p-1$ allows us to construct a supersolution to (4.1.1) when $q > p$ which vanishes identically on the boundary of Ω.

Lemma 4.5.1 *Consider $G_0 \in W^{1,\infty}(\Omega)$ satisfying* (4.1.4) *and let G be the corresponding solution to* (4.1.1)-(4.1.3) *with $q = p-1$. Setting $g(t,x) := G(\ell[G_0]^{2-p}t, x)/\ell[G_0]$ for $(t,x) \in \bar{Q}$, the parameter $\ell[G_0]$ being defined in* (4.1.7)*, g is a solution to* (4.1.1)-(4.1.2) *with initial condition $G_0/\ell[G_0]$ and $q = p-1$ such that $|\nabla g| \le 1$ in $\bar{Q}$. Moreover, g is a supersolution to* (4.1.1) *in Q for any $q > p-1$.*

Proof. Owing to the definition (4.1.7) of $\ell[G_0]$, we clearly have $|\nabla g| \le 1$ in $\bar{Q}$.
Next, let $\phi \in C^2(Q)$ and consider $(t_0, x_0) \in Q$ where $g - \phi$ has a local minimum. Since g is 1-Lipschitz continuous with respect to the space variable, this property implies that

$$|\nabla \phi(t_0, x_0)| \le 1\,. \tag{4.5.1}$$

Moreover, introducing $\psi(t,x) := \ell[G_0]\,\phi(\ell[G_0]^{p-2}t, x)$ for $(t,x) \in \bar{Q}$, the function $G - \psi$ has a local minimum at $(t_0\ell[G_0]^{2-p}, x_0)$, so that

$$\partial_t\psi(t_0\ell[G_0]^{2-p}, x_0) - \Delta_p\psi(t_0\ell[G_0]^{2-p}, x_0) - |\nabla\psi(t_0\ell[G_0]^{2-p}, x_0)|^{p-1} \ge 0\,,$$

and thus

$$\partial_t\phi(t_0, x_0) - \Delta_p\phi(t_0, x_0) - |\nabla\phi(t_0, x_0)|^{p-1} \ge 0\,. \tag{4.5.2}$$

Hence, g is a supersolution to (4.1.1) with $q = p-1$. In a similar way, it can be shown that g is also a subsolution and therefore a solution to (4.1.1) with $q = p-1$.
Furthermore, we infer from (4.5.1), (4.5.2), and the property $q > p-1$ that

$$\partial_t\phi(t_0, x_0) - \Delta_p\phi(t_0, x_0) - |\nabla\phi(t_0, x_0)|^{q} \ge |\nabla\phi(t_0, x_0)|^{p-1}\left(1 - |\nabla\phi(t_0, x_0)|^{q-p+1}\right) \ge 0\,,$$

which completes the proof of Lemma 4.5.1. ■

Proposition 4.5.2 *Assume that $q > p$ and consider an initial condition u_0 satisfying* (4.1.4) *for which there is $G_0 \in W^{1,\infty}(\Omega)$ satisfying* (4.1.4) *such that*

$$u_0(x) \le \frac{G_0(x)}{\ell[G_0]}\,, \qquad x \in \bar{\Omega}\,. \tag{4.5.3}$$

Then there is a unique solution u to (4.1.1)-(4.1.3) *in the sense of Definition 4.1.1 and it satisfies*

$$u(t,x) \le g(t,x) := \frac{1}{\ell[G_0]}\, G\left(\frac{t}{\ell[G_0]^{p-2}}, x\right)\,, \qquad (t,x) \in [0,\infty) \times \bar{\Omega}\,, \tag{4.5.4}$$

where G denotes the solution to (4.1.1)-(4.1.2) *with initial condition G_0 and $q = p-1$.*

Proof. On the one hand, the solution U to the p-Laplacian equation (4.4.19)-(4.4.21) is clearly a subsolution to (4.1.1) in Q. On the other hand, the function g defined in (4.5.4) is a supersolution to (4.1.1) in Q by Lemma 4.5.1 and is thus also a supersolution to (4.4.19). Since $U = g = 0$ on $(0,\infty)\times\partial\Omega$ and $U(0,x) = u_0(x) \le g(0,x)$ for $x \in \bar\Omega$ by (4.5.3), the comparison principle [A4.10, Theorem 8.2] applied to the p-Laplacian equation (4.4.19) ensures that $U \le g$ in $[0,\infty)\times\bar\Omega$. This property and the simultaneous vanishing of U and g on $(0,\infty)\times\partial\Omega$ allow us to use the classical Perron method to establish the existence of a solution u to (4.1.1)-(4.1.3) in the sense of Definition 4.1.1 which satisfies (4.5.4). The uniqueness next follows from the comparison principle [A4.10, Theorem 8.2]. ∎

4.5.2 Large time behaviour

We first recall that Lemma 4.4.2 is also valid in that case. It next readily follows from Lemma 4.4.3 and (4.5.4) that

$$\begin{aligned} 0 \ \le\ & u(t,x) = u(t,x) - u(t,x_0) \le g(t,x) \le \frac{L_1}{(1+t)^{1/(p-2)}}\, |x - x_0|\,, \\ & (t,x,x_0) \in [0,\infty)\times\bar\Omega\times\partial\Omega\,. \end{aligned}$$

The convergence proof is then the same as that performed in Section 4.4.3 for $q \in (p-1,p]$.

4.5.3 Blowup

Let us first recall that, by a weak solution to (4.1.1)-(4.1.3), we mean a nonnegative function $u \in C([0,\infty)\times\bar\Omega)$ which belongs to $L^\infty(0,T;W^{1,\infty}(\Omega))$ and satisfies

$$\frac{d}{dt}\int_\Omega u(t,x)\ \psi(x)\ dx = \int_\Omega \big(-|\nabla u(t,x)|^{p-2}\ \nabla u(t,x)\cdot\nabla\psi(x) + |\nabla u(t,x)|^q\ \psi(x)\big)\ dx \tag{4.5.5}$$

for any $\psi \in H_0^1(\Omega)$ and $T > 0$. We now show that such a solution cannot exist for all times if $q > p$ and u_0 is sufficiently large.

Proposition 4.5.3 *Assume that $q > p$ and define $r := q/(q-p)$. There is a positive real number κ depending on Ω, p, and q such that, if $\|u_0\|_{r+1} > \kappa$, then* (4.1.1)-(4.1.3) *has no global weak solution.*

Proof. We argue as in [A4.15, Theorem 2.4] and use classical approximation arguments to deduce from (4.5.5) and Hölder's and Young's inequalities that

$$\begin{aligned} \frac{1}{r+1}\ \frac{d}{dt}\|u\|_{r+1}^{r+1} &= \int_\Omega u^r\ |\nabla u|^q\ dx - \frac{q}{q-p}\ \int_\Omega u^{r-1}\ |\nabla u|^p\ dx \\ &\ge \int_\Omega u^r\ |\nabla u|^q\ dx - \frac{q}{q-p}\ |\Omega|^{(q-p)/q}\ \left(\int_\Omega u^r\ |\nabla u|^q\ dx\right)^{p/q} \end{aligned}$$

$$
\begin{aligned}
&\geq \int_\Omega u^r\ |\nabla u|^q\ dx - \frac{p}{q}\int_\Omega u^r\ |\nabla u|^q\ dx - \left(\frac{q}{q-p}\right)^{p/(q-p)}\ |\Omega| \\
&\geq \frac{q-p}{q}\int_\Omega u^r\ |\nabla u|^q\ dx - \left(\frac{q}{q-p}\right)^{p/(q-p)}\ |\Omega| \\
&\geq \frac{q-p}{q}\left(\frac{q-p}{q-p+1}\right)^q \int_\Omega \left|\nabla\left(u^{(q-p+1)/(q-p)}\right)\right|^q\ dx \\
&\quad - \left(\frac{q}{q-p}\right)^{p/(q-p)}\ |\Omega|.
\end{aligned}
$$

We now use the Poincaré inequality to obtain that

$$\frac{1}{r+1}\ \frac{d}{dt}\|u\|_{r+1}^{r+1} \geq \kappa_1 \int_\Omega u^{r+q}\ dx - \kappa_2$$

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$ depending only on Ω, p, and q. Since $q > 1$, we use again Hölder's inequality to deduce

$$\frac{1}{r+1}\ \frac{d}{dt}\|u\|_{r+1}^{r+1} \geq \frac{\kappa_1}{|\Omega|^{(q-1)/(r+1)}}\ \|u\|_{r+1}^{r+q} - \kappa_2\,.$$

Since $q > 1$, this clearly contradicts the global existence as soon as $\|u_0\|_{r+1}$ is sufficiently large. ∎

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Article 5: Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions

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Abstract

In this paper we consider quasilinear Keller-Segel type systems of two kinds in higher dimensions. In the case of a nonlinear diffusion system we prove an optimal (with respect to possible nonlinear diffusions generating explosion in finite time of solutions) finite-time blowup result. In the case of a cross-diffusion system we give results which are optimal provided one assumes some proper non-decay of a nonlinear chemical sensitivity. Moreover, we show that once we do not assume the above mentioned non-decay, our result cannot be as strong as in the case of nonlinear diffusion without nonlinear cross-diffusion terms. To this end we provide an example, interesting by itself, of global-in-time unbounded solutions to the nonlinear cross-diffusion Keller-Segel system with chemical sensitivity decaying fast enough, in a range of parameters in which there is a finite-time blowup result in a corresponding case without nonlinear cross-diffusion.

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5.1 Introduction

This work deals with radially symmetric nonnegative solution couples (u, v) of the parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{5.1.1}$$

in a ball $\Omega = B_R \subset \mathbb{R}^n$, where $n \geq 3$, $R > 0$, and the initial data are supposed to satisfy $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ such that $u_0 > 0$ and $v_0 > 0$ in $\bar{\Omega}$.
Moreover, we assume that $\phi, \psi \in C^2([0,\infty))$ and that there is $\beta \in C^2([0,\infty))$ such that

$$\phi(s) > 0, \qquad \psi(s) = s\beta(s), \quad \text{and} \quad \beta(s) > 0 \quad \text{for } s \in [0,\infty) \tag{5.1.2}$$

are satisfied.
Systems of this kind were introduced in [A5.12] to describe the motion of cells which are diffusing and moving towards the gradient of a substance called chemoattractant, the latter being produced by the cells themselves. In particular, the essentiality of both nonlinear diffusion as well as nonlinear chemosensitivity were emphasized in [A5.9] where it was explained that they can be used to model the so-called volume-filling effect. The Keller-Segel system has been studied extensively by many authors and the main issue of the investigation was chemotactic collapse of cells interpreted as finite-time blowup of the component u of a solution to (5.1.1). It is however worth to be underlined that despite the fact that the original Keller-Segel model was a system of parabolic equations the main results concerning the finite-time blowup of solutions to (5.1.1) were usually proved for its parabolic-elliptic simplification. There were a few methods introduced to investigate the phenomenon of finite-time explosion of solutions in that case. Two main methods among them being the change of variables leading to a reduction of the parabolic-elliptic simplification of (5.1.1) to a single equation obeying a maximum principle introduced in [A5.11] and the so-called moment method making strong use of the fact that the second equation of the parabolic-elliptic simplification of (5.1.1) is a Poisson equation, see [A5.2, A5.13]. Those two methods and their ramifications led to a variety of results concerning appearance of chemotactic collapse in both semilinear (i.e. $\phi \equiv 1$, $\psi(s) = s$) and quasilinear Keller-Segel systems. In particular, there have been characterized values of initial mass distinguishing between finite-time blowup and global existence of bounded solutions to the two-dimensional semilinear version of (5.1.1) in both radially symmetric ond non-radial settings (see [A5.13, A5.14, A5.15]). Moreover, it has been shown that in higher dimensions a finite-time blowup of solutions to the semilinear version of (5.1.1) can

occur independently of the initial mass provided that the initial data are concentrated enough [A5.13]. Finally, in the case of a quasilinear system, for any space dimension n there have been identified critical nonlinearities such that if ϕ and ψ satisfy the supercritical relation, then solutions to (5.1.1) stay bounded for any time while for those satisfying the subcritical relation solutions blow up in finite-time independently of the magnitude of initial mass provided the data are concentrated enough, see [A5.6].

However, all those results are available only for a parabolic-elliptic simplification of (5.1.1). In the case of the original fully parabolic version the investigation of chemotactic collapse turned out to be a much more challenging issue. So far the only two existing results in the literature stating the occurrence of finite-time blowup of solutions to (5.1.1) are those in [A5.8], where an example of a special solution to the semilinear version of (5.1.1) in dimension $n = 2$ blowing up in a finite-time is shown, and the result in [A5.5] where the explosion of solutions to the one-dimensional Keller-Segel system with appropriately weak diffusion of cells and sufficiently fast diffusion of chemoattractant is shown. The breakthrough has been made recently in [A5.19]. Introducing a new method M. Winkler shows there that in dimensions $n \geq 3$ generic solutions to the semilinear version of (5.1.1) blow up in finite time independently of the size of initial mass. In the present paper we generalize his method to the quasilinear case. This way, to the best of our knowledge, we obtain a first result concerning a finite-time blowup of solutions to the fully parabolic quasilinear Keller-Segel system in higher dimensions. So far the only result in that direction was achieved in dimension $n = 1$ and only for large initial masses in [A5.5]. Moreover, the result concerning a chemotactic collapse in the case where $\beta(u) \equiv 1$ is optimal. Namely, we show that in dimension $n \geq 3$, for $\phi(u) \leq Cu^p, p < 1 - \frac{2}{n}$ and some constant $C > 0$, independently of the size of initial mass, one can find generic radially symmetric initial data leading to finite-time blowup (see Corollary 5.1.4). This result is optimal in view of the result in [A5.16] guaranteeing global existence of bounded solutions to (5.1.1) with $\psi \equiv 1$ for ϕ satisfying $\phi(u) \geq Cu^q, q > 1 - \frac{2}{n}$ for some constant $C > 0$. Moreover, in Corollary 5.1.5 we prove that in the case of full nonlinear cross-diffusion we obtain a result at least as good as in the parabolic-elliptic case, compare [A5.6]. Furthermore, it is also an optimal result for a fully parabolic problem when restricting ourselves to polynomial nonlinearities, see [A5.17]. Theorem 5.1.1, which is our main achievement, shows that restricting ourselves to the case of $\psi(u)$ not decaying when u is large, we obtain the result which is a counterpart of the existence of global-in-time solutions in [A5.4]. Finally, we show (see Theorem 5.1.6) that without assuming a lack of decay of $\psi(u)$ one cannot expect the existence of critical exponents distinguishing between boundedness of solutions and finite-time blowup. It turns out that the possible asymptotic behavior of solutions to the nonlinear cross-diffusion system (5.1.1) can be more complicated. We show that under the proper choice of parameters (corresponding to the choice of parameters which yield finite-time blowup in a semilinear case) one can construct global-in-time radially symmetric solutions admitting infinite-time blowup. This result seems to be quite interesting by itself since the phenomenon of infinite-time blowup does not seem to be that often met in parabolic equations.

To be more precise when formulating our finite-time blowup results we have to introduce the following notation. Suppose that there exist positive constants s_0, a, and b such that the functions

$$G(s) := \int_{s_0}^{s}\int_{s_0}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)}\,d\tau\,d\sigma, \quad s > 0, \qquad \text{and} \qquad H(s) := \int_{0}^{s} \frac{\sigma\phi(\sigma)}{\psi(\sigma)}\,d\sigma, \quad s \geq 0, \tag{5.1.3}$$

fulfill

$$G(s) \leq a\, s^{2-\alpha}, \; s \geq s_0, \qquad \text{with some } \alpha > \frac{2}{n}, \tag{5.1.4}$$

as well as

$$H(s) \leq \gamma \cdot G(s) + b(s+1), \; s > 0, \qquad \text{with some } \gamma \in \left(0, \frac{n-2}{n}\right). \tag{5.1.5}$$

We remark that H in (5.1.3) is well-defined due to the positivity of β in $[0, \infty)$.
It is well-known that the function

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega G(u) \tag{5.1.6}$$

is a Liapunov functional for (5.1.1) with dissipation rate

$$\mathcal{D}(u,v) := \int_\Omega v_t^2 + \int_\Omega \psi(u) \cdot \Big| \frac{\phi(u)}{\psi(u)}\nabla u - \nabla v \Big|^2. \tag{5.1.7}$$

More precisely, any classical solution to (5.1.1) satisfies

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t), v(\cdot,t)) = -\mathcal{D}(u(\cdot,t), v(\cdot,t)) \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)), \tag{5.1.8}$$

where $T_{max}(u_0, v_0) \in (0, \infty]$ denotes the maximal existence time of (u, v) (see [A5.18, Lemma 2.1]).

In order to prove our result of finite-time blowup, we need to impose the additional condition that there exists $c_0 > 0$ such that

$$\psi(s) \geq c_0\, s, \quad s \geq 0, \tag{5.1.9}$$

which in view of (5.1.2) means that $\beta(s) \geq c_0 > 0$ for $s \geq 0$.
Then we have the following result for blowup in finite time.

Theorem 5.1.1 *Suppose that $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, assume that (5.1.4), (5.1.5), and (5.1.9) are satisfied, and let $m > 0$ and $A > 0$ be given. Then there exist positive constants $T(m, A)$ and $K(m)$ such that for any*

$$(u_0, v_0) \in \mathcal{B}(m,A) \;:=\; \Big\{ (u_0,v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega) \;\Big|\; u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \bar\Omega,\ \textstyle\int_\Omega u_0 = m,\ \|v_0\|_{W^{1,2}(\Omega)} \leq A, \text{ and } \mathcal{F}(u_0,v_0) \leq -K(m)\cdot(1+A^2) \Big\}, \tag{5.1.10}$$

the corresponding solution (u, v) of (5.1.1) blows up at the finite time $T_{max}(u_0, v_0) \in (0, \infty)$, where $T_{max}(u_0, v_0) \leq T(m, A)$.

Moreover, the set $\mathcal{B}(m, A)$ has the following properties.

Theorem 5.1.2 *Let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, let $\mathcal{B}(m, A)$ be as defined in* (5.1.10), *and assume that* (5.1.4) *is fulfilled.*

(i) Then for any $m > 0$ there exists $A > 0$ such that $\mathcal{B}(m, A) \neq \emptyset$.

(ii) Suppose that (5.1.4) *holds with some $\alpha > \frac{4}{n+2}$ and, moreover, let $p \in (1, \frac{2n}{n+2})$ such that $p > 2 - \alpha$. Then for any $m > 0$ and $A > 0$, the set $\mathcal{B}(m, A)$ is dense in the space of all radially symmetric positive functions in $C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ with respect to the topology in $L^p(\Omega) \times W^{1,2}(\Omega)$. In particular, given positive radial functions $(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ and $\varepsilon > 0$, there exist positive radial $(u_{0\varepsilon}, v_{0\varepsilon}) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ such that*

$$\|u_{0\varepsilon} - u_0\|_{L^p(\Omega)} + \|v_{0\varepsilon} - v_0\|_{W^{1,2}(\Omega)} < \varepsilon$$

and the solution $(u_\varepsilon, v_\varepsilon)$ of (5.1.1) *with initial data $(u_{0\varepsilon}, v_{0\varepsilon})$ blows up in finite time.*

Furthermore, we state three corollaries which cover interesting special cases. Corollary 5.1.3 is an immediate consequence of Theorem 5.1.1 while Corollary 5.1.4 follows since (5.1.5) is satisfied which, in the case that ϕ is decreasing, is deduced in view of the possibility of choosing $s_0 > e^{\frac{1}{\gamma}}$ and integration by parts, and, in case of $s^q/\phi(s) \to c$ as $s \to \infty$, is implied by [A5.18, Corollary 5.2]. Moreover, Corollary 5.1.5 follows from Theorem 5.1.1, because [A5.18, Corollary 5.2] shows that the functions ϕ and ψ given in Corollary 5.1.5 satisfy (5.1.4) and (5.1.5). Corollary 5.1.4 is optimal in view of the results given in [A5.16].

Corollary 5.1.3 *Assume that $\psi(s) = s$ for $s \geq 0$ and that* (5.1.4) *and* (5.1.5) *are fulfilled. Moreover, let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and let $m > 0$ and $A > 0$ be given. Then there exist positive constants $T(m, A)$ and $K(m)$ such that for any $(u_0, v_0) \in \mathcal{B}(m, A)$ the corresponding solution (u, v) of* (5.1.1) *blows up at the finite time $T_{max}(u_0, v_0) \leq T(m, A)$.*

Corollary 5.1.4 *Assume that $\psi(s) = s$ for $s \geq 0$ and that $\phi(s) \leq Cs^q$, $s \geq 1$, for some $q < 1 - \frac{2}{n}$ and $C > 0$. Furthermore, suppose that either ϕ is a decreasing function or that there exists $c > 0$ such that $s^q/\phi(s) \to c$ as $s \to \infty$. Let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and let $m > 0$ and $A > 0$ be given. Then there exist positive constants $T(m, A)$ and $K(m)$ such that for any $(u_0, v_0) \in \mathcal{B}(m, A)$ the corresponding solution (u, v) of* (5.1.1) *blows up at the finite time $T_{max}(u_0, v_0) \leq T(m, A)$.*

Corollary 5.1.5 *Assume that $\phi(s) = (s+1)^{-p}$ and $\psi(s) = s(s+1)^{q-1}$, $s \geq 0$, with $q \geq 1$ and $p \in \mathbb{R}$ such that $p + q > \frac{2}{n}$. Moreover, let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and let $m > 0$ and $A > 0$ be given. Then there exist positive constants $T(m, A)$ and $K(m)$ such that for any $(u_0, v_0) \in \mathcal{B}(m, A)$ the corresponding solution (u, v) of* (5.1.1) *blows up at the finite time $T_{max}(u_0, v_0) \leq T(m, A)$.*

In view of [A5.17] the latter result is optimal in the case $q \geq 1$, while in view of [A5.18] it remains an interesting question whether Corollary 5.1.5 can be extended to the case $q < 1$. In the following theorem, in particular, we provide a negative answer to this question. However, it still remains open to find critical exponents (if possible) distinguishing between finite- and infinite-time blowup of solutions when $q < 1$.

Theorem 5.1.6 *Let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$. Moreover, assume that $\lim_{s\to\infty} \phi(s) = 0$, that there exists a positive constant $D > 0$ such that for any $s > 0$*

$$\frac{\beta(s)}{\phi(s)} \leq D \tag{5.1.11}$$

and that there exist constants $D_1 > 0$ and $\gamma_1 > n$ such that for any $s > 0$

$$\beta(s) \leq D_1 s^{-\gamma_1}. \tag{5.1.12}$$

Assume also that (5.1.4) and (5.1.5) hold. Then there exists a radially symmetric global-in-time solution (u, v) to (5.1.1) blowing up in infinite time with respect to the L^∞-norm.

Remark 5.1.7 *Notice that for $\alpha \in (\frac{2}{n}, 1)$ in (5.1.4), choosing $\phi(u) = \beta(u)$ we make sure that (5.1.4) and (5.1.5) are satisfied mutually with (5.1.11) and (5.1.12) indicating that the assumptions of Theorem 5.1.6 are not contradictory.*

5.2 Preliminaries

In this section we state some known results concerning local existence of solutions to (5.1.1) as well as some useful properties of the solutions.

Lemma 5.2.1 *Suppose that $(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ are radially symmetric and positive in $\bar{\Omega}$, and let $q \in (n, \infty)$. Then there exist $T_{max}(u_0, v_0) \in (0, \infty]$ and a classical solution (u, v) of (5.1.1) in $\Omega \times (0, T_{max}(u_0, v_0))$, where u and v are radially symmetric functions and satisfy*

$$u \in C^0([0, T_{max}(u_0, v_0)); C^0(\bar{\Omega})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}(u_0, v_0))),$$
$$v \in C^0([0, T_{max}(u_0, v_0)); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}(u_0, v_0))).$$

Moreover,

$$\text{either } T_{max}(u_0, v_0) = \infty, \qquad \text{or } \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max}(u_0, v_0)$$

is fulfilled, equation (5.1.8) holds and we have

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 \qquad \text{for all } t \in (0, T_{max}(u_0, v_0)), \tag{5.2.1}$$

$$\int_\Omega v(x,t)dx \leq \max\left\{ \int_\Omega u_0, \int_\Omega v_0 \right\} \qquad \text{for all } t \in (0, T_{max}(u_0, v_0)). \tag{5.2.2}$$

Proof. The claims concerning existence and regularity of the solution follow from well-known parabolic regularity theory and fixed point arguments, and the extensibility criterion also is proved by standard arguments. For details, we refer the reader to [A5.1, A5.10, A5.20]. Moreover, the energy equation (5.1.8) is proved in [A5.18, Lemma 2.1] and the mass identities (5.2.1) and (5.2.2) immediately follow from integrating the first and second equation in (5.1.1), respectively, by using the Neumann boundary conditions along with an ODE comparison. Conservation of radial symmetry is a consequence of uniqueness of solutions and the adequate form of equations in (5.1.1). ∎

Next, we state a consequence of the Gagliardo-Nirenberg and the Young inequalities which will be used in forthcoming proofs and which is given in [A5.19, Lemma 2.2] (see [A5.7] for details of the proof).

Lemma 5.2.2 *There is $C > 0$ such that*

$$\|\varphi\|_{L^2(\Omega)} \le C\|\nabla\varphi\|_{L^2(\Omega)}^{\frac{n}{n+2}}\|\varphi\|_{L^1(\Omega)}^{\frac{2}{n+2}} + C\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{5.2.3}$$

In addition, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|\varphi\|^2_{L^2(\Omega)} \le \varepsilon\|\nabla\varphi\|^2_{L^2(\Omega)} + C(\varepsilon)\|\varphi\|^2_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{5.2.4}$$

The following pointwise upper bound for the function v will be an important ingredient to prove finite-time blowup. The result is given in [A5.19, Lemma 3.2] and its proof is exactly the same as the one performed in [A5.19, Section 3] since there only the second equation in (5.1.1) is used.

Lemma 5.2.3 *Let $p \in (1, \frac{n}{n-1})$. Then there is $C(p) > 0$ such that whenever $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are positive in $\bar{\Omega}$ and radially symmetric, the solution of* (5.1.1) *satisfies*

$$v(r,t) \le C(p) \cdot \Big(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)}\Big) \cdot r^{-\frac{n-p}{p}} \tag{5.2.5}$$

for all $(r,t) \in (0,R) \times (0, T_{max}(u_0,v_0))$.

5.3 Finite-time blowup: estimates for the Liapunov functional

In this section, we estimate the Liapunov functional $\mathcal{F}$ in terms of the dissipation rate $\mathcal{D}$ and frequently use the ideas from [A5.19, Section 4], where the case $\phi(u) = 1$ and $\psi(u) = u$ is studied. In order to be able to handle the more general system (5.1.1), we introduce new estimates in Lemma 5.3.4 along with a more careful choice of some constants and the use of the terms contained in $\mathcal{F}$ which were not used in [A5.19].
Following the ansatz of [A5.19], in view of the previous section we fix $m > 0$, $M > 0$, $B > 0$, and $\kappa > n-2$ and assume that

$$\int_\Omega u = m \qquad \text{and} \qquad \int_\Omega v \le M \tag{5.3.1}$$

and

$$v(x) \leq B|x|^{-\kappa} \qquad \text{for all } x \in \Omega \tag{5.3.2}$$

are satisfied. Moreover, we define the space

$$\mathcal{S}(m, M, B, \kappa) := \Bigg\{ (u,v) \in C^1(\bar{\Omega}) \times C^2(\bar{\Omega}) \ \Bigg| \ u \text{ and } v \text{ are positive and radially symmetric satisfying } \tfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \text{ (5.3.1), and (5.3.2)} \Bigg\}. \tag{5.3.3}$$

The goal of this section is to prove that the inequality

$$\frac{\mathcal{F}(u,v)}{\mathcal{D}^{\theta}(u,v) + 1} \geq -C(m, M, B, \kappa) \qquad \text{for all } (u,v) \in \mathcal{S}(m, M, B, \kappa) \tag{5.3.4}$$

holds with some constants $\theta \in (0,1)$ and $C(m, M, B, \kappa) > 0$ (see Theorem 5.3.6). Here it will be important to state precisely the dependence of C on M and B.
The main ingredient of the proof of (5.3.4) is the following estimate of $\int_\Omega uv$.

Lemma 5.3.1 *Let* (5.1.5) *and* (5.1.9) *be fulfilled. Then there are* $C(m, \kappa) > 0$ *and*

$$\theta := \frac{1}{1 + \frac{n}{(2n+4)\kappa}} \in \left(\frac{1}{2}, 1\right) \tag{5.3.5}$$

such that all $(u,v) \in \mathcal{S}(m, M, B, \kappa)$ *satisfy*

$$\int_\Omega uv \leq C(m,\kappa) \cdot \left(1 + M^2 + B^{\frac{2n+4}{n+4}}\right) \cdot \left(\left\| \Delta v - v + u \right\|_{L^2(\Omega)}^{2\theta} + \left\| \frac{\phi(u)}{\sqrt{\psi(u)}} \nabla u - \sqrt{\psi(u)} \nabla v \right\|_{L^2(\Omega)} + 1 \right) + \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega G(u). \tag{5.3.6}$$

Lemma 5.3.1 is a generalization of [A5.19, Lemma 4.1] and our proof, which will be given after proving several claims in the forthcoming lemmata, is based on the ideas given in [A5.19, Section 4] along with some additional estimates in order to cope with the more general functions ϕ and ψ.
For notational convenience, we abbreviate

$$f := -\Delta v + v - u \tag{5.3.7}$$

and

$$g := \left(\frac{\phi(u)}{\sqrt{\psi(u)}} \nabla u - \sqrt{\psi(u)} \nabla v \right) \cdot \frac{x}{|x|}, \qquad x \neq 0, \tag{5.3.8}$$

for $(u,v) \in \mathcal{S}(m, M, B, \kappa)$.
The first step towards the proof of Lemma 5.3.1 is the following estimate which is completely similar to [A5.19, Lemma 4.2]. But as our different choice of the constants and their precise dependence on M are important for the sequel, we give the proof for the reader's convenience.

Lemma 5.3.2 *For any* $\varepsilon \in (0,1)$ *there exists* $C(\varepsilon) > 0$ *such that for all* $(u,v) \in \mathcal{S}(m,M,B,\kappa)$

$$\int_\Omega uv \le (1+\varepsilon)\int_\Omega |\nabla v|^2 + C(\varepsilon)\cdot\big(1+M^2\big)\cdot\left(\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1\right) \tag{5.3.9}$$

is fulfilled.

Proof. Multiplying (5.3.7) by v and integrating by parts over Ω we have

$$\int_\Omega uv = \int_\Omega |\nabla v|^2 + \int_\Omega v^2 - \int_\Omega fv. \tag{5.3.10}$$

Now given $\varepsilon \in (0,1)$, by Lemma 5.2.2 and (5.3.1) we can fix $c_1 = C_1\cdot(1+M) > 0$ and $c_2 = C_2(\varepsilon)\cdot M^2 > 0$ such that

$$\|v\|_{L^2(\Omega)} \le c_1\cdot\left(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1\right) \tag{5.3.11}$$

and

$$\int_\Omega v^2 \le \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2 + c_2. \tag{5.3.12}$$

Applying the Cauchy-Schwarz inequality along with (5.3.11) and Young's inequality (with exponents $\frac{2n+4}{n}$ and $\frac{2n+4}{n+4}$), we obtain $c_3 = C_3(\varepsilon)\cdot(1+M^{\frac{2n+4}{n+4}}) > 0$ such that

$$\begin{aligned} -\int_\Omega fv &\le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le c_1\cdot(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1)\cdot\|f\|_{L^2(\Omega)} \\ &\le \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2 + c_3\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_1\|f\|_{L^2(\Omega)}. \end{aligned} \tag{5.3.13}$$

Since $\frac{2n+4}{n+4} > 1$, we use Young's inequality once more and deduce that

$$c_1\|f\|_{L^2(\Omega)} \le c_1\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_1$$

is satisfied. Combining the latter inequality with (5.3.10), (5.3.12), and (5.3.13), the claimed estimate (5.3.9) is proved, where we use $\frac{2n+4}{n+4} < 2$ to deduce the estimate $(1+M^2)$ in (5.3.9). ∎

In view of Lemma 5.3.2, the next step is to estimate $\int |\nabla v|^2$. This is first done in the annulus $\Omega\setminus B_{r_0}$, where the value of r_0 will be fixed in Lemma 5.3.5 below. Since in [A5.19, Lemma 4.3] only equation (5.3.7) is used we could simply repeat its proof. However we give it in details in order to state the exact dependence of the constants on M and B which will be of importance further.

Lemma 5.3.3 *For any* $r_0 \in (0,R)$ *and* $\varepsilon \in (0,1)$*, there exists a constant* $C(\varepsilon,m,\kappa) > 0$ *such that all* $(u,v) \in \mathcal{S}(m,M,B,\kappa)$ *satisfy*

$$\int_{\Omega\setminus B_{r_0}} |\nabla v|^2 \le \varepsilon\int_\Omega uv + \varepsilon\int_\Omega |\nabla v|^2 + C(\varepsilon,m,\kappa)\cdot\left(1+M^{\frac{2n+4}{n+4}} + B^{\frac{2n+4}{n+4}}\right)\cdot\left\{r_0^{-\frac{2n+4}{n}\kappa}\right.$$

$$+\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}\Bigg\}. \tag{5.3.14}$$

Proof. Let $\alpha_1 \in (0,1)$ be arbitrary. As $v > 0$, a multiplication of (5.3.7) by v^{α_1} and an integration by parts over Ω implies

$$\alpha_1 \int_\Omega v^{\alpha_1 - 1}|\nabla v|^2 \le \alpha_1 \int_\Omega v^{\alpha_1-1}|\nabla v|^2 + \int_\Omega v^{\alpha_1+1} = \int_\Omega u v^{\alpha_1} + \int_\Omega f v^{\alpha_1}. \tag{5.3.15}$$

Using next (5.3.2) and $\alpha_1 \in (0,1)$, we obtain

$$\alpha_1 \int_\Omega v^{\alpha_1-1}|\nabla v|^2 \ge \alpha_1 B^{\alpha_1-1} r_0^{(1-\alpha_1)\kappa} \cdot \int_{\Omega\setminus B_{r_0}} |\nabla v|^2,$$

whence (5.3.15) yields

$$\int_{\Omega\setminus B_{r_0}} |\nabla v|^2 \quad\le\quad \frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega u v^{\alpha_1} \;+\; \frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega f v^{\alpha_1}. \tag{5.3.16}$$

In view of $\alpha_1 \in (0,1)$ and Young's inequality, for any $\eta > 0$ there is $c_1(\eta, B) = C_1(\eta)\cdot B > 0$ such that

$$\frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} v^{\alpha_1}(r) \le \eta v(r) + c_1(\eta,B) r_0^{-\kappa} \qquad \text{for all } r \in (0,R). \tag{5.3.17}$$

The choice $\eta := \varepsilon$ implies

$$\begin{aligned}
\frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega u v^{\alpha_1} \quad &\le \quad \varepsilon \int_\Omega uv + c_1(\varepsilon,B) r_0^{-\kappa} \int_\Omega u \\
&= \quad \varepsilon \int_\Omega uv + c_1(\varepsilon,B) m r_0^{-\kappa} \\
&\le \quad \varepsilon \int_\Omega uv + c_1(\varepsilon,B) m R^{\frac{n+4}{n}\kappa} r_0^{-\frac{2n+4}{n}\kappa}
\end{aligned} \tag{5.3.18}$$

due to (5.3.1) and $u \ge 0$.
Furthermore, using (5.3.17) with $\eta := 1$ along with the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}
\frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega f v^{\alpha_1} \quad &\le \quad \int_\Omega |f| v + c_1(1,B) r_0^{-\kappa} \int_\Omega |f|, \\
&\le \quad \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + c_1(1,B) r_0^{-\kappa} \sqrt{|\Omega|} \|f\|_{L^2(\Omega)}.
\end{aligned} \tag{5.3.19}$$

Since by Lemma 5.2.2 and (5.3.1), there exists $c_2(M) = C_2 \cdot (1+M) > 0$ such that

$$\|v\|_{L^2(\Omega)} \le c_2(M)\cdot\Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1\Big) \le c_2(M)\cdot\Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + R^\kappa r_0^{-\kappa}\Big),$$

from (5.3.19) we infer

$$\frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega f v^{\alpha_1} \le c_3(M,B,\kappa) \cdot \Big(\|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + r_0^{-\kappa} \|f\|_{L^2(\Omega)} \Big)$$

with some $c_3(M,B,\kappa) = C_3(\kappa) \cdot (1+M+B) > 0$. Applying Young's inequality,

$$c_3(M,B,\kappa) \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} \le \varepsilon \|\nabla v\|_{L^2(\Omega)}^2 + c_4(\varepsilon, M, B, \kappa) \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}$$

and

$$c_3(M,B,\kappa) r_0^{-\kappa} \|f\|_{L^2(\Omega)} \le c_3(M,B,\kappa) \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_3(M,B,\kappa) r_0^{-\frac{2n+4}{n}\kappa}$$

hold with some $c_4(\varepsilon, M, B, \kappa) = C_4(\varepsilon,\kappa) \cdot \big(1 + M^{\frac{2n+4}{n+4}} + B^{\frac{2n+4}{n+4}}\big) > 0$. Thus, (5.3.19) finally turns into

$$\begin{aligned} \frac{B^{1-\alpha_1}}{\alpha_1} r_0^{-(1-\alpha_1)\kappa} \int_\Omega f v^{\alpha_1} \quad &\le \quad \varepsilon \int_\Omega |\nabla v|^2 + \Big(c_4(\varepsilon, M, B, \kappa) + c_3(M,B,\kappa) \Big) \cdot \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} \\ &\quad + c_3(M,B,\kappa) r_0^{-\frac{2n+4}{n}\kappa}. \end{aligned}$$

In conjunction with (5.3.16) and (5.3.18), the claim (5.3.14) is proved. ■

Next we prove a corresponding estimate of ∇v on the ball B_{r_0}. Our proof is based on ideas from [A5.19, Lemma 4.4] which are generalized to the problem (5.1.1). We recall that G and H are defined in (5.1.3) and remark that the following proof is the only place where we use the assumption (5.1.9). Moreover, it is important that r_0 can be chosen arbitrarily small in order to obtain a subquadratic power of $\|f\|_{L^2(\Omega)}$ in Lemma 5.3.5.

Lemma 5.3.4 *Assume that* (5.1.5) *and* (5.1.9) *are satisfied. Then there exist* $\mu = \mu(\gamma) \in (0,2)$ *and* $C(m) > 0$ *such that for all* $r_0 \in (0,R)$ *and* $(u,v) \in \mathcal{S}(m,M,B,\kappa)$

$$\begin{aligned} \int_{B_{r_0}} |\nabla v|^2 \quad &\le \quad \mu \int_\Omega G(u) + C(m) \cdot \Bigg\{ r_0 \cdot \Big\| \Delta v - v + u \Big\|_{L^2(\Omega)}^2 \\ &\quad + \Bigg\| \frac{\phi(u)}{\sqrt{\psi(u)}} \nabla u - \sqrt{\psi(u)} \nabla v \Bigg\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}^2 + 1 \Bigg\} \qquad (5.3.20) \end{aligned}$$

is fulfilled.

Proof. As (5.1.5) implies $(\frac{4(n-1)}{n-2} - 2)\gamma < 2$, we can fix $\delta \in (0, \frac{2n-2}{R}]$ small enough such that

$$\mu := \left(\frac{4(n-1)}{n-2} e^{\delta R} - 2 \right) \cdot \gamma \in (0,2) \qquad (5.3.21)$$

is fulfilled. As u and v are radially symmetric, (5.3.7) and (5.3.8) yield the identities

$$(r^{n-1} v_r)_r = -r^{n-1} u - r^{n-1} f + r^{n-1} v \qquad (5.3.22)$$

and

$$v_r = \frac{\phi(u)}{\psi(u)} u_r - \frac{g}{\sqrt{\psi(u)}} \ . \tag{5.3.23}$$

Multiplying (5.3.22) by $r^{n-1}v_r$ and using (5.3.23) as well as Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2}\left((r^{n-1}v_r)^2\right)_r &= -r^{2n-2}uv_r - r^{2n-2}fv_r + r^{2n-2}vv_r \\ &\le -r^{2n-2}\frac{u\phi(u)}{\psi(u)}u_r + r^{2n-2}\frac{u}{\sqrt{\psi(u)}}g + \frac{\delta}{2}(r^{n-1}v_r)^2 + \frac{1}{2\delta}r^{2n-2}f^2 \\ &\quad +\frac{1}{2}r^{2n-2}(v^2)_r \qquad \text{for all } r\in(0,R). \end{aligned} \tag{5.3.24}$$

Defining $y(r) := (r^{n-1}v_r)^2$, $r \in [0, R]$, we obtain

$$y_r \le -2r^{2n-2}\frac{u\phi(u)}{\psi(u)}u_r + 2r^{2n-2}\frac{u}{\sqrt{\psi(u)}}g + \delta y + \frac{1}{\delta}r^{2n-2}f^2 + r^{2n-2}(v^2)_r, \qquad r\in(0,R),$$

along with $y(0) = 0$ due to the regularity of v. Thus, an integration implies

$$\begin{aligned} r^{2n-2}v_r^2(r) = y(r) &\le -2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\frac{u(\rho)\phi(u(\rho))}{\psi(u(\rho))}u_r(\rho)\,d\rho \\ &\quad +2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\frac{u(\rho)}{\sqrt{\psi(u(\rho))}}g(\rho)\,d\rho \\ &\quad +\frac{1}{\delta}\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}f^2(\rho)d\rho \\ &\quad +\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}(v^2)_r(\rho)d\rho \end{aligned} \tag{5.3.25}$$

for all $r \in (0, R)$. Integrating by parts and using the nonnegativity of H (defined in (5.1.3)), we obtain

$$\begin{aligned} &-2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\frac{u(\rho)\phi(u(\rho))}{\psi(u(\rho))}u_r(\rho)\,d\rho \\ &\quad= 4(n-1)\int_0^r e^{\delta(r-\rho)}\rho^{2n-3}H(u(\rho))\,d\rho \\ &\qquad -2\delta\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}H(u(\rho))\,d\rho - 2r^{2n-2}H(u(r)) \\ &\quad\le 4(n-1)e^{\delta R}\int_0^r \rho^{2n-3}H(u(\rho))\,d\rho - 2r^{2n-2}H(u(r)), \qquad r\in(0,R). \end{aligned} \tag{5.3.26}$$

Next, denoting by ω_n the $(n-1)$-dimensional measure of the sphere ∂B_1 and applying the Cauchy-Schwarz inequality as well as (5.1.9), we deduce that

$$2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\frac{u(\rho)}{\sqrt{\psi(u(\rho))}}g(\rho)\,d\rho$$

$$
\begin{aligned}
&\leq 2\left(\int_0^R \rho^{n-1}\frac{u^2(\rho)}{\psi(u(\rho))}\,d\rho\right)^{\frac{1}{2}}\cdot\left(\int_0^r e^{2\delta(r-\rho)}\cdot\rho^{3n-3}g^2(\rho)\,d\rho\right)^{\frac{1}{2}}\\
&\leq 2\left(\frac{1}{c_0}\int_0^R \rho^{n-1}u(\rho)\,d\rho\right)^{\frac{1}{2}}\cdot\left(e^{2\delta R}r^{2n-2}\int_0^R \rho^{n-1}g^2(\rho)\,d\rho\right)^{\frac{1}{2}}\\
&\leq \frac{2e^{\delta R}}{w_n\sqrt{c_0}}\cdot\sqrt{m}\cdot r^{n-1}\cdot\|g\|_{L^2(\Omega)},\qquad r\in(0,R). \qquad (5.3.27)
\end{aligned}
$$

Similarly, we estimate the third term on the right-hand side of (5.3.25) according to

$$
\begin{aligned}
\frac{1}{\delta}\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}f^2(\rho)\,d\rho &\leq \frac{e^{\delta R}}{\delta}\cdot r^{n-1}\cdot\int_0^R \rho^{n-1}f^2(\rho)\,d\rho\\
&= \frac{e^{\delta R}}{\delta\omega_n}\cdot r^{n-1}\cdot\|f\|^2_{L^2(\Omega)}\qquad\text{for all } r\in(0,R). \qquad (5.3.28)
\end{aligned}
$$

As $\delta \leq \frac{2n-2}{R}$ yields $(2n-2)\rho^{2n-3} \geq \delta\rho^{2n-2}$ for all $\rho \in (0,R)$, integrating by parts we furthermore arrive at

$$
\begin{aligned}
\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}(v^2)_r(\rho)\,d\rho &= r^{2n-2}v^2(r)\\
&\quad -\int_0^r e^{\delta(r-\rho)}\cdot\left[(2n-2)\rho^{2n-3}-\delta\rho^{2n-2}\right]\cdot v^2(\rho)\,d\rho\\
&\leq r^{2n-2}v^2(r)\qquad\text{for all } r\in(0,R). \qquad (5.3.29)
\end{aligned}
$$

Hence, (5.3.25)-(5.3.29) imply that there is a constant $c_1(m) > 0$ such that

$$
\begin{aligned}
r^{2n-2}v_r^2(r) &\leq 4(n-1)e^{\delta R}\int_0^r \rho^{2n-3}H(u(\rho))\,d\rho - 2r^{2n-2}H(u(r))\\
&\quad +\frac{c_1(m)}{\omega_n}r^{n-1}\|g\|_{L^2(\Omega)} + \frac{c_1(m)}{\omega_n}r^{n-1}\|f\|^2_{L^2(\Omega)} + r^{2n-2}v^2(r),\qquad r\in(0,R).
\end{aligned}
$$

Multiplying this inequality by $\omega_n r^{1-n}$ and integrating over $r \in (0, r_0)$, we have

$$
\begin{aligned}
\int_{B_{r_0}}|\nabla v|^2 &= \omega_n\int_0^{r_0} r^{n-1}v_r^2(r)\,dr\\
&\leq 4(n-1)e^{\delta R}\omega_n\int_0^{r_0} r^{1-n}\int_0^r \rho^{2n-3}H(u(\rho))\,d\rho\,dr\\
&\quad -2\omega_n\int_0^{r_0} r^{n-1}H(u(r))\,dr + c_1(m)r_0\|g\|_{L^2(\Omega)}\\
&\quad +c_1(m)r_0\|f\|^2_{L^2(\Omega)} + \omega_n\int_0^{r_0} r^{n-1}v^2(r)\,dr\\
&\leq 4(n-1)e^{\delta R}\omega_n\int_0^{r_0} r^{1-n}\int_0^r \rho^{2n-3}H(u(\rho))\,d\rho\,dr\\
&\quad -2\int_{B_{r_0}} H(u) + c_1(m)R\|g\|_{L^2(\Omega)}
\end{aligned}
$$

$$+c_1(m)r_0\|f\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}. \tag{5.3.30}$$

Finally, Fubini's theorem, $n \geq 3$, the nonnegativity of H, (5.1.5), and (5.3.21) yield

$$\begin{aligned}
&4(n-1)e^{\delta R}\omega_n \int_0^{r_0} r^{1-n} \int_0^r \rho^{2n-3} H(u(\rho))\,d\rho\,dr - 2\int_{B_{r_0}} H(u) \\
&= 4(n-1)e^{\delta R}\omega_n \int_0^{r_0} \left(\int_\rho^{r_0} r^{1-n}\,dr\right) \rho^{2n-3} H(u(\rho))\,d\rho - 2\int_{B_{r_0}} H(u) \\
&= \frac{4(n-1)}{n-2} e^{\delta R}\omega_n \int_0^{r_0} \left(\rho^{2-n} - r_0^{2-n}\right) \rho^{2n-3} H(u(\rho))\,d\rho - 2\int_{B_{r_0}} H(u) \\
&\leq \frac{4(n-1)}{n-2} e^{\delta R}\omega_n \int_0^{r_0} \rho^{n-1} H(u(\rho))\,d\rho - 2\int_{B_{r_0}} H(u) \\
&= \left(\frac{4(n-1)}{n-2}e^{\delta R} - 2\right)\int_{B_{r_0}} H(u) \leq \left(\frac{4(n-1)}{n-2}e^{\delta R} - 2\right)\int_\Omega H(u) \\
&\leq \left(\frac{4(n-1)}{n-2}e^{\delta R} - 2\right)\int_\Omega (\gamma G(u) + b(u+1)) = \mu\int_\Omega G(u) + c_2(m)
\end{aligned}$$

with some $c_2(m) > 0$. Upon a combination with (5.3.30), the claim is proved. ∎

The final step towards the proof of (5.3.6) is now a combination of Lemma 5.3.3 and Lemma 5.3.4. The proof is very similar to the one given in [A5.19, Lemma 4.5], but as we have to choose some constants in a different way, we give the proof for completeness of our arguments.

Lemma 5.3.5 *Suppose that* (5.1.5) *and* (5.1.9) *are fulfilled and let* $\theta \in (\frac{1}{2}, 1)$ *and* $\mu \in (0, 2)$ *be as defined in* (5.3.5) *and* (5.3.21), *respectively. Then for any* $\varepsilon \in (0, \frac{1}{2})$ *there exists* $C(\varepsilon, m, \kappa) > 0$ *such that*

$$\begin{aligned}
\int_\Omega |\nabla v|^2 \;\leq\; & C(\varepsilon, m, \kappa) \cdot \left(1 + M^2 + B^{\frac{2n+4}{n+4}}\right) \cdot \Bigg(\Big\|\Delta v - v + u\Big\|^{2\theta}_{L^2(\Omega)} \\
&+ \Big\|\frac{\phi(u)}{\sqrt{\psi(u)}}\nabla u - \sqrt{\psi(u)}\nabla v\Big\|_{L^2(\Omega)} + 1\Bigg) \\
&+ \frac{\varepsilon}{1-2\varepsilon}\int_\Omega uv + \frac{\mu}{1-2\varepsilon}\int_\Omega G(u)
\end{aligned} \tag{5.3.31}$$

is fulfilled for all $(u, v) \in \mathcal{S}(m, M, B, \kappa)$.

Proof. We fix $\varepsilon \in (0, \frac{1}{2})$ and set $\beta := \frac{(2n+4)\kappa}{n}$ which implies $\theta = \frac{\beta}{\beta+1}$. Next we define $r_0 := \min\{\frac{R}{2}, \|f\|^{-\frac{2}{\beta+1}}_{L^2(\Omega)}\} \in (0, R)$. Hence, by Lemma 5.3.3 there is $c_1 = C_1(\varepsilon, m, \kappa) \cdot \big(1 + M^{\frac{2n+4}{n+4}} + B^{\frac{2n+4}{n+4}}\big) > 0$ such that

$$\int_{\Omega \setminus B_{r_0}} |\nabla v|^2 \leq \varepsilon\int_\Omega uv + \varepsilon\int_\Omega |\nabla v|^2 + c_1 \cdot \left(r_0^{-\beta} + \|f\|^{\frac{2n+4}{n+4}}_{L^2(\Omega)}\right). \tag{5.3.32}$$

Applying next Lemma 5.3.4, we get a constant $c_2 = c_2(m)$ such that

$$\int_{B_{r_0}} |\nabla v|^2 \le \mu \int_\Omega G(u) + c_2 \cdot \Big(r_0 \|f\|^2_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + 1 \Big). \tag{5.3.33}$$

Adding both inequalities, we deduce that

$$\begin{aligned}(1-\varepsilon)\int_\Omega |\nabla v|^2 \quad &\le \quad \varepsilon \int_\Omega uv + \mu \int_\Omega G(u) + c_1 r_0^{-\beta} + c_1 \|f\|^{\frac{2n+4}{n+4}}_{L^2(\Omega)} \\ &\quad + c_2 r_0 \|f\|^2_{L^2(\Omega)} + c_2(\|g\|_{L^2(\Omega)} + 1) + c_2 \|v\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.3.34}$$

Next, by Lemma 5.2.2 and (5.3.1) there exists $c_3 = C_3(\varepsilon, m) \cdot M^2 > 0$ such that

$$c_2 \|v\|^2_{L^2(\Omega)} \le \varepsilon \int_\Omega |\nabla v|^2 + c_3,$$

which inserted into (5.3.34) yields

$$(1-2\varepsilon)\int_\Omega |\nabla v|^2 \le \varepsilon \int_\Omega uv + \mu \int_\Omega G(u) + c_2(\|g\|_{L^2(\Omega)} + 1) + c_3 + I, \tag{5.3.35}$$

where we set

$$I := c_1 r_0^{-\beta} + c_1 \|f\|^{\frac{2n+4}{n+4}}_{L^2(\Omega)} + c_2 r_0 \|f\|^2_{L^2(\Omega)}.$$

In case of $\|f\|_{L^2(\Omega)} \le (\frac{2}{R})^{\frac{\beta+1}{2}}$, we have $r_0 = \frac{R}{2}$ and conclude that

$$I \le c_1 \cdot \Big(\frac{2}{R}\Big)^\beta + c_1 \cdot \Big(\frac{2}{R}\Big)^{\frac{\beta+1}{2}\cdot\frac{2n+4}{n+4}} + c_2 \cdot \frac{R}{2} \cdot \Big(\frac{2}{R}\Big)^{\beta+1},$$

which in conjunction with (5.3.35) proves (5.3.31) in this case.
Furthermore, in the case $\|f\|_{L^2(\Omega)} > (\frac{2}{R})^{\frac{\beta+1}{2}}$ we have $r_0 = \|f\|^{-\frac{2}{\beta+1}}_{L^2(\Omega)}$ and therefore

$$I \le c_1 \|f\|^{\frac{2\beta}{\beta+1}}_{L^2(\Omega)} + c_1 \|f\|^{\frac{2n+4}{n+4}}_{L^2(\Omega)} + c_2 \|f\|^{2-\frac{2}{\beta+1}}_{L^2(\Omega)} = (c_1 + c_2)\|f\|^{\frac{2\beta}{\beta+1}}_{L^2(\Omega)} + c_1 \|f\|^{\frac{2n+4}{n+4}}_{L^2(\Omega)}.$$

In view of $\kappa > n-2$ and $n \ge 3$, we calculate

$$\frac{\beta}{\frac{n+2}{2}} = \frac{2}{n+2} \cdot \frac{(2n+4)\kappa}{n} > \frac{4(n-2)}{n} \ge \frac{4}{3} > 1$$

which implies that $2\theta = \frac{2\beta}{\beta+1} > \frac{2n+4}{n+4}$. Applying once more Young's inequality, we obtain

$$I \le (2c_1 + c_2)\|f\|^{\frac{2\beta}{\beta+1}}_{L^2(\Omega)} + c_1,$$

which inserted into (5.3.35) proves (5.3.31) in the case $\|f\|_{L^2(\Omega)} > (\frac{2}{R})^{\frac{\beta+1}{2}}$ and thereby completes the proof. ■

Next, we complete the proof of the announced estimate (5.3.6).

Proof of Lemma 5.3.1.
Let $\mu \in (0,2)$ be as defined in Lemma 5.3.4. In view of $\mu < 2$ there exists $\eta \in (0, \frac{1}{2})$ such that $\mu(1-\eta) < 1$. Keeping this value of η fixed, we moreover fix $\varepsilon \in (0, \frac{1}{4})$ small enough such that

$$\frac{\mu(1+\varepsilon-\eta)}{1-3\varepsilon-\varepsilon^2+\varepsilon\eta} \leq 1 \quad \text{and} \qquad \frac{\eta(1-2\varepsilon)}{1-3\varepsilon-\varepsilon^2+\varepsilon\eta} \leq \frac{1}{2}. \tag{5.3.36}$$

An application of Lemma 5.3.2 implies the existence of $c_1 = C_1 \cdot (1+M^2) > 0$ such that

$$\int_\Omega uv \leq \eta \int_\Omega |\nabla v|^2 + (1+\varepsilon-\eta) \int_\Omega |\nabla v|^2 + c_1 \cdot \left(\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1 \right).$$

Furthermore, by Lemma 5.3.5 there is $c_2 = C_2(m,\kappa) \cdot \left(1 + M^2 + B^{\frac{2n+4}{n+4}}\right) > 0$ such that

$$\begin{aligned} \int_\Omega uv \quad &\leq \quad \eta \int_\Omega |\nabla v|^2 + \frac{\varepsilon(1+\varepsilon-\eta)}{1-2\varepsilon} \int_\Omega uv + \frac{\mu(1+\varepsilon-\eta)}{1-2\varepsilon} \int_\Omega G(u) \\ &\qquad + c_2(1+\varepsilon-\eta) \cdot \left(\|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)} + 1 \right) + c_1 \cdot \left(\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1 \right). \end{aligned}$$

A rearrangement of the terms yields

$$\begin{aligned} \int_\Omega uv \quad &\leq \quad \frac{\eta(1-2\varepsilon)}{1-3\varepsilon-\varepsilon^2+\varepsilon\eta} \int_\Omega |\nabla v|^2 + \frac{\mu(1+\varepsilon-\eta)}{1-3\varepsilon-\varepsilon^2+\varepsilon\eta} \int_\Omega G(u) \\ &\qquad + c_3 \cdot \left(\|f\|_{L^2(\Omega)}^{2\theta} + \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + \|g\|_{L^2(\Omega)} + 1 \right) \end{aligned}$$

with some $c_3 = C_3(m,\kappa) \cdot \left(1 + M^2 + B^{\frac{2n+4}{n+4}}\right) > 0$. As $\frac{2n+4}{n+4} < 2\theta$ (which has been shown in Lemma 5.3.5), a further application of the Young inequality along with (5.3.36) implies (5.3.6). ■

The final result of this section is to show that the Liapunov functional $\mathcal{F}$ can be estimated according to (5.3.4). The proof uses the idea of [A5.19, Theorem 5.1] as a basic ingredient, but in fact our estimates also make use of the other terms which are contained in $\mathcal{F}$.

Theorem 5.3.6 *Assume that* (5.1.5) *and* (5.1.9) *are satisfied and let* $\theta \in (\frac{1}{2}, 1)$ *be as defined in* (5.3.5). *Then there exists* $C(m,\kappa) > 0$ *such that*

$$\mathcal{F}(u,v) \geq -C(m,\kappa) \cdot \left(1 + M^2 + B^{\frac{2n+4}{n+4}}\right) \cdot \left(\mathcal{D}^\theta(u,v) + 1\right) \tag{5.3.37}$$

is fulfilled for all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$, *where* $\mathcal{F}$ *and* $\mathcal{D}$ *are given in* (5.1.6) *and* (5.1.7), *respectively.*

Proof. In view of (5.3.7), (5.3.8), and $\theta > \frac{1}{2}$, an application of Young's inequality to (5.3.6) implies the existence of $c_1 = C_1(m,\kappa)\cdot\left(1+M^2+B^{\frac{2n+4}{n+4}}\right) > 0$ such that

$$\int_\Omega uv \le c_1\left(\left(\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)}\right)^\theta+1\right)+\frac{1}{2}\int_\Omega|\nabla v|^2+\int_\Omega G(u).$$

Hence, we conclude that

$$\begin{aligned}\mathcal{F}(u,v) &= \frac{1}{2}\int_\Omega|\nabla v|^2+\frac{1}{2}\int_\Omega v^2-\int_\Omega uv+\int_\Omega G(u)\\ &\ge -c_1\cdot\left((\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)})^\theta+1\right).\end{aligned}$$

As (5.1.7), (5.3.7), and (5.3.8) imply $\mathcal{D}(u,v)=\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)}$, the proof is complete. ∎

5.4 Finite-time blowup: proof of the main results

In view of Theorem 5.3.6 and $\theta \in (0,1)$, we derive an ODI for the function $y(t) := -\mathcal{F}(u(\cdot,t),v(\cdot,t))$ with superlinear nonlinearity. This shows that the solution (u,v) blows up in finite time if $-\mathcal{F}(u_0,v_0)$ is large. The following result and its proof are completely the same as in [A5.19, Lemma 5.2], so that we confine ourselves to giving only a sketch of the main ideas of the proof.

Lemma 5.4.1 *Suppose that* (5.1.5) *and* (5.1.9) *are fulfilled, let* $\theta \in (\frac{1}{2},1)$ *be as defined in* (5.3.5) *and let* $m>0, A>0$ *and* $\kappa > n-2$. *Then there exist* $K=K(m,A,\kappa)=k(m,\kappa)\cdot(1+A^2)>0$ *and* $C=C(m,A,\kappa)>0$ *such that for any*

$$(u_0,v_0)\in\widetilde{\mathcal{B}}(m,A,\kappa) := \Big\{(u_0,v_0)\in C^0(\bar\Omega)\times W^{1,\infty}(\Omega)\ \Big|\ u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \bar\Omega,\ \textstyle\int_\Omega u_0 = m,\ \|v_0\|_{W^{1,2}(\Omega)}\le A, \text{ and } \mathcal{F}(u_0,v_0)\le -K\Big\} \tag{5.4.1}$$

the corresponding solution (u,v) *of* (5.1.1) *satisfies*

$$\mathcal{F}(u(\cdot,t),v(\cdot,t)) \le \frac{\mathcal{F}(u_0,v_0)}{(1-Ct)^{\frac{\theta}{1-\theta}}} \qquad \text{for all } t\in(0,T_{max}(u_0,v_0)). \tag{5.4.2}$$

In particular, (u,v) *blows up in finite time* $T_{max}(u_0,v_0)\le\frac{1}{C}$.

Proof. We only give a sketch of the main ideas and refer to [A5.19, Lemma 5.2] for further details.
We fix $c_1>0$ such that

$$\|\varphi\|_{L^1(\Omega)}\le c_1\|\varphi\|_{W^{1,2}(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega).$$

Moreover, in view of $\kappa > n-2$ and Lemma 5.2.3, there is $c_2 = c_2(\kappa) > 0$ such that for any $(u_0, v_0) \in \widetilde{\mathcal{B}}(m, A, \kappa)$ the solution (u, v) to (5.1.1) fulfills

$$v(r,t) \le c_2 \cdot \Big(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)}\Big) \cdot r^{-\kappa} \tag{5.4.3}$$

for all $(r,t) \in (0,R) \times (0, T_{max}(u_0,v_0))$. Setting $B := c_2(m + c_1A + A)$ and $M := \max\{m, c_1A\}$, Lemma 5.2.1 and (5.4.3) imply that $(u(\cdot,t), v(\cdot,t)) \in \mathcal{S}(m, M, B, \kappa)$ for all $t \in (0, T_{max}(u_0,v_0))$ provided that $(u_0,v_0) \in \widetilde{\mathcal{B}}(m,A,\kappa)$. In view of Theorem 5.3.6 and our definition of B and M, there is a constant $c_3 = C_3(m,\kappa) \cdot \big(1 + A^2\big)$ such that

$$\mathcal{F}(u(\cdot,t), v(\cdot,t)) \ge -c_3 \cdot \Big(\mathcal{D}^\theta(u(\cdot,t), v(\cdot,t)) + 1\Big) \tag{5.4.4}$$

is satisfied for all $t \in (0, T_{max}(u_0,v_0))$ provided that $(u_0,v_0) \in \widetilde{\mathcal{B}}(m,A,\kappa)$. Hence, we set $K(m,A,\kappa) = 2c_3$, $C(m,A,\kappa) = \frac{1-\theta}{2c_3\theta}$, and $y(t) := -\mathcal{F}(u(\cdot,t), v(\cdot,t))$, $t \in [0, T_{max}(u_0,v_0))$, for $(u_0,v_0) \in \widetilde{\mathcal{B}}(m,A,\kappa)$. As y is nondecreasing by (5.1.8) and therefore satisfies $y(t) \ge 2c_3$ for $t \in (0, T_{max}(u_0,v_0))$, (5.4.4) and (5.1.8) imply

$$y'(t) \ge \left(\frac{y(t)}{2c_3}\right)^{\frac{1}{\theta}} \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)),$$

which implies (5.4.2). ∎

The proof of Theorem 5.1.1 is now immediate.

Proof of Theorem 5.1.1.
We fix an arbitrary $\kappa > n-2$. Then the claim directly follows from Lemma 5.4.1 by defining $K(m) := k(m,\kappa)$ and $T(m,A) := \frac{1}{C(m,A,\kappa)}$, where $k(m,\kappa)$ and $C(m,A,\kappa)$ are provided in Lemma 5.4.1. ∎

Let us next show that the set $\mathcal{B}(m,A)$ defined in (5.1.10) has the properties claimed in Theorem 5.1.2. Since the condition

$$\mathcal{F}(u_0,v_0) \le -K(m) \cdot (1 + A^\tau) \tag{5.4.5}$$

in (5.1.10) is given with $\tau = 2$, we can use the functions constructed in [A5.18, Lemma 4.1] to deduce that $\mathcal{B}(m,A) \ne \emptyset$ without any additional restriction on α (which is given in (5.1.4)). In case of $\tau > 2$, this is not possible. Moreover, as (5.4.5) cannot be imposed for $\tau < 2$ in view of the Liapunov functional $\mathcal{F}$, the condition (5.4.5) with $\tau = 2$ seems to be optimal for defining $\mathcal{B}(m,A)$.

Proof of Theorem 5.1.2.
Part (ii) of the claim immediately follows from [A5.19, Lemma 6.1]. In fact, given $m > 0$, $p \in (1, \frac{2n}{n+2})$ as well as radial and positive functions $u \in C^0(\bar{\Omega})$ and $v \in W^{1,\infty}(\Omega)$ with

$\int_\Omega u = m$, sequences $(u_k)_{k\in\mathbb{N}} \subset C^0(\bar\Omega)$ and $(v_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ of radially symmetric positive functions with $\int_\Omega u_k = m$ for all $k \in \mathbb{N}$ are constructed, which satisfy

$$u_k \to u \quad \text{in } L^p(\Omega), \quad v_k \to v \quad \text{in } W^{1,2}(\Omega), \quad \text{and} \quad \int_\Omega u_k v_k \to \infty \qquad \text{as } k \to \infty. \tag{5.4.6}$$

Combining this with (5.1.4) and our additional condition $p > 2 - \alpha$, we find some $C > 0$ such that

$$\frac{1}{2}\int_\Omega |\nabla v_k|^2 + \frac{1}{2}\int_\Omega v_k^2 + \int_\Omega G(u_k) \le C \qquad \text{for all } k \in \mathbb{N}.$$

Thus, (5.4.6) implies $\mathcal{F}(u_k, v_k) \to -\infty$ as $k \to \infty$ which proves part (ii) of the claim.

In view of part (ii), it is sufficient to prove part(i) of the claim in the case $\alpha \in (\frac{2}{n}, 1)$. To this end we notice that, given $m > 0$ and

$$\gamma_2 \in ((1-\alpha)n, n-2), \tag{5.4.7}$$

by [A5.18, Lemma 4.1] there exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ there are radial and positive functions $u_\eta, v_\eta \in C^\infty(\bar\Omega)$ with $\int_\Omega u_\eta = m$ satisfying

$$\int_\Omega |\nabla v_\eta|^2 \le c_1 \eta^{-(-n+2\gamma_2+2)}, \quad \int_\Omega v_\eta^2 \le c_1 \eta^{-(-n+2\gamma_2)},$$
$$\int_\Omega G(u_\eta) \le c_1 \eta^{-(1-\alpha)n}, \quad \int_\Omega u_\eta v_\eta \ge c_2 \eta^{-\gamma_2}$$

for all $\eta \in (0, \eta_0)$ with positive constants c_1 and c_2. Hence, (5.4.7) implies that there are $c_3, c_4 > 0$ and $\eta_1 \in (0, \eta_0)$ such that

$$\|v_\eta\|_{W^{1,2}(\Omega)} \le A_\eta := c_3 \eta^{-(\gamma_2+1-\frac{n}{2})} \qquad \text{and} \qquad \mathcal{F}(u_\eta, v_\eta) \le -c_4 \eta^{-\gamma_2} \qquad \text{for all } \eta \in (0, \eta_1)$$

are fulfilled. Since $\gamma_2 < n-2$ implies $\gamma_2 > 2(\gamma_2 + 1 - \frac{n}{2})$, we conclude that there exist $\eta_2 \in (0, \eta_1)$ and $c_5 > 0$ such that

$$\mathcal{F}(u_\eta, v_\eta) \le -K(m)\left(1 + A_\eta^2\right) \qquad \text{for all } \eta \in (0, \eta_2).$$

Hence, $(u_\eta, v_\eta) \in \mathcal{B}(m, A_\eta)$ for η small enough. ∎

5.5 Unbounded global-in-time solutions

The last section is devoted to the proof of Theorem 5.1.6. To this end we provide the following lemma.

Lemma 5.5.1 *Let $\Omega \subset \mathbb{R}^n$ with some $n \ge 2$. Moreover assume that* (5.1.11) *and* (5.1.12) *are satisfied. Then there exists $p > n$ such that for any solution (u, v) to* (5.1.1) *and any $T \in (0, \infty)$ with $T \le T_{max}(u_0, v_0)$ there is $C > 0$ such that u admits the estimate*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le C, \qquad t \in \left(\frac{T}{2}, T\right). \tag{5.5.1}$$

Before proving the above lemma let us show how to infer Theorem 5.1.6 from it.
Proof of Theorem 5.1.6.
We fix $T \in (0, \infty)$ with $T \le T_{max}(u_0, v_0)$ and first use the second equation in (5.1.1). By a standard regularity result in the theory of parabolic equations, see [A5.10, Lemma 4.1] for example, (5.5.1) yields a uniform estimate of the L^∞-norm of ∇v on $(\frac{T}{2}, T)$. Then by (5.1.11) and [A5.3, Theorem 2.2] we arrive at the uniform estimate of $\|u\|_{L^\infty(\Omega)}$ on $(\frac{T}{2}, T)$. Hence, in view of Lemma 5.2.1, we have shown the existence of a global-in-time solution to (5.1.1) whatever initial data we start with. On the other hand choosing $\Omega = B_R$ and radially symmetric initial data, since (5.1.5) and (5.1.4) are satisfied, we conclude with the use of [A5.18, Theorem 5.1] that the solutions we arrived at are unbounded. ■

Next we complete this section by proving Lemma 5.5.1.
Proof of Lemma 5.5.1.
Multiplying the first equation of (5.1.1) by u^{p-1}, $p \in (n, \gamma_1]$, and the second one by Δv, we arrive at

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + (p-1)\int_\Omega \phi(u)\,|\nabla u|^2\,u^{p-2}dx = (p-1)\int_\Omega u^{p-1}\beta(u)\nabla v\nabla u\,dx, \qquad (5.5.2)$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |\Delta v|^2\,dx + \int_\Omega |\nabla v|^2 dx \le \frac{1}{2}\int_\Omega u^2 dx. \qquad (5.5.3)$$

Since

$$u^{p-1}\beta(u) = u^{\frac{p-2}{2}} u^{\frac{p}{2}} \sqrt{\beta(u)}\sqrt{\beta(u)},$$

in view of (5.1.11) we infer from (5.5.2) that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{p-1}{2}\int_\Omega \phi(u)\,|\nabla u|^2\,u^{p-2}dx \le C\int_\Omega u^p\beta(u)|\nabla v|^2 dx. \qquad (5.5.4)$$

Next adding (5.5.4) and (5.5.3) and applying (5.1.12) we arrive at

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u^p dx + \int_\Omega |\nabla v|^2 dx\right) &\le C\left(\int_\Omega u^p dx\right)^{\frac{2}{p}} + C\int_\Omega |\nabla v|^2 \\ &\le C\left(\int_\Omega u^p dx + \int_\Omega |\nabla v|^2 dx + 1\right), \qquad (5.5.5)\end{aligned}$$

which in turn, by Grönwall's lemma, yields the claimed estimate of $\|u\|_{L^p(\Omega)}$. ■

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Article 6: Competitive exclusion in a two-species chemotaxis model

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Abstract

We consider a mathematical model for the spatio-temporal evolution of two biological species in a competitive situation. Besides diffusing, both species move toward higher concentrations of a chemical substance which is produced by themselves. The resulting system consists of two parabolic equations with Lotka-Volterra-type kinetic terms and chemotactic cross-diffusion, along with an elliptic equation describing the behavior of the chemical.
We study the question in how far the phenomenon of competitive exclusion occurs in such a context. We identify parameter regimes for which indeed one of the species dies out asymptotically, whereas the other reaches its carrying capacity in the large time limit.

Key words: chemotaxis, stability of solutions, asymptotic behavior, competitive exclusion
MSC 2010: 92C17, 35K55, 35B35, 35B40

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6.1 Introduction

We consider two biological species which compete for the resources and migrate towards a higher concentration of a chemical produced by themselves. Here the movement of the two populations is governed by diffusion and chemotaxis. We further assume that the populations proliferate, that the mutual competition between them takes place according to the classical Lotka-Volterra dynamics and that the chemical signal diffuses much faster than the two populations. Denoting the population densities by $u(x,t)$ and $v(x,t)$ and the concentration of the chemoattractant by $w(x,t)$, classical models (see [A6.11]) lead to the system

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ t>0,\\ -\Delta w + \lambda w = ku+v, & x\in\Omega,\ t>0, \end{cases} \tag{6.1.1}$$

under homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t>0, \tag{6.1.2}$$

and initial conditions

$$u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), \qquad x\in\Omega, \tag{6.1.3}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 1$, with smooth boundary, where d_i, χ_i, μ_i, a_i $(i=1,2)$, λ and k are positive parameters.

In order to describe the competition of two species, the associated Lotka-Volterra ODE system

$$\begin{cases} u' = \mu_1 u(1-u-a_1 v), & t>0,\\ v' = \mu_2 v(1-v-a_2 u), & t>0, \end{cases}$$

has been studied extensively. It is well-known that if

$$a_1 > 1 > a_2 \geq 0 \tag{6.1.4}$$

and both species are initially positive then the second population outcompetes the first in the sense that $u(t)\to 0$ and $v(t)\to 1$ as $t\to\infty$. A proof of this result and of extensions to systems with more populations is given in [A6.23, Theorem 2.1]. It is the objective of the present work to investigate in how far this phenomenon, usually referred to as *competitive exclusion*, can be observed also in cases when both species move toward increasing concentrations of a signal which they produce themselves.
The influence of chemotaxis on the dynamics of biological species competing for resources like nutrients or space is for instance pointed out in [A6.2, A6.6, A6.7, A6.17]. Particular fields of relevance include economically important situations when different bacteria interact with crop plants, where beyond standard kinetics, the respective overall competitive

fitnesses are crucially affected by chemotaxis and motility, see [A6.1, A6.18, A6.22]. Derivations of related mathematical models can be found in [A6.8, A6.10, A6.13] and some basic mathematical aspects such as the global existence of solutions to models which involve both chemotaxis and competition are addressed in [A6.9, A6.24]. Moreover, for some particular models the existence and stability of steady states reflecting either competitive exclusion or coexistence have already been studied analytically, see [A6.3, A6.4, A6.19, A6.24]; however, to the best of our knowledge the literature does not provide any qualitative information on the solution behavior in the context of competitive exclusion when chemotaxis as well as competitive terms involving both species are present.

Concerning the problem considered in this paper, in case of $a_1, a_2 \in [0,1)$ it has been shown in [A6.16] that (6.1.1)-(6.1.3) possesses a unique positive steady state and conditions on the parameters μ_i and χ_i are established which ensure its global asymptotic stability. In contrast to this result of coexistence of the species we shall show here that in presence of (6.1.4) competitive exclusion will take place, provided that the influence of chemotaxis is sufficiently small.

In order to state our results in this direction, let us introduce the ratios

$$q_1 := \frac{\chi_1}{\mu_1} \qquad \text{and} \qquad q_2 := \frac{\chi_2}{\mu_2}. \tag{6.1.5}$$

It turns out that in our analysis, besides the number k these parameters will play the role of key parameters with regard to the effect in question. In particular, we shall see that if both q_1 and q_2 are sufficiently small then competitive exclusion occurs for any solution (u, v, w) with $v \not\equiv 0$.

More precisely, in addition to (6.1.4) our overall assumptions are

$$\begin{aligned} &k, q_1 \text{ and } q_2 \text{ are nonnegative and such that } q_1 \le a_1, q_2 < \tfrac{1}{2} \text{ and} \\ &kq_1 + \max\left\{q_2\ ,\ \frac{a_2 - a_2 q_2}{1-2q_2}\ ,\ \frac{kq_2 - a_2 q_2}{1-2q_2}\right\} < 1. \end{aligned} \tag{6.1.6}$$

Observe that these can be rewritten in separate conditions for k, q_2 and q_1 in such a way that we require

$$\begin{aligned} &k \ge 0, \\ &q_2 \in [0, \tfrac{1}{2}) \text{ is such that } q_2 < \begin{cases} \frac{1-a_2}{2-a_2} & \text{if } k \le \frac{a_2(2-a_2)}{1-a_2}, \\ \frac{1}{2-a_2+k} & \text{if } k > \frac{a_2(2-a_2)}{1-a_2}, \end{cases} \\ &q_1 \in [0, a_1] \text{ satisfies } kq_1 < 1 - \max\left\{q_2\ ,\ \frac{a_2 - a_2 q_2}{1-2q_2}\ ,\ \frac{kq_2 - a_2 q_2}{1-2q_2}\right\}. \end{aligned} \tag{6.1.7}$$

Here, the latter hypothesis (6.1.7) itself is equivalent to saying that $kq_1 + q_2 < 1$ and

$$\begin{cases} kq_1 + (2-a_2)q_2 + a_2 - 2kq_1q_2 < 1 & \text{if } kq_2 < a_2, \\ kq_1 + (2-a_2+k)q_2 - 2kq_1q_2 < 1 & \text{if } kq_2 \ge a_2. \end{cases} \tag{6.1.8}$$

Prescribing the above conditions, we obtain the following main result on competitive exclusion.

Theorem 6.1.1 *Assume* (6.1.4), *and suppose that* k *and the numbers* q_1 *and* q_2 *defined in* (6.1.5) *satisfy* (6.1.6). *Then for any choice of nonnegative initial data* $u_0 \in C^0(\bar{\Omega})$ *and* $v_0 \in C^0(\bar{\Omega})$ *satisfying* $v_0 \not\equiv 0$, *the problem* (6.1.1)-(6.1.3) *possesses a uniquely determined global-in-time classical solution* (u, v, w) *such that* $u \geq 0, v > 0$ *and* $w > 0$ *in* $\bar{\Omega} \times (0, \infty)$ *and*

$$u(\cdot, t) \to 0, \quad v(\cdot, t) \to 1 \quad \text{and} \quad w(\cdot, t) \to \frac{1}{\lambda} \qquad \text{as } t \to \infty, \tag{6.1.9}$$

uniformly with respect to $x \in \Omega$. *Moreover, either* $u \equiv 0$ *in* $\bar{\Omega} \times [0, \infty)$ *or* $u > 0$ *in* $\bar{\Omega} \times (0, \infty)$ *is satisfied.*

Let us illustrate how the condition (6.1.6) becomes easier to handle in some special cases.

Remark 6.1.2 *i) In the prototypical case when* $\chi_1 = \chi_2 \equiv \chi$ *and* $\mu_1 = \mu_2 \equiv \mu$, (6.1.6) *reduces to the condition that* $q := \frac{\chi}{\mu}$ *satisfies* $q < \frac{1}{k+1}$ *and*

$$q < \begin{cases} \frac{2+k-a_2-\sqrt{(k+2-a_2)^2-8k(1-a_2)}}{4k} & \text{if } a_2 > kq \\ \frac{2+2k-a_2-\sqrt{(2k+2-a_2)^2-8k}}{4k} & \text{if } a_2 \leq kq. \end{cases} \tag{6.1.10}$$

ii) If in the above case we moreover have $k = 1$ *then* (6.1.10) *becomes*

$$q < \begin{cases} \frac{4-a_2-\sqrt{8-8a_2+a_2^2}}{4} & \text{if } a_2 \leq q \\ \frac{1-a_2}{2} & \text{if } a_2 > q. \end{cases} \tag{6.1.11}$$

We observe that the first case can only occur if $a_2 < \frac{4-a_2-\sqrt{8-8a_2+a_2^2}}{4}$ *is satisfied, which is equivalent to* $a_2 < \frac{1}{3}$ *in view of* $a_2 \in [0, 1)$. *Hence, the first case in* (6.1.11) *is equivalent to*

$$a_2 < \frac{1}{3} \quad \text{and} \quad a_2 \leq q < \frac{4 - a_2 - \sqrt{8 - 8a_2 + a_2^2}}{4}.$$

The second case in (6.1.11) *is equivalent to*

$$q < \min\left\{a_2, \frac{1-a_2}{2}\right\} = \begin{cases} a_2 & \text{if } a_2 < \frac{1}{3} \\ \frac{1-a_2}{2} & \text{if } a_2 \in [\frac{1}{3}, 1). \end{cases}$$

Combining both cases we conclude

$$q < \begin{cases} \frac{4-a_2-\sqrt{8-8a_2+a_2^2}}{4} & \text{if } a_2 < \frac{1}{3} \\ \frac{1-a_2}{2} & \text{if } a_2 \in \left[\frac{1}{3}, 1\right). \end{cases} \tag{6.1.12}$$

iii) In the limit case $k = 0$, (6.1.10) *requires that*

$$q < \frac{1 - a_2}{2 - a_2} \tag{6.1.13}$$

iv) Finally, in the borderline case $a_2 = 0$, (6.1.13) reads

$$q < \frac{1}{2} \tag{6.1.14}$$

and is thus consistent with the conditions already found in [A6.15, Theorem 5.1].

Remark 6.1.3 *The global existence statement in Theorem 6.1.1 remains valid if* (6.1.6) *is replaced with the weaker requirement that $kq_1 + q_2 < 1$. In fact, Lemma 6.2.2 below will show that in this case the interplay of diffusion and kinetics in* (6.1.1) *is strong enough to overbalance chemotactic cross-diffusion in such a way that all solutions are global and remain bounded.*

The plan of this paper is as follows. In Section 6.2 we show the local existence of a solution along with its positivity properties and prove the existence of a global bounded solution once $kq_1 + q_2 < 1$ is satisfied. Section 6.3 contains relations between the possible limits of u and v which are established by using comparison methods in combination with some algebraic inequalities. In particular we show that $v(t) \to 1$ if $u(t) \to 0$ is satisfied. In Section 6.4 we then prove that u converges to 0 in the cases $kq_2 < a_2$ and $kq_2 \geq a_2$, respectively, and complete the proof of Theorem 6.1.1. The final Section 6.5 contains our conclusions and a discussion.

6.2 Preliminaries: boundedness

In this section we state some basic properties of the solutions to (6.1.1)-(6.1.3) and give a criterion for their boundedness. We start with the local existence of a solution and its positivity properties.

Lemma 6.2.1 *Suppose that $u_0, v_0 \in C^0(\bar{\Omega})$ are nonnegative such that $v_0 \not\equiv 0$. Then there exists $T_{max} \in (0, \infty]$ and a unique classical solution (u, v, w) of* (6.1.1)-(6.1.3) *which is nonnegative and belongs to $C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$. Moreover, $v > 0$ and $w > 0$ in $\bar{\Omega} \times (0, T_{max})$ and either $u \equiv 0$ in $\bar{\Omega} \times [0, T_{max})$ or $u > 0$ in $\bar{\Omega} \times (0, T_{max})$ are satisfied. Furthermore, we have the following extensibility criterion:*

$$\text{If } T_{max} < \infty, \text{ then } \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{6.2.1}$$

Proof. The local existence and regularity of the solution as well as the extensibility criterion (6.2.1) can be proved by a slight adaption of well-known methods. We thus may confine ourselves with an outline of the proof and refer the reader e.g. to [A6.20], where details are given in a closely related situation.
For small $T \in (0, 1)$, in the space

$$X := C^0([0,T]; C^0(\bar{\Omega})) \times C^0([0,T]; C^0(\bar{\Omega}))$$

we consider the closed set

$$S := \Big\{ (u,v) \in X \ \Big| \ \|u\|_{L^\infty((0,T);L^\infty(\Omega))} \leq R+1 \text{ and } \|v\|_{L^\infty((0,T);L^\infty(\Omega))} \leq R+1 \Big\},$$

where $R := \|u_0 + v_0\|_{L^\infty(\Omega)}$. For $(u,v) \in S$, we introduce a mapping Φ on S by letting $w \in \bigcap_{1<p<\infty} L^\infty((0,T); W^{2,p}(\Omega))$ denote the (weak) solution of

$$\begin{cases} -\Delta w + \lambda w = ku + v, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6.2.2}$$

and then defining

$$\begin{aligned} \Phi(u,v)(t) &:= \begin{pmatrix} \Phi_1(u,v)(t) \\ \Phi_2(u,v)(t) \end{pmatrix} \\ &:= \begin{pmatrix} e^{d_1 t\Delta} u_0 + \int_0^t e^{d_1(t-s)\Delta} \Big[-\chi_1 \nabla \cdot (u\nabla w) + f_1(u,v) \Big](s) ds \\ e^{d_2 t\Delta} v_0 + \int_0^t e^{d_2(t-s)\Delta} \Big[-\chi_2 \nabla \cdot (v\nabla w) + f_2(u,v) \Big](s) ds \end{pmatrix} \end{aligned}$$

for $t \in [0,T]$, where $(e^{\tau\Delta})_{\tau \geq 0}$ denotes the Neumann heat semigroup, and where

$$f_1(u,v) := \mu_1 u(1-u-a_1 v) \quad \text{and} \quad f_2(u,v) := \mu_2 v(1-v-a_2 u), \qquad u \in \mathbb{R},\ v \in \mathbb{R}.$$

Then by a straightforward reasoning involving standard elliptic regularity properties and known smoothing estimates for the heat semigroup ([A6.14]), it is possible to show that if $T = T(R)$ is sufficiently small then Φ is a contraction on S. The accordingly existing fixed point (u,v) of Φ, along with w as gained from (6.2.2), can then, again by standard regularity arguments, shown to be smooth in $\bar\Omega \times (0,T)$ and continuous in $\bar\Omega \times [0,T]$ in all its components, and to solve (6.1.1) classically in $\Omega \times (0,T)$. Since the choice of T depends on R only, (6.2.1) is now immediate.
An application of the strong maximum principle to the first and second equation of (6.1.1) implies the claim concerning the positivity of u and v. Hence, $ku + v$ is positive in $\bar\Omega \times (0, T_{max})$ and the strong elliptic maximum principle applied to the third equation of (6.1.1) yields positivity also of w.
Finally, taking differences $U := u_1 - u_2$ and $V := v_1 - v_2$ of two supposedly existing solutions (u_i, v_i, w_i) in $\Omega \times (0,T)$ for some $T > 0$, $i \in \{1,2\}$, upon testing the equations for U and V obtained from (6.1.1) by U and V, respectively, in a straightforward manner one can derive an inequality of the form

$$\frac{1}{2}\frac{d}{dt}\Big\{ \int_\Omega U^2 + \int_\Omega V^2 \Big\} \leq C(T')\Big\{ \int_\Omega U^2 + \int_\Omega V^2 \Big\} \qquad \text{for all } t \in (0,T'),$$

valid for any fixed $T' \in (0,T)$ and some $C(T') > 0$ depending on the bounded quantities $\|u_i\|_{L^\infty(\Omega\times(0,T'))}$ and $\|v_i\|_{L^\infty(\Omega\times(0,T'))}$, $i \in \{1,2\}$. This clearly implies uniqueness. ∎

We now let $\mathcal{L}_j = \mathcal{L}_j(x,t)$, $j \in \{1,2\}$, the parabolic operators

$$\mathcal{L}_j \varphi := d_j \Delta\varphi - \chi_j \nabla w(x,t) \cdot \nabla\varphi, \qquad (x,t) \in \Omega \times (0, T_{max}), \tag{6.2.3}$$

for $\varphi \in C^2(\Omega)$. Then the first and third equation of (6.1.1) show that

$$u_t - \mathcal{L}_1 u \;=\; u \cdot \Big\{ -\chi_1 \Delta w + \mu_1(1-u-a_1 v) \Big\}$$

$$= \quad u \cdot \Big\{\mu_1 - (\mu_1 - k\chi_1)u - (a_1\mu_1 - \chi_1)v - \lambda\chi_1 w\Big\} \qquad (6.2.4)$$

in $\Omega \times (0, T_{max})$. Similarly, the second and third equation of (6.1.1) imply

$$\begin{aligned} v_t - \mathcal{L}_2 v &= v \cdot \Big\{ - \chi_2 \Delta w + \mu_2(1 - v - a_2 u)\Big\} \\ &= v \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)v + (k\chi_2 - a_2\mu_2)u - \lambda\chi_2 w\Big\} \end{aligned} \qquad (6.2.5)$$

in $\Omega \times (0, T_{max})$. The final result of this section asserts boundedness of the solution once the ratios q_1 and q_2 defined in (6.1.5) are small enough.

Lemma 6.2.2 *Assume that*

$$kq_1 + q_2 < 1. \qquad (6.2.6)$$

Then $T_{max} = \infty$ and both u and v are bounded in $\Omega \times (0, \infty)$.

Proof. According to the fact that u, v and w are all nonnegative by Lemma 6.2.1, we have

$$\begin{aligned} &\mathcal{P}_1 u := u_t - \mathcal{L}_1 u - u \cdot \Big\{\mu_1 - (\mu_1 - k\chi_1)u + \chi_1 v\Big\} \le 0 \quad \text{in } \Omega \times (0, T_{max}) \qquad \text{and} \\ &\mathcal{P}_2 v := v_t - \mathcal{L}_2 v - v \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)v + k\chi_2 u\Big\} \le 0 \quad \text{in } \Omega \times (0, T_{max}), \end{aligned} \qquad (6.2.7)$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (6.2.3). We now observe that (6.2.6) is equivalent to

$$(\mu_1 - k\chi_1)(\mu_2 - \chi_2) > k\chi_1\chi_2$$

and hence to

$$\frac{\mu_1 - k\chi_1}{\chi_1} > \frac{k\chi_2}{\mu_2 - \chi_2}.$$

We can thus pick $\xi > 0$ large enough such that

$$\xi \ge \max\Big\{\|u_0\|_{L^\infty(\Omega)} \ , \ \frac{\mu_2 - \chi_2}{k\chi_2}\|v_0\|_{L^\infty(\Omega)}\Big\} \qquad (6.2.8)$$

and that

$$\frac{\mu_1 - k\chi_1 - \frac{\mu_1}{\xi}}{\chi_1} > \frac{k\chi_2 + \frac{\mu_2}{\xi}}{\mu_2 - \chi_2},$$

which enables us to find $A > 0$ fulfilling

$$\frac{\mu_1 - k\chi_1 - \frac{\mu_1}{\xi}}{\chi_1} > A > \frac{k\chi_2 + \frac{\mu_2}{\xi}}{\mu_2 - \chi_2}. \qquad (6.2.9)$$

Then the constant functions defined by

$$\overline{u}(x,t) := \xi \quad \text{and} \quad \overline{v}(x,t) := A\xi, \qquad (x,t) \in \bar{\Omega} \times [0, T_{max}),$$

satisfy

$$\overline{u}(x,0)=\xi\geq u_0(x) \quad \text{and} \quad \overline{v}(x,0)=A\xi>\frac{k\chi_2}{\mu_2-\chi_2}\cdot\xi\geq v_0(x) \qquad \text{for all } x\in\Omega \quad (6.2.10)$$

by (6.2.8). Moreover, (6.2.9) warrants that

$$\mathcal{P}_1\overline{u}=-\xi\cdot\Big\{\mu_1-(\mu_1-k\chi_1)\xi+\chi_1\cdot A\xi\Big\}>0 \qquad \text{in } \Omega\times(0,T_{max})$$

and

$$\mathcal{P}_2\overline{v}=-A\xi\cdot\Big\{\mu_2-(\mu_2-\chi_2)\cdot A\xi+k\chi_2\xi\Big\}>0 \qquad \text{in } \Omega\times(0,T_{max}).$$

In view of (6.2.7) and (6.2.10), the comparison principle for cooperative reaction-diffusion systems (see for instance [A6.14, Proposition 52.22]) allows us to conclude that $u\leq\overline{u}$ and $v\leq\overline{v}$ in $\Omega\times(0,T_{max})$, which by Lemma 6.2.1 entails that $T_{max}=\infty$ and that u and v are globally bounded. ■

6.3 Some technical inequalities

According to the above boundedness result, under the assumption (6.1.6) we know that

$$\begin{aligned} L_1 &:= \limsup_{t\to\infty}\Big(\max_{x\in\bar\Omega}u(x,t)\Big), \\ L_2 &:= \limsup_{t\to\infty}\Big(\max_{x\in\bar\Omega}v(x,t)\Big), \qquad \text{and} \\ l_2 &:= \liminf_{t\to\infty}\Big(\min_{x\in\bar\Omega}v(x,t)\Big) \end{aligned} \qquad (6.3.1)$$

define finite real numbers satisfying

$$L_1\geq 0 \qquad \text{and} \qquad 0\leq l_2\leq L_2.$$

Proving Theorem 6.1.1 then amounts to verifying that $L_1=0$ and $L_2=l_2=1$, because the large time behavior of w is then uniquely determined according to the following.

Lemma 6.3.1 *For each $t\in(0,T_{max})$, we have*

$$\min_{y\in\bar\Omega}v(y,t)\leq\lambda w(x,t)\leq k\cdot\max_{y\in\bar\Omega}u(y,t)+\max_{y\in\bar\Omega}v(y,t) \qquad \textit{for all } x\in\bar\Omega. \qquad (6.3.2)$$

Proof. The proof repeats a standard elliptic comparison argument: If $\varphi\in C^2(\bar\Omega)$ denotes an arbitrary function satisfying $\frac{\partial\varphi}{\partial\nu}<0$ on $\partial\Omega$, then for any $\varepsilon>0$, at each point $x_0\in\bar\Omega$ where $z:=w(\cdot,t)+\varepsilon\varphi$ attains its maximum we necessarily have $x_0\in\Omega$ and hence $\Delta z(x_0)\leq 0$. Since $\Delta z=\lambda z-ku-v+\varepsilon(\Delta\varphi-\lambda\varphi)$, this implies that

$$\lambda z(x)\leq\lambda z(x_0) \quad \leq \quad ku(x_0,t)+v(x_0,t)-\varepsilon(\Delta\varphi-\lambda\varphi)(x_0)$$

$$\leq \quad k \cdot \max_{y\in\bar\Omega} u(y,t) + \max_{y\in\bar\Omega} v(y,t) + \varepsilon \cdot \max_{y\in\bar\Omega} |\Delta\varphi(y) - \lambda\varphi(y)|.$$

Taking $\varepsilon \searrow 0$ we arrive at the right inequality in (6.3.2), whereas the left can be seen similarly on dropping the nonnegative term $k \cdot \min_{y\in\bar\Omega} u(y,t)$. ∎

A first trivial observation linking the asymptotic of (u, v, w) to L_1, L_2 and l_2 then is the following.

Lemma 6.3.2 *Assume* (6.2.6). *Then for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that*

$$u(x,t) \leq L_1 + \varepsilon \qquad \textit{for all } x \in \bar\Omega \textit{ and } t \geq t_\varepsilon, \tag{6.3.3}$$

that

$$l_2 - \varepsilon \leq v(x,t) \leq L_2 + \varepsilon \qquad \textit{for all } x \in \bar\Omega \textit{ and } t \geq t_\varepsilon, \tag{6.3.4}$$

and that

$$l_2 - \varepsilon \leq \lambda w(x,t) \leq k(L_1+\varepsilon) + (L_2+\varepsilon) \qquad \textit{for all } x \in \bar\Omega \textit{ and } t \geq t_\varepsilon. \tag{6.3.5}$$

Proof. That (6.3.3) and (6.3.4) can be achieved for suitably large t_ε is an immediate consequence of the definitions in (6.3.1). Then applying Lemma 6.3.1 for fixed $t \geq t_\varepsilon$ we readily obtain (6.3.5). ∎

Next we compare u with a suitable spatially homogeneous function and obtain an upper bound for L_1 in terms of l_2.

Lemma 6.3.3 *Assume* (6.1.6). *Then the numbers L_1 and l_2 defined in* (6.3.1) *fulfill the relation*

$$(1 - kq_1)L_1 \leq (1 - a_1 l_2)_+ \,. \tag{6.3.6}$$

Proof. If $u \equiv 0$ in $\bar\Omega \times [0,\infty)$, then (6.3.6) is fulfilled in view of $L_1 = 0$. Otherwise, according to (6.2.4), taking $\mathcal{L}_1$ as in (6.2.3) we recall that

$$u_t = \mathcal{L}_1 u + u \cdot \Big\{ \mu_1 - (\mu_1 - k\chi_1)u - (a_1\mu_1 - \chi_1)v - \lambda\chi_1 w \Big\} \qquad \text{in } \Omega \times (0,\infty),$$

where (6.1.6) ensures that $a_1\mu_1 - \chi_1 \geq 0$. Thus, if for fixed $\varepsilon > 0$ we take t_ε as given by Lemma 6.3.2, then (6.3.4) and (6.3.5) yield

$$-(a_1\mu_1 - \chi_1)v \leq -(a_1\mu_1 - \chi_1)\cdot(l_2 - \varepsilon) \qquad \text{in } \Omega \times (t_\varepsilon,\infty)$$

and

$$-\lambda\chi_1 w \leq -\chi_1 \cdot (l_2 - \varepsilon) \qquad \text{in } \Omega \times (t_\varepsilon,\infty),$$

and therefore we obtain

$$u_t \leq \mathcal{L}_1 u + u \cdot \Big\{ \mu_1 - (\mu_1 - k\chi_1)u - a_1\mu_1(l_2 - \varepsilon) \Big\} \qquad \text{in } \Omega \times (t_\varepsilon,\infty).$$

Since $\mathcal{L}_1$ annihilates spatially homogeneous functions, a parabolic comparison argument hence implies that

$$u(x,t) \le \overline{u}(t) \qquad \text{for all } x \in \bar{\Omega} \text{ and } t \ge t_\varepsilon, \tag{6.3.7}$$

where $\overline{u}$ denotes the solution of the initial-value problem

$$\begin{cases} \overline{u}' = \overline{u} \cdot \Big\{ \mu_1 - (\mu_1 - k\chi_1)\overline{u} - a_1\mu_1(l_2 - \varepsilon) \Big\}, & t > t_\varepsilon, \\ \overline{u}(t_\varepsilon) = \max_{x\in\bar{\Omega}} u(x,t_\varepsilon). \end{cases}$$

Since $u(\cdot, t_\varepsilon)$ is positive in $\bar{\Omega}$ by Lemma 6.2.1, it is clear that

$$\overline{u}(t) \to \max\Big\{0 \ , \ \frac{\mu_1 - a_1\mu_1(l_2-\varepsilon)}{\mu_1 - k\chi_1}\Big\} \qquad \text{as } t \to \infty,$$

which in conjunction with (6.3.7) yields the inequality

$$\limsup_{t\to\infty} \Big(\max_{x\in\bar{\Omega}} u(x,t) \Big) \le \max\Big\{0 \ , \ \frac{\mu_1 - a_1\mu_1(l_2-\varepsilon)}{\mu_1 - k\chi_1}\Big\}.$$

Taking $\varepsilon \searrow 0$ now shows that indeed (6.3.6) must be valid. ■

In order to study the large time behavior of v we need to distinguish two cases depending on the sign of $kq_2 - a_2$. We again use comparison arguments involving spatially homogeneous functions and first give the result for $kq_2 < a_2$.

Lemma 6.3.4 *Suppose that* (6.1.6) *holds, and that* $kq_2 < a_2$. *Then*

$$(1-q_2)L_2 \le (1-q_2 l_2)_+ \tag{6.3.8}$$

and

$$(1-q_2)l_2 \ge 1 - a_2L_1 - q_2L_2. \tag{6.3.9}$$

Proof. The procedure is similar to that in Lemma 6.3.3: Given $\varepsilon > 0$, we take $t_\varepsilon > 0$ as provided by Lemma 6.3.2. We recall that by (6.2.5) we have

$$v_t = \mathcal{L}_2 v + v \cdot \Big\{ \mu_2 - (\mu_2 - \chi_2)v - (a_2\mu_2 - k\chi_2)u - \lambda\chi_2 w \Big\} \qquad \text{in } \Omega \times (0,\infty) \tag{6.3.10}$$

with $\mathcal{L}_2$ given by (6.2.3). Since $a_2\mu_2 - k\chi_2$ is nonnegative according to our hypothesis $kq_2 < a_2$, using that $u \ge 0$ we can estimate

$$-(a_2\mu_2 - k\chi_2)u \le 0 \qquad \text{in } \Omega \times (0,\infty),$$

whereas by (6.3.5),

$$-\lambda\chi_2 w \le -\chi_2 \cdot (l_2 - \varepsilon) \qquad \text{in } \Omega \times (t_\varepsilon, \infty).$$

Thus, (6.3.10) implies that

$$v_t \le \mathcal{L}_2 v + v \cdot \Big\{ \mu_2 - (\mu_2 - \chi_2)v - \chi_2(l_2 - \varepsilon) \Big\} \qquad \text{in } \Omega \times (t_\varepsilon, \infty),$$

whence by comparison we find that

$$v(x,t) \le \overline{v}(t) \qquad \text{for all } x \in \bar{\Omega} \text{ and } t \ge t_\varepsilon,$$

if we let $\overline{v}$ denote the solution of

$$\begin{cases} \overline{v}' = \overline{v} \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)\overline{v} - \chi_2(l_2 - \varepsilon)\Big\}, & t > t_\varepsilon, \\ \overline{v}(t_\varepsilon) = \max_{x\in\bar{\Omega}} v(x,t_\varepsilon). \end{cases}$$

In light of the long time asymptotics of $\overline{v}$, this entails that

$$\limsup_{t\to\infty} \Big(\max_{x\in\bar{\Omega}} v(x,t)\Big) \le \max\Big\{0 \ , \ \frac{\mu_2 - \chi_2(l_2 - \varepsilon)}{\mu_2 - \chi_2}\Big\}$$

for any $\varepsilon > 0$ and hence

$$L_2 \le \max\Big\{0 \ , \ \frac{\mu_2 - \chi_2 l_2}{\mu_2 - \chi_2}\Big\},$$

which proves (6.3.8).

Similarly, (6.3.9) can be obtained by going back to (6.3.10) and using (6.3.3) and (6.3.5) to estimate

$$-(a_2\mu_2 - k\chi_2)u \ge -(a_2\mu_2 - k\chi_2)\cdot(L_1 + \varepsilon) \qquad \text{in } \Omega \times (t_\varepsilon, \infty)$$

and

$$-\lambda\chi_2 w \ge -\chi_2 \cdot (kL_1 + L_2 + (k+1)\varepsilon) \qquad \text{in } \Omega \times (t_\varepsilon, \infty),$$

again because $k\chi_2 \le a_2\mu_2$. We thereupon obtain

$$\begin{aligned} v_t - \mathcal{L}_2 v &\ge v \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)v - (a_2\mu_2 - k\chi_2)\cdot(L_1+\varepsilon) - \chi_2(kL_1 + L_2 + (k+1)\varepsilon)\Big\} \\ &= v \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)v - a_2\mu_2 \cdot (L_1+\varepsilon) - \chi_2(L_2+\varepsilon)\Big\} \qquad \text{in } \Omega \times (t_\varepsilon, \infty), \end{aligned}$$

whence

$$v(x,t) \ge \underline{v}(t) \qquad \text{for all } x \in \bar{\Omega} \text{ and } t \ge t_\varepsilon \tag{6.3.11}$$

by the comparison principle, where

$$\begin{cases} \underline{v}' = \underline{v} \cdot \Big\{\mu_2 - (\mu_2 - \chi_2)\underline{v} - a_2\mu_2(L_1+\varepsilon) - \chi_2(L_2+\varepsilon)\Big\}, & t > t_\varepsilon, \\ \underline{v}(t_\varepsilon) = \min_{x\in\bar{\Omega}} v(x,t_\varepsilon). \end{cases} \tag{6.3.12}$$

Now an important observation, singling out the particular steady state solution $(\tilde{u}, \tilde{v}, \tilde{w}) \equiv (1, 0, \frac{k}{\lambda})$ for which (6.3.9) does not hold, is that $\underline{v}(t_\varepsilon)$ is positive thanks to the positivity of v in $\bar{\Omega} \times (0, \infty)$ asserted by Lemma 6.2.1. Consequently, $\underline{v}$ again approaches the larger of the equilibria of (6.3.12), that is, we have

$$\underline{v}(t) \to \max\Big\{0 \ , \ \frac{\mu_2 - a_2\mu_2(L_1+\varepsilon) - \chi_2(L_2+\varepsilon)}{\mu_2 - \chi_2}\Big\} \qquad \text{as } t \to \infty,$$

which in the limit $\varepsilon \searrow 0$ clearly implies (6.3.9). ■

In case of $kq_2 \ge a_2$ we proceed in a similar way.

Lemma 6.3.5 *Assume* (6.1.6), *and suppose that* $kq_2 \geq a_2$. *Then*

$$(1-q_2)L_2 \leq \Big(1+(kq_2-a_2)L_1-q_2l_2\Big)_+ \tag{6.3.13}$$

and

$$(1-q_2)l_2 \geq 1-kq_2L_1-q_2L_2. \tag{6.3.14}$$

Proof. Again using (6.2.5) as a starting point, given $\varepsilon > 0$ we take $t_\varepsilon > 0$ as given by Lemma 6.3.2 and estimate

$$(k\chi_2-a_2\mu_2)u \leq (k\chi_2-a_2\mu_2)\cdot(L_1+\varepsilon) \qquad \text{in } \Omega\times(t_\varepsilon,\infty)$$

and

$$-\lambda\chi_2 w \leq -\chi_2\cdot(l_2-\varepsilon) \qquad \text{in } \Omega\times(t_\varepsilon,\infty).$$

We thereupon obtain from the identity

$$v_t = \mathcal{L}_2 v + v\cdot\Big\{\mu_2-(\mu_2-\chi_2)v+(k\chi_2-a_2\mu_2)u-\lambda\chi_2 w\Big\} \qquad \text{in } \Omega\times(0,\infty), \tag{6.3.15}$$

as obtained in (6.2.5), that

$$v_t \leq \mathcal{L}_2 v + v\cdot\Big\{\mu_2-(\mu_2-\chi_2)v+(k\chi_2-a_2\mu_2)(L_1+\varepsilon)-\chi_2(l_2-\varepsilon)\Big\} \quad \text{in } \Omega\times(t_\varepsilon,\infty).$$

By comparison with spatially homogeneous ODE solutions in the same manner as in Lemma 6.3.4, we thereby derive the inequality

$$\limsup_{t\to\infty}\Big(\max_{x\in\bar\Omega} v(x,t)\Big) \leq \max\Big\{0\ ,\ \frac{\mu_2+(k\chi_2-a_2\mu_2)(L_1+\varepsilon)-\chi_2(l_2-\varepsilon)}{\mu_2-\chi_2}\Big\},$$

which on taking $\varepsilon \searrow 0$ yields (6.3.13).
Similarly, inserting the lower estimates

$$(k\chi_2-a_2\mu_2)u \geq 0 \qquad \text{in } \Omega\times(0,\infty)$$

and

$$-\lambda\chi_2 w \geq -\chi_2\cdot(kL_1+L_2+(k+1)\varepsilon) \qquad \text{in } \Omega\times(t_\varepsilon,\infty)$$

into (6.3.15) shows that

$$v_t \geq \mathcal{L}_2 v + v\cdot\Big\{\mu_2-(\mu_2-\chi_2)v-\chi_2(kL_1+L_2+(k+1)\varepsilon)\Big\} \qquad \text{in } \Omega\times(t_\varepsilon,\infty),$$

which on comparison entails that

$$\liminf_{t\to\infty}\Big(\min_{x\in\bar\Omega} v(x,t)\Big) \geq \frac{\mu_2-\chi_2(kL_1+L_2+(k+1)\varepsilon)}{\mu_2-\chi_2}$$

and thereby proves (6.3.14). ∎

Using the estimates shown in this section, we are now able to prove that $v(t) \to 1$ in $L^\infty(\Omega)$ as $t \to \infty$ if we assume $L_1 = 0$.

Lemma 6.3.6 *Assume that* (6.1.6) *holds, and suppose that* $L_1 = 0$. *Then* $L_2 = l_2 = 1$.

Proof. We first observe that $q_2 l_2 \le 1$, for otherwise by either (6.3.8) or by (6.3.13) in combination with $L_1 = 0$ we would have $L_2 = 0$ and hence could draw the conclusion that $l_2 = 0$ which is absurd in view of (6.3.9) and (6.3.14).
Accordingly, in light of the hypothesis $L_1 = 0$, Lemma 6.3.4 and Lemma 6.3.5 show that in both cases $kq_2 < a_2$ and $kq_2 \ge a_2$, the inequalities

$$(1 - q_2)L_2 \le 1 - q_2 l_2 \tag{6.3.16}$$

and

$$(1 - q_2)l_2 \ge 1 - q_2 L_2 \tag{6.3.17}$$

hold, which on subtraction imply

$$(1 - q_2)(L_2 - l_2) \le q_2(L_2 - l_2).$$

Since (6.1.6) implies that $q_2 < \frac{1}{2}$, this asserts that $L_2 \le l_2$ and hence $L_2 = l_2$. Therefore, once more applying (6.3.16) shows that $L_2 \le 1$, while similarly (6.3.17) entails that $l_2 \ge 1$. This completes the proof. ■

6.4 Asymptotic behavior

According to Lemmas 6.2.1, 6.2.2, 6.3.2 and 6.3.6, in order to prove Theorem 6.1.1 it remains to show that $L_1 = 0$ is indeed valid. This will be done by considering again two cases depending on the sign of $kq_2 - a_2$.

6.4.1 The case $kq_2 < a_2$

Combining Lemmas 6.3.3 and 6.3.4, we complete the proof of Theorem 6.1.1 for $kq_2 < a_2$.

Lemma 6.4.1 *Suppose that* (6.1.6) *holds as well as* $kq_2 < a_2$. *Then* $L_1 = 0$.

Proof. Let us suppose on the contrary that L_1 be positive. Then Lemma 6.3.3 says that

$$(1 - kq_1)L_1 \le 1 - a_1 l_2 \tag{6.4.1}$$

and hence

$$l_2 < \frac{1}{a_1}. \tag{6.4.2}$$

Here we observe that by (6.4.2) we also have $1 - q_2 l_2 > 1 - \frac{q_2}{a_1} \ge 1 - \frac{1}{a_1} > 0$ due to the fact that (6.1.6) entails that $q_2 \le 1$. Consequently, Lemma 6.3.4 asserts that

$$(1 - q_2)L_2 \le 1 - q_2 l_2 \tag{6.4.3}$$

and

$$(1-q_2)l_2 \geq 1 - a_2 L_1 - q_2 L_2. \qquad (6.4.4)$$

Now combining (6.4.4) with (6.4.1) yields

$$\begin{aligned} q_2 L_2 &\geq 1 - a_2 L_1 - (1-q_2)l_2 \\ &\geq 1 - a_2 \cdot \frac{1 - a_1 l_2}{1 - kq_1} - (1-q_2)l_2 \\ &= 1 - \frac{a_2}{1-kq_1} + \Big(\frac{a_1 a_2}{1-kq_1} - 1 + q_2\Big) \cdot l_2, \end{aligned}$$

which in light of (6.4.3) shows that

$$\frac{1-q_2}{q_2} \cdot \Big\{ 1 - \frac{a_2}{1-kq_1} + \Big(\frac{a_1 a_2}{1-kq_1} - 1 + q_2\Big) \cdot l_2 \Big\} \leq (1-q_2)L_2 \leq 1 - q_2 l_2.$$

Thus, necessarily

$$\frac{1-q_2}{q_2} \cdot \Big(1 - \frac{a_2}{1-kq_1}\Big) - 1 \leq \Big\{ -\frac{1-q_2}{q_2} \cdot \Big(\frac{a_1 a_2}{1-kq_1} - 1 + q_2\Big) - q_2 \Big\} \cdot l_2,$$

which on multiplication by $(1-kq_1)q_2$ can be seen to be equivalent to

$$\Big\{1 - a_1 a_2 - kq_1 - (2 - a_1 a_2)q_2 + 2kq_1 q_2\Big\} \cdot l_2 \geq 1 - a_2 - kq_1 - (2-a_2)q_2 + 2kq_1 q_2. \quad (6.4.5)$$

Since according to (6.1.8),

$$I := 1 - a_2 - kq_1 - (2-a_2)q_2 + 2kq_1 q_2$$

is positive, (6.4.5) is thus only possible if also

$$J := 1 - a_1 a_2 - kq_1 - (2 - a_1 a_2)q_2 + 2kq_1 q_2$$

is positive. Therefore, (6.4.5) implies that

$$l_2 \geq \frac{I}{J},$$

which in conjunction with (6.4.2) says that $a_1 I < J$, that is,

$$a_1 - a_1 a_2 - a_1 kq_1 - a_1(2-a_2)q_2 + 2a_1 kq_1 q_2 < 1 - a_1 a_2 - kq_1 - (2 - a_1 a_2)q_2 + 2kq_1 q_2.$$

A simple rearrangement thus yields

$$(a_1 - 1) \cdot (1 - kq_1 - 2q_2 + 2kq_1 q_2) < 0,$$

which is incompatible with the assumption $I > 0$, because $a_1 > 1$ and

$$1 - kq_1 - 2q_2 + 2kq_1 q_2 = I + a_2(1-q_2) \geq I > 0$$

thanks to the fact that $q_2 < 1$ by (6.1.6). This contradiction shows that actually L_1 must vanish. ∎

6.4.2 The case $kq_2 \geq a_2$

Finally, a combination of Lemmas 6.3.3 and 6.3.5 completes the proof of Theorem 6.1.1 also for $kq_2 \geq a_2$ like in the preceding section. The details of the proof are given in the following Lemma.

Lemma 6.4.2 *Let* (6.1.6) *hold, and assume that* $kq_2 \geq a_2$*. Then* $L_1 = 0$.

Proof. If L_1 was positive, again Lemma 6.3.3 would yield

$$l_2 < \frac{1}{a_1} \tag{6.4.6}$$

and

$$L_1 \leq \frac{1 - a_1 l_2}{1 - kq_1}. \tag{6.4.7}$$

On the other hand, since (6.4.6) and $kq_2 \geq a_2$ imply $1 + (kq_2 - a_2)L_1 - q_2 l_2 > 1 - \frac{q_2}{a_1} \geq 1 - \frac{1}{a_1} > 0$, Lemma 6.3.5 says that

$$(1 - q_2)L_2 \leq 1 + (kq_2 - a_2)L_1 - q_2 l_2, \tag{6.4.8}$$

which combined with (6.4.7) implies

$$\begin{aligned} (1 - q_2)L_2 &\leq 1 + (kq_2 - a_2) \cdot \frac{1 - a_1 l_2}{1 - kq_1} - q_2 l_2 \\ &= \Big(1 + \frac{kq_2 - a_2}{1 - kq_1}\Big) - \Big\{\frac{a_1(kq_2 - a_2)}{1 - kq_1} + q_2\Big\} \cdot l_2, \end{aligned} \tag{6.4.9}$$

because $kq_2 \geq a_2$. Moreover, the second statement in Lemma 6.3.5 asserts that

$$(1 - q_2)l_2 \geq 1 - kq_2 L_1 - q_2 L_2,$$

which in view of (6.4.7) and (6.4.9) becomes

$$\begin{aligned} (1 - q_2)l_2 \geq\ & 1 - kq_2 \cdot \frac{1 - a_1 l_2}{1 - kq_1} - \frac{q_2}{1 - q_2} \cdot \Big(1 + \frac{kq_2 - a_2}{1 - kq_1}\Big) \\ & + \frac{q_2}{1 - q_2} \cdot \Big\{\frac{a_1(kq_2 - a_2)}{1 - kq_1} + q_2\Big\} \cdot l_2. \end{aligned}$$

When multiplied by $(1 - kq_1)(1 - q_2)$, this yields

$$\begin{aligned} &\Big\{(1 - kq_1)(1 - q_2)^2 - a_1 kq_2(1 - q_2) - a_1 q_2(kq_2 - a_2) - q_2^2(1 - kq_1)\Big\} \cdot l_2 \\ &\quad \geq (1 - kq_1)(1 - q_2) - kq_2(1 - q_2) - (1 - kq_1)q_2 - q_2(kq_2 - a_2), \end{aligned}$$

which can be simplified so as to become

$$J \cdot l_2 \geq I,$$

where

$$I := 1 - kq_1 - (2 + k - a_2)q_2 + 2kq_1q_2$$

is positive thanks to (6.1.8), and hence also

$$J := 1 - kq_1 - (2 - a_1a_2 + ka_1)q_2 + 2kq_1q_2$$

must be positive. We thus have $l_2 \geq \frac{I}{J}$, whence we may conclude using (6.4.6) that $a_1 I < J$, that is,

$$a_1 - a_1kq_1 - (2a_1 + ka_1 - a_1a_2)q_2 + 2a_1kq_1q_2 < 1 - kq_1 - (2 - a_1a_2 + ka_1)q_2 + 2kq_1q_2.$$

However, this is equivalent to

$$(a_1 - 1)(1 - kq_1)(1 - 2q_2) < 0,$$

which contradicts (6.1.6), because clearly $kq_1 < 1$ and $q_2 < \frac{1}{2}$. ∎

6.5 Conclusions and discussion

In this paper we have considered two biological species which compete for the same resources and migrate chemotactically towards a higher concentration of a chemical substance, which they produce. The problem is modeled by using a system of three partial differential equations: two nonlinear parabolic equations to describe the evolution of the biological species and a linear PDE to model the behavior of the chemical. This chemical diffuses considerably faster than the living organism, and it is thus assumed that the evolution of the chemical signal is governed by an elliptic equation.

The system contains several parameters which measure different aspects in the system: chemotaxis effects, competition, diffusion, chemical production and decay. In the case when competition is absent, it is known that due to chemotaxis, the considered system may produce finite-time blow-up ([A6.5]), while if on the other hand chemotactic effects are blinded out, then the competitive terms keep the solution bounded and guarantee their global existence. A natural and challenging question has been posed in the literature for such systems: Which are the constraints and the threshold values that decide between driving the system toward global existence, or enforcing blow-up? This question remains open even in the case of a single species. In that case the competitive term is simplified to a logistic growth function (cf. [A6.15] for partial results).

A second question concerns the influence of chemotaxis effects on the stability of the homogeneous steady states determined by the competitive terms. The presence of a large number of parameters in the system makes this question difficult to answer. In the case where the competitive terms are weak in the sense that in (6.1.1) we have $a_i \in [0, 1)$ for $i = 1, 2$, a partial answer is given in [A6.16] within some range of the chemotactic parameters. In this paper we have studied the problem under the assumption that when compared to the latter setting, one of the species is significantly more aggressive in terms of competition.

In this framework, characterized by the assumption (6.1.4), we have seen that if in (6.1.5), both ratios q_i, $i = 1, 2$, between the chemotactic sensitivities χ_i and the competition parameters μ_i are suitably small then all nontrivial solutions will be global in time and bounded, and that they approach the homogeneous steady state in which the aggressive subpopulation is at its carrying capacity and the less aggressive species has died out. This inter alia shows that the phenomenon of (asymptotic) extinction of one species, known to be valid for the associated Lotka-Volterra ODE system without diffusion and chemotaxis, persists also in such systems with chemotactic interaction, provided the latter is sufficiently weak. Global existence of solutions is obtained under the assumption $q_1 k + q_2 < 1$. In that case competition prevents blow-up but extra assumptions are required to prove the stability claim in (6.1.9).

We do not know in how far the set (6.1.6) of hypotheses under which our results have been derived is optimal. After all, in some known borderline cases our approach yields requirements which are consistent with assumptions made in the literature for correspondingly simplified models (cf. the discussion in Remark 6.1.2). In light of results from the literature on corresponding single-species systems, it seems natural to conjecture that for suitably large values of q_i, solutions may exhibit more colorful dynamics. Indeed, in such a setting numerical simulations indicate that chaotic behavior may occur ([A6.12]). It is conceivable that some solutions may even blow up in finite time, but a substantial influence of the space dimension n on the occurrence of such explosion phenomena is most likely to be expected: In the single-species case, for instance, although some examples of high-dimensional blow-up phenomena despite logistic-type growth restrictions have been found for $n \geq 5$ ([A6.21]), it is known that blow-up never occurs when $n \leq 2$ ([A6.15]). In particular, the detection of explosions must thus be restricted to the case $n \geq 3$ in which even numerical approaches seem delicate. As opposed to this, our assumptions in this paper are completely independent of n, and moreover they are fully explicit; thereby our results reveal, in a quantitative manner, a stability feature of the competitive exclusion phenomenon with respect to chemotactic interaction.

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Article 7:
On a multiscale model involving cell contractivity and its effects on tumor invasion

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Abstract

Cancer cell migration is an essential feature in the process of tumor spread and establishing of metastasis. It characterizes the invasion observed on the level of the cell population, but it is also tightly connected to the events taking place on the subcellular level. These are conditioning the motile and proliferative behavior of the cells, but are also influenced by it. In this work we propose a multiscale model linking these two levels and aiming to assess their interdependence. On the subcellular, microscopic scale it accounts for integrin binding to soluble and insoluble components present in the peritumoral environment, which is seen as the onset of biochemical events leading to changes in the cell's ability to contract and modify its shape. On the macroscale of the cell population this leads to modifications in the diffusion and haptotaxis performed by the tumor cells and implicitly to changes in the tumor environment. We prove the (local) well posedness of our model and perform numerical simulations in order to illustrate the model predictions.

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7.1 Introduction

Tumor cells are able to migrate through the surrounding tissue and degrade it on their way toward blood vessels and distal organs where they initiate and develop further tumors, a process known as metastasis [A7.14]. According to the structure of the peritumoral environment, the movement of cancer cells is diffusion- or transport-dominated and also influenced by two mechanisms: *chemotaxis* and *haptotaxis*. The former defines the cell motion in response to a chemoattractant (or chemorepellent) concentration. As such gradients may lack in the solution, the differences in the concentration of an adhesive molecule e.g., along an extracellular matrix (ECM) can be relevant instead. The cells need to adhere to the ECM fibers in order to be able to move [A7.1], hence they will migrate from a region of low concentration of relevant adhesive molecules to an area with a higher concentration, a process called *haptotaxis* [A7.7]. Thereby, the contact with the surrounding tissue stimulates the production of proteolytic enzymes (matrix degrading enzymes (MDEs) like matrix metalloproteinases), which degrade the tissue fibers [A7.15], thus creating interstices to be occupied during the migration process toward neighboring blood vessels.
When characterizing tumor migration, the spatial scales of interest range from the subcellular level to the macroscopic one (tissue and cell populations), while the time scales stretch from seconds (or even shorter) at the intracellular level up to months for the doubling times of tumors.

Most of the existing models for cancer invasion can be assigned to three categories:
Microscopic models are concerned with the events at the subcellular level initiating and controling (tumor) cell migration. These processes are usually characterized with systems of ordinary differential equations (ODEs) for the concentrations of the involved biochemical substances. For instance, some of these models focus on the expression of MDEs and proteolysis [A7.6], whereas others emphasize cell polarization and onset of lamellipod protrusion [A7.24], a crucial step in integrin-mediated haptotactic motility.
In the *mesoscopic framework*, cell migration is characterized by Boltzmann-like kinetic transport equations for the cell density function, in which the integral operators characterize innovations of the cell velocities instead of modeling particle collisions as in gas theory. This approach has been introduced by Othmer, Dunbar & Alt [A7.28] in order to provide a description of cell dispersal via velocity jump processes. It was extended e.g., by Hillen [A7.17] to model the mesenchymal motion of cancer cells and the subsequent tissue modification. Bellomo et al. [A7.5] proposed a general framework for such kinetic models on the mesoscopic level (also allowing for the inclusion of the “cell state” to reflect dynamics on the microlevel) that they called the *kinetic theory of active particles* (KTAP).

Macroscopic descriptions can be derived from mesoscopic models by means of averaging processes leading to evolution equations for the moments of the cell distribution function. This was done, at least formally, in, e.g., [A7.17] in the context of mesenchymal motion of tumor cells, whereas rigorous results on hyperbolic and parabolic limits of kinetic equations for chemotaxis were obtained, e.g., in [A7.8] and [A7.29] respectively. Further models for cell population migration that rely only on mass balance equations were proposed by Anderson et al. [A7.2] and Chaplain & Lolas [A7.9], for example.

Combining two or all three of these modeling levels leads to a *multiscale setting*, which has received increasing interest over the last decade. Many – in particular those involving couplings between micro and mesoscales – align to the general KTAP by Bellomo et al. In [A7.32, A7.20, A7.21] multiscale models for bacterial dispersal and respectively for cancer cell migration through tissue networks have been deduced and analyzed. On the subcellular level the latter account for integrin binding to ECM fibers or to proteolytic rests resulting from the degradation of such fibers, whereas the behavior of individual cells on the mesoscopic scale is described via a Boltzmann-type transport equation for the cell density function. This in turn is further coupled with an integro-differential ODE for the ECM fiber density and a reaction-diffusion equation (RD-PDE) for the chemoattractant concentration. Bridging the gap between the scales, the macroscopic fiber density influences the vector field of subcellular states. A related model for glioma invasion focusing on haptotaxis and the interaction between tumor cells and brain tissue via integrin binding on the microlevel was studied in [A7.12]. Due to its high dimensionality and the large differences between the scales, the numerical handling of such a micro-meso-macro model is a challenging issue. A way out is to use adequate scalings to obtain macroscopic limits, as in [A7.12, A7.34]. Another way uses a nonparametric density estimation technique from statistics to assess the density of cells directly on the macrolevel, without needing to deduce the corresponding reaction-diffusion (transport) equations (RD(T)-PDEs), but only relying on simulations of the involved basic stochastic processes [A7.30, A7.31, A7.32].

Yet another way to avoid the difficulties with the numerics of a full micro-meso-macro model is to directly connect the microscopic and the macroscopic levels, leading to a much simplified (but still multiscale) approach, which concentrates on the population evolution at the macroscopic level and uses systems of RD(T)-PDEs. These are coupled with ODEs modeling processes inside or on the surface of a cell. The coefficients in the macroscopic formulation (for, e.g., diffusion, chemotaxis, haptotaxis) can depend in a nonlinear way on the solutions and even on the microscale dynamics. In this work we use such an approach: motivated by the more complex micro-meso-macro setting in [A7.20, A7.21] we propose a micro-macro model for the influence of integrin binding dynamics on tumor invasion by way of a contractivity function. The latter captures the effects of subcellular dynamics on the ability of a cell to polarize and modify its shape by restructuring its cytoskeleton. The integrins on the cell surface bind (reversibly) to insoluble (ECM fibers) and soluble (proteolytic rests of ECM fibers) ligands which are present in the peritumoral environment, hence initiating a whole network of intracellular signaling cascades (see e.g., [A7.23] and the references therein), the outcome of which are – as already mentioned – changes in the cell's flexibility. These, however, are expected to need some time to happen, which is modeled by a time lag in the equation characterizing the evolution of the contractivity function. Further, these events on the subcellular level have consequences for the cell's migratory behavior, influencing both its diffusive spread and the haptotaxis, which we model by letting the respective coefficients depend on the cell contractivity function, see equation (7.2.5) below.

The paper has the following structure: In Section 7.2 we introduce our multiscale model characterizing the evolution of cancer cell density, concentration of proteolytic rests, density of tissue fibers, contractivity function, and concentrations of integrins bound to ECM fibers and to fiber residuals degraded during the interaction with tumor cells. The proof of the existence and uniqueness of a solution to this system is done in Section 7.3, followed in Section 7.4 by a nondimensionalization preliminary to the numerical simulations performed in Section 7.5. Finally, a discussion of the results is provided in Section 7.6.

7.2 The Model

7.2.1 The subcellular level

We provide a simplified description of the events on the subcellular level by considering as in [A7.20, A7.21, A7.25] merely the integrin binding dynamics on the cell surface. For the sake of completeness we recall here the corresponding kinetic model for the binding of ECM-proteins v and proteolytic products l to free integrins denoted by R. The reversible binding of integrins to ECM-proteins leads to a complex Rv according to the equation

$$v + R \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} Rv.$$

The corresponding equation for the formation and dissociation of complexes Rl of integrin and proteolytic product reads

$$l + R \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} Rl.$$

We denote the concentrations of integrins of an individual cell bound to ECM-molecules by y_1 and the concentration of integrins of the same cell bound to the proteolytic product l by y_2. The total concentration of integrins (bound or unbound) of each cell is assumed to be conserved and given by $R_0 \in \mathbb{R}_+$. Thus, $R_0 - y_1 - y_2$ is the concentration of unbound integrins on the cell's surface. Hence, one has $\boldsymbol{y} = (y_1, y_2) \in Y$ with

$$Y := \{(y_1, y_2) \in (0, R_0)^2 \mid y_1 + y_2 < R_0\}. \tag{7.2.1}$$

The state equations for the cell surface dynamics then read

$$\frac{\partial \boldsymbol{y}}{\partial t} = \boldsymbol{G}(\boldsymbol{y}, v(t, \boldsymbol{x}), l(t, \boldsymbol{x})) \tag{7.2.2}$$

with the mapping $\boldsymbol{G} : Y \times [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ defined by

$$\boldsymbol{G}(\boldsymbol{y}, v, l) := \begin{pmatrix} k_1(R_0 - y_1 - y_2)v - k_{-1}y_1 \\ k_2(R_0 - y_1 - y_2)l - k_{-2}y_2 \end{pmatrix}. \tag{7.2.3}$$

Thereby, the coefficients k_1 and k_2 denote the binding rates between integrins and ECM fibers and between integrins and proteolytic residuals, respectively, while k_{-1} and k_{-2} are the corresponding detaching rates.
On the other hand, since contractivity is the outcome of a sequence of biochemical processes initiated by binding of integrins, activation of multiple signaling proteins and involving e.g., actin polimerization, restructuring of the cell's cytoskeleton, formation of protrusions, polarization etc. (see e.g., [A7.19] and the references therein), it is reasonable to assume that it depends on some delay corresponding to the time passed between integrin binding and the effects on the cell's ability to reorganize its shape by contraction. This leads to an equation of the form

$$\kappa_t = -q\kappa + H(\boldsymbol{y}(t-\tau)) \qquad \text{in } (0,T) \times \Omega \tag{7.2.4}$$

where H is a contractivity source term depending on the subcellular dynamics and their above mentioned subsequent effects and with τ denoting a constant delay in taking influence on the contractivity. Another choice for the time lag is e.g., to use a distributed one, as in the next section. A precise form for the function H is provided at the end of Subsection 7.2.2 below. [10]

7.2.2 The macroscopic level

The evolution of cancer cell density $c(t, \boldsymbol{x})$ is influenced by the random motility $\mathbf{J_{random}}$ and the directional flow $\mathbf{J_{directional}}$. The former characterizes cell diffusion into the tissue and is given by

$$\mathbf{J_{random}} = -\varphi(\kappa, c, v)\nabla c$$

with the random motility function $\varphi(\kappa, c, v)$ depending on the contractivity function κ, on the cell density itself, and on the density $v(t, \boldsymbol{x})$ of tissue fibers, as the spread of cancer cells is conditioned by their neighbors and surroundings.
On the other hand, $\mathbf{J_{directional}}$ corresponds to the cancer cell flux due to spatial gradients of stimulating chemotactic and haptotactic responses:

$$\mathbf{J_{directional}} = \mathbf{J_{chemotaxis}} + \mathbf{J_{haptotaxis}} = f(c, l)c\nabla l + \psi(\kappa, v)c\nabla v$$

where $f(c, l)$ and $\psi(\kappa, v)$ are the chemotactic and haptotactic functions, respectively. As in [A7.20, A7.21, A7.25], in our present model the role of the chemoattractant is played

[10] It should be proportional to the amount of integrins bound to ECM fibers and inversely proportional to the amount of integrins bound to the chemoattractant molecules, as the latter hints on the cell successfully following the chemoattractant gradient, hence there is less need of enhancing contractivity –via shape change, restructuring of the cytoskeleton, reorientation etc.

by the proteolytic residuals following the degradation of tissue by the cells performing mesenchymal motion [A7.13, A7.18]. We denote with $l(t,\boldsymbol{x})$ their concentration.
Then, due to the equilibrium of fluxes we obtain the first equation in system (7.2.5) below. Thereby, the last term on the right-hand side models cell proliferation with crowding effects and the proliferation rate and the carrying capacity are denoted by μ_c and K_c, respectively.

The ECM fibers are supposed to be degraded through interaction with the cancer cells with the rate δ_v. They also reestablish and remodel themselves while competing with the diffusive cancer cells for space. This is described by a term similar to the proliferation in the equation for cancer cells with the corresponding production rate μ_v and carrying capacity K_v. Crowding effects are accounted for as well. Further, the ECM does not diffuse, but can be only degraded by the cells producing matrix degrading enzymes. These considerations lead to the second equation in system (7.2.5).
The chemoattractant concentration satisfies a reaction-diffusion equation with a source term reflecting the degradation of tissue fibers under the influence of the migrating tumor cells, along with a simple decay term. The diffusion constant, production and decay rates are denoted with α, δ_l and β, respectively.

Finally, combining the equations on micro and macro levels, we obtain the following system of equations:

$$\begin{cases} c_t = \nabla\cdot(\varphi(\kappa,c,v)\nabla c) - \nabla\cdot(\psi(\kappa,v)c\nabla v) - \nabla\cdot(f(c,l)c\nabla l) \\ \qquad +\mu_c c\left(1-\frac{c}{K_c}-\eta_1\frac{v}{K_v}\right), \\ v_t = -\delta_v cv + \mu_v v\left(1-\eta_2\frac{c}{K_c}-\frac{v}{K_v}\right), \\ l_t = \alpha\Delta l + \delta_l cv - \beta l, \\ \boldsymbol{y}_t = \boldsymbol{G}(v,l,\boldsymbol{y}), \\ \kappa_t = -q\kappa + H(\boldsymbol{y}(t-\tau)) \end{cases} \tag{7.2.5}$$

in $(0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth enough boundary and with $n\in\{1,2,3\}$. Here $\eta_1,\eta_2\in(0,1)$ are parameters characterizing growth reduction due to the competition between the cancer cells and the tissue fibers (see e.g., [A7.16]).
We further assume the boundary conditions

$$\frac{\partial c}{\partial\boldsymbol{\nu}} = \frac{\partial v}{\partial\boldsymbol{\nu}} = \frac{\partial l}{\partial\boldsymbol{\nu}} = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{7.2.6}$$

where $\boldsymbol{\nu}$ denotes the outward unit normal vector on $\partial\Omega$, and the initial conditions

$$\begin{matrix} c(0,\boldsymbol{x}) = c_0(\boldsymbol{x}), & v(0,\boldsymbol{x}) = v_0(\boldsymbol{x}), & l(0,\boldsymbol{x}) = l_0(\boldsymbol{x}), \\ \kappa(0,\boldsymbol{x}) = \kappa_0(\boldsymbol{x}), & \boldsymbol{y}(t,\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}), & \end{matrix} \quad t\in(-\infty,0],\ \boldsymbol{x}\in\Omega. \tag{7.2.7}$$

In our model we consider

$$\varphi(\kappa,c,v) = \frac{D_c\kappa}{1+\frac{cv}{K_cK_v}},\ \psi(\kappa,v) = \frac{D_H\kappa v}{K_v+v},\ f(c,l) = \frac{D_k}{1+\frac{cl}{K_c\lambda}},\ H(\boldsymbol{y}) = \frac{My_1}{R_0+y_2} \tag{7.2.8}$$

for the random motility, haptotaxis and chemotaxis functions and for the function modeling the influence of the integrin binding on the contractivity, respectively. Here K_c and K_v are the carrying capacities for the cancer cells and ECM, respectively, and λ is an appropriate reference variable for the proteolytic rests.

7.3 Local Existence

7.3.1 The case with distributed delay

In this case we start by considering a distributed delay

$$\kappa(t,\boldsymbol{x}) = \int_0^\infty q e^{-qu} \tilde{H}(\boldsymbol{y}(t-u,\boldsymbol{x}))\,du, \qquad (t,\boldsymbol{x}) \in (0,T)\times\Omega, \tag{7.3.1}$$

for the characterization of the cell contractivity. Using the transformation $s = t - u$, (7.3.1) is equivalent to

$$\kappa(t,\boldsymbol{x}) = \int_{-\infty}^t q e^{-q(t-s)} \tilde{H}(\boldsymbol{y}(s,\boldsymbol{x}))\,ds, \qquad (t,\boldsymbol{x}) \in (0,T)\times\Omega.$$

In view of (7.2.7) this means that κ fulfills (7.2.4) with $\tau = 0$, $H(\boldsymbol{y}) := q\tilde{H}(\boldsymbol{y})$ and the initial condition

$$\kappa(0,\boldsymbol{x}) = \int_{-\infty}^0 q e^{qs} \tilde{H}(\boldsymbol{y}_0(\boldsymbol{x}))\,ds = \tilde{H}(\boldsymbol{y}_0(\boldsymbol{x})) =: \kappa_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega.$$

Hence, the distributed delay corresponds to the case $\tau = 0$ for the problem (7.2.5)-(7.2.7).

Thus, we fix $\tau = 0$, $p \in (\frac{n+2}{2}, \infty)$ and define the spaces

$$\begin{aligned} X &:= \left\{ u \in L^p(0,T;W^{2,p}(\Omega)) \;:\; u_t \in L^p(0,T;L^p(\Omega)) \right\}, \\ Z &:= L^{2p}(0,T;W^{1,2p}(\Omega)), \qquad V := C^1(0,T;C^0(\bar{\Omega})). \end{aligned}$$

Then we have the following local existence result for the case with distributed delay.

Theorem 7.3.1 *Assume* $\tau = 0$, $p \in (\frac{n+2}{2}, \infty)$,

$$\begin{aligned} & c_0, v_0, l_0 \in W^{2,p}(\Omega),\ \kappa_0 \in W^{1,2p}(\Omega),\ \boldsymbol{y}_0 \in (W^{1,2p}(\Omega))^2, \quad \frac{\partial c_0}{\partial \boldsymbol{\nu}} = \frac{\partial v_0}{\partial \boldsymbol{\nu}} = \frac{\partial l_0}{\partial \boldsymbol{\nu}} = 0 \\ & \text{on } \partial\Omega, \quad 0 < c_0 < K_c,\ 0 < v_0 < K_v,\ l_0 > 0,\ \kappa_0 > 0 \text{ and } \boldsymbol{y}_0 \in Y \text{ for all } \boldsymbol{x} \in \bar{\Omega} \end{aligned} \tag{7.3.2}$$

together with (7.2.3) *and let*

$$\begin{aligned} & H \in C^1(\bar{Y}), f \in C^1([0,\infty)^2), \varphi \in C^1([0,\infty)^3), \psi \in C^1([0,\infty)^2) \text{ be} \\ & \text{nonnegative such that for any } 0 < a < b < \infty \text{ there exists } \delta_{a,b} > 0 \text{ with} \\ & \varphi(\kappa, c, v) \geq \delta_{a,b} \text{ for all } (\kappa, c, v) \in [a,b] \times [0,b]^2. \end{aligned} \tag{7.3.3}$$

Then there is $T > 0$ such that there exists a unique solution to (7.2.5)-(7.2.7) satisfying

$$\begin{aligned} &c, l \in X,\ v \in X \cap V,\ \kappa \in Z \cap V,\ \boldsymbol{y} \in Z^2 \cap V^2 \text{ such that } 0 \leq c \leq K_c,\\ &0 < v \leq K_v,\ l \geq 0,\ \kappa > 0 \text{ and } \boldsymbol{y} \in Y \text{ for all } (t, \boldsymbol{x}) \in [0, T) \times \bar{\Omega}. \end{aligned} \tag{7.3.4}$$

Proof. We define

$$X_0 := \left\{ c \in X \ :\ c \geq 0,\ \|c\|_X \leq \gamma := \|c_0\|_{W^{2,p}(\Omega)} + 1,\ \frac{\partial c}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } (0, T) \times \partial\Omega \right\},$$

fix $T_0 \in (0, \infty)$ such that $c_0 \in X_0$ for all $T \in (0, T_0]$ and define the map $\mathcal{F} : X_0 \to X_0$ with $\mathcal{F}(\tilde{c}) = c$, where c is defined in the following way: Given $\tilde{c} \in X_0$, we let v, l, κ, $\boldsymbol{y}$ and c denote the solutions of the problems

$$\begin{cases} v_t = -\delta_v \tilde{c} v + \mu_v v \left(1 - \eta_2 \frac{\tilde{c}}{K_c} - \frac{v}{K_v}\right) & \text{in } (0, T) \times \Omega,\\ v(0, \boldsymbol{x}) = v_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{7.3.5}$$

$$\begin{cases} l_t = \alpha \Delta l + \delta_l \tilde{c} v - \beta l & \text{in } (0, T) \times \Omega,\\ \frac{\partial l}{\partial \boldsymbol{\nu}} = 0 & \text{on } (0, T) \times \partial\Omega,\\ l(0, \boldsymbol{x}) = l_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{7.3.6}$$

$$\begin{cases} \boldsymbol{y}_t = \boldsymbol{G}(v, l, \boldsymbol{y}) & \text{in } (0, T) \times \Omega,\\ \boldsymbol{y}(0, \boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{7.3.7}$$

$$\begin{cases} \kappa_t = -q\kappa + H(\boldsymbol{y}(t)) & \text{in } (0, T) \times \Omega,\\ \kappa(0, \boldsymbol{x}) = \kappa_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{7.3.8}$$

$$\begin{cases} c_t = \nabla \cdot (\varphi(\kappa, \tilde{c}, v) \nabla c) - \nabla \cdot (\psi(\kappa, v) c \nabla v) - \nabla \cdot (f(\tilde{c}, l) c \nabla l) \\ \qquad + \mu_c c \left(1 - \frac{\tilde{c}}{K_c} - \eta_1 \frac{v}{K_v}\right) & \text{in } (0, T) \times \Omega,\\ \frac{\partial c}{\partial \boldsymbol{\nu}} = 0 & \text{on } (0, T) \times \partial\Omega,\\ c(0, \boldsymbol{x}) = c_0(\boldsymbol{x}) & \text{in } \Omega. \end{cases} \tag{7.3.9}$$

In order to obtain a unique solution of (7.2.5)-(7.2.7) for $T \in (0, T_0]$ small enough, we proceed in several steps. In view of the highly nonlinear couplings between the equations contained in (7.2.5), we use the approximations (7.3.5)-(7.3.9) which can be solved consecutively for given $\tilde{c}$. In Step 1, we provide estimates and regularity properties for the solutions to (7.3.5)-(7.3.9) which allow us to deduce that the above mapping $\mathcal{F}$ is well-defined for $T > 0$ small enough. In Step 2, we then iteratively define a sequence of solutions to (7.3.5)-(7.3.9) and prove its convergence to a weak solution of the original problem (7.2.5)-(7.2.7) by using the estimates from the first step and standard compactness arguments. Finally, the uniqueness of solutions to (7.2.5)-(7.2.7) is shown in Step 3 by the aid of Gronwall's lemma.

Step 1: Estimates
For given $\tilde{c} \in X_0$, (7.3.5) is an ODE of Bernoulli type which is explicitly solvable. Using (7.3.2) along with the nonnegativity of $\tilde{c}$, an ODE comparison principle implies that

$$0 \le v \le K_v \qquad \text{in } [0,T] \times \bar{\Omega} \tag{7.3.10}$$

is fulfilled, since 0 and K_v are constant sub- and supersolutions to (7.3.5), respectively. As X is continuously embedded into $C^0([0,T]\times\bar{\Omega})$ due to $p > \frac{n+2}{2}$ (see [A7.22, Lemma II.3.3]) and $\|\tilde{c}\|_X \le \gamma$, we obtain from (7.3.10) that $v_t \ge -C_1 v$ with some $C_1 > 0$ depending on γ. In view of $T \le T_0$ and (7.3.2) this implies

$$v(t,\boldsymbol{x}) \ge e^{-C_1 T_0} \left(\min_{\boldsymbol{x} \in \bar{\Omega}} v_0(\boldsymbol{x}) \right) =: C_2 > 0, \qquad (t,\boldsymbol{x}) \in (0,T) \times \Omega. \tag{7.3.11}$$

Hence, $z := \frac{1}{v}$ is uniformly bounded in $(0,T) \times \Omega$ and satisfies the linear ODE

$$z_t = \left(-\mu_v + \tilde{c} \left(\delta_v + \eta_2 \frac{\mu_v}{K_c} \right) \right) z + \frac{\mu_v}{K_v} \qquad \text{in } (0,T) \times \Omega. \tag{7.3.12}$$

In view of (7.3.2), (7.3.10), (7.3.11) and $\tilde{c} \in X_0$, we therefore conclude that v fulfills

$$C_2 \le v \le K_v \quad \text{in } [0,T] \times \bar{\Omega}, \quad \|v\|_X + \|v\|_V \le C_3, \quad \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } (0,T) \times \partial\Omega \tag{7.3.13}$$

with some C_3 depending on γ and T_0, where $v \in V$ as $\tilde{c}, c_0$ are continuous due to $p > \frac{n+2}{2}$ and v_t is continuous and uniformly bounded by (7.3.5).

In view of $\tilde{c} \in X_0$ and (7.3.13), we have that $\tilde{c}v$ is uniformly bounded in $(0,T) \times \Omega$. Hence, by [A7.22, Theorem IV.9.1] (and the remark at the end of Section IV.9 concerning the Neumann problem) there is a unique solution l of (7.3.6) which satisfies

$$l \ge 0 \quad \text{in } (0,T) \times \Omega \qquad \text{and} \qquad \|l\|_X \le C_4 \tag{7.3.14}$$

with some C_4 depending on γ and T_0, where the nonnegativity of l follows from the comparison principle and the nonnegativity of $\tilde{c}$, v and l_0.

Now (7.3.7) is a linear ODE for $\boldsymbol{y}$. As furthermore $\boldsymbol{G}$ satisfies the subtangential condition with respect to Y for all nonnegative v and l, we obtain that Y is a positive invariant set for (7.3.7). Thus, in view of (7.3.2), (7.3.13) and (7.3.14) we deduce that there is a unique solution $\boldsymbol{y}$ of (7.3.7) such that

$$\boldsymbol{y}(t,\boldsymbol{x}) \in Y \quad \text{for } (t,\boldsymbol{x}) \in [0,T] \times \bar{\Omega} \qquad \text{and} \qquad \|\boldsymbol{y}\|_{Z^2} + \|\boldsymbol{y}\|_{V^2} \le C_5 \tag{7.3.15}$$

hold with some constant C_5 depending on γ and T_0, since X is continuously embedded into Z and $C^0([0,T]\times\bar{\Omega})$ due to $p > \frac{n+2}{2}$ (see [A7.22, Lemma II.3.3]) and as the continuity of v and l along with (7.3.7) imply $\boldsymbol{y} \in V^2$.

As (7.3.8) is a linear ODE for κ and H is nonnegative, we deduce from (7.3.2), (7.3.13)-(7.3.15) and the comparison principle that (7.3.8) has a unique solution which fulfills

$$\begin{aligned} &0 < C_6 := e^{-qT_0}\left(\min_{\boldsymbol{x}\in\bar{\Omega}} \kappa_0(\boldsymbol{x})\right) \le \kappa(t,\boldsymbol{x}) \le C_7 \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\Omega, \\ &\|\kappa\|_Z + \|\kappa\|_V \le C_8 \end{aligned} \tag{7.3.16}$$

with some constants depending on γ and T_0, as (7.3.8) is a linear ODE and $H(\boldsymbol{y}) \in C^0([0,T]\times\bar{\Omega})$ and $\kappa_0 \in C^0(\bar{\Omega})$ hold in view of the continuity of H, (7.3.15) and the continuous embedding of $W^{1,2p}(\Omega)$ into $C^0(\bar{\Omega})$ for $p > \frac{n+2}{2}$.

Finally, (7.3.9) is a linear parabolic equation for c, of the form

$$c_t = a_{ii}c_{x_ix_i} + a_ic_{x_i} + ac,$$

where

$$\begin{aligned} a_{ii} &:= \varphi(\kappa,\tilde{c},v), \\ a_i &:= \frac{\partial\varphi}{\partial\kappa}(\kappa,\tilde{c},v)\kappa_{x_i} + \frac{\partial\varphi}{\partial c}(\kappa,\tilde{c},v)\tilde{c}_{x_i} + \frac{\partial\varphi}{\partial v}(\kappa,\tilde{c},v)v_{x_i} - \psi(\kappa,v)v_{x_i} - f(\tilde{c},l)l_{x_i}, \\ a &:= -\psi(\kappa,v)v_{x_ix_i} - \frac{\partial\psi}{\partial\kappa}(\kappa,v)\kappa_{x_i}v_{x_i} - \frac{\partial\psi}{\partial v}(\kappa,v)(v_{x_i})^2 - f(\tilde{c},l)l_{x_ix_i} \\ &\quad -\frac{\partial f}{\partial c}(\tilde{c},l)\tilde{c}_{x_i}l_{x_i} - \frac{\partial f}{\partial l}(\tilde{c},l)(l_{x_i})^2 + \mu_c\left(1 - \frac{\tilde{c}}{K_c} - \eta_1\frac{v}{K_v}\right). \end{aligned}$$

In view of (7.3.3) and (7.3.13)-(7.3.16) and due to the continuous embedding of X into Z, a_{ii} is continuous in $[0,T]\times\bar{\Omega}$ and there are positive constants C_9 and C_{10} depending on γ and T_0 such that

$$C_9 \le a_{ii} \le C_{10} \quad \text{for } (t,\boldsymbol{x}) \in [0,T]\times\bar{\Omega}, \quad \|a_i\|_{L^{2p}((0,T)\times\Omega)} + \|a\|_{L^p((0,T)\times\Omega)} \le C_{10}. \tag{7.3.17}$$

Hence, by Theorem IV.9.1, its proof and the remark at the end of Section IV.9 in [A7.22], there is some $T_1 \le T_0$ such that for any $T \in (0,T_1]$ there is a unique solution c of (7.3.9) fulfilling

$$\|c\|_X \le \|c_0\|_{W^{2,p}(\Omega)} + \varepsilon(T), \tag{7.3.18}$$

where $\varepsilon(T) \to 0$ as $T \searrow 0$. Hence, there exists $T_2 \in (0,T_1]$ such that $\|c\|_X \le \gamma$ for all $T \in (0,T_2]$. As c satisfies the boundary condition (7.2.6) and the comparison principle implies $c \ge 0$ in $(0,T)\times\Omega$, we conclude that $\mathcal{F}$ is a well-defined self-mapping for $T \in (0,T_2]$.

Step 2: Existence

For $m \ge 1$ let v_m, l_m, $\boldsymbol{y}_m$, κ_m and c_m denote the solutions to (7.3.5)-(7.3.9) with $\tilde{c} := c_{m-1}$. In particular, we have $c_m = \mathcal{F}(c_{m-1})$. Due to Step 1, we have $\|c_m\|_X \le \gamma$ for all $m \in \mathbb{N}$. As $X = L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega))$ is reflexive, X is continuously embedded into $C^\alpha([0,T]\times\bar{\Omega})$ for $0 < \alpha < 1 - \frac{n+2}{2p}$ (see [A7.22, Lemma II.3.3]) and X is compactly embedded into $L^p(0,T;W^{1,2p}(\Omega))$ due to $p > \frac{n+2}{2}$ and the Aubin-Lions lemma (see [A7.33, Theorem III.2.1]). Hence, there exists a subsequence of $(c_m)_{m\in\mathbb{N}}$ (not relabeled) and $c \in X$ such that

$$\begin{aligned} &c_m \rightharpoonup c \text{ weakly in } X,\\ &c_m \to c \text{ strongly in } L^p(0,T;W^{1,2p}(\Omega)) \text{ and in } C^0([0,T]\times\bar{\Omega}) \end{aligned} \tag{7.3.19}$$

for $m \to \infty$. In particular, as X is continuously embedded into Z, we have

$$\nabla c_m \to \nabla c \text{ a.e. in } (0,T)\times\Omega \quad \text{and} \quad \nabla c_m \rightharpoonup \nabla c \text{ weakly in } L^{2p}((0,T)\times\Omega) \tag{7.3.20}$$

for $m \to \infty$ up to a further choice of a subsequence.

In a similar way, since v_m fulfills (7.3.13) for all $m \in \mathbb{N}$, there is a subsequence such that

$$\begin{aligned} &v_m \rightharpoonup v \text{ weakly in } X,\\ &v_m \to v \text{ strongly in } L^p(0,T;W^{1,2p}(\Omega)) \text{ and in } C^0([0,T]\times\bar{\Omega}),\\ &\nabla v_m \to \nabla v \text{ a.e. in } (0,T)\times\Omega \quad \text{and} \quad \nabla v_m \rightharpoonup \nabla v \text{ weakly in } L^{2p}((0,T)\times\Omega) \end{aligned} \tag{7.3.21}$$

for $m \to \infty$. Hence, in view of (7.3.19) and (7.3.21) and as v_m solves (7.3.5) with $\tilde{c} = c_{m-1}$ for $m \in \mathbb{N}$, v is a solution to

$$\int_0^T \int_\Omega -v\Phi_t \, d\boldsymbol{x}dt = \int_0^T \int_\Omega \left[-\delta_v cv + \mu_v v\left(1 - \eta_2\frac{c}{K_c} - \frac{v}{K_v}\right)\right]\Phi \, d\boldsymbol{x}dt$$

for all $\Phi \in C_0^\infty((0,T)\times\Omega)$ so that v solves the second equation of (7.2.5) in the weak sense. As its right-hand side is continuous in $[0,T]\times\bar{\Omega}$, we deduce that $v \in V$ is a classical solution of this equation. Due to (7.3.21) and (7.3.13), v further satisfies (7.2.6), (7.2.7) and (7.3.4).

Using (7.3.14), we obtain a further subsequence such that

$$\begin{aligned} &l_m \rightharpoonup l \text{ weakly in } X,\\ &l_m \to l \text{ strongly in } L^p(0,T;W^{1,2p}(\Omega)) \text{ and in } C^0([0,T]\times\bar{\Omega}),\\ &\nabla l_m \to \nabla l \text{ a.e. in } (0,T)\times\Omega \quad \text{and} \quad \nabla l_m \rightharpoonup \nabla l \text{ weakly in } L^{2p}((0,T)\times\Omega) \end{aligned} \tag{7.3.22}$$

for $m \to \infty$. In view of (7.3.14) and (7.3.21) and since l_m satisfies (7.3.6) with $\tilde{c} = c_{m-1}$ for $m \in \mathbb{N}$, this implies that l is a weak solution to the third equation of (7.2.5) such that (7.2.6), (7.2.7) and (7.3.4) are fulfilled.

Moreover, (7.3.15) also implies that $(\boldsymbol{y}_m)_m$ is uniformly bounded in $(W^{1,2p}((0,T)\times\Omega))^2$ and this space is compactly embedded into $(C^0([0,T]\times\bar{\Omega}))^2$, due to $p > \frac{n+2}{2}$. Thus, we can choose another subsequence such that

$$\begin{aligned} &\boldsymbol{y}_m \rightharpoonup \boldsymbol{y} \text{ weakly in } Z^2 \text{ and in } (W^{1,2p}((0,T)\times\Omega))^2,\\ &\boldsymbol{y}_m \to \boldsymbol{y} \text{ strongly in } (C^0([0,T]\times\bar{\Omega}))^2 \end{aligned} \tag{7.3.23}$$

for $m \to \infty$. Combined with (7.3.21), (7.3.22) and (7.3.7) for $m \in \mathbb{N}$, this implies that $\boldsymbol{y}$ is a weak solution to the fourth equation of (7.2.5). As its right-hand side is continuous in $[0,T] \times \bar{\Omega}$, we deduce that $\boldsymbol{y} \in V^2$ is a classical solution of this equation. Furthermore, (7.2.7) and (7.3.4) are satisfied due to (7.3.15).

In a similar way, by (7.3.16) we obtain a subsequence such that

$$\begin{aligned} &\kappa_m \rightharpoonup \kappa \text{ weakly in } Z \text{ and in } W^{1,2p}((0,T)\times\Omega), \\ &\kappa_m \to \kappa \text{ strongly in } C^0([0,T]\times\bar{\Omega}) \end{aligned} \tag{7.3.24}$$

for $m \to \infty$. Together with (7.3.23) and (7.3.8) for $m \in \mathbb{N}$, this implies that κ is a weak solution to (7.2.4). Due to the continuity in $[0,T] \times \bar{\Omega}$ of the right hand side in (7.2.4), we deduce that $\kappa \in V$ is a classical solution of this equation. Furthermore, (7.2.7) and (7.3.4) hold, due to (7.3.16).

Now c_m is a weak solution to

$$\begin{cases} \partial_t c_m = \nabla\cdot(\varphi(\kappa_m, c_{m-1}, v_m)\nabla c_m) - \nabla\cdot(\psi(\kappa_m, v_m) c_m \nabla v_m) \\ \qquad\qquad -\nabla\cdot(f(c_{m-1}, l_m) c_m \nabla l_m) + \mu_c c_m \left(1 - \frac{c_{m-1}}{K_c} - \eta_1 \frac{v_m}{K_v}\right) \text{ in } (0,T)\times\Omega, \\ \frac{\partial c_m}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{7.3.25}$$

which satisfies the initial condition $c_m(0,\boldsymbol{x}) = c_0(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$ and $m \in \mathbb{N}$. Hence, by letting $m \to \infty$ in each of the integral terms involved in the weak formulation of (7.3.25) and by using (7.3.19)-(7.3.24), we conclude that c is a weak solution to the first equation of (7.2.5) such that (7.2.6) and (7.2.7) are fulfilled. In view of (7.3.19) and $c_m \geq 0$ we further have $c \in X \cap C^0([0,T]\times\bar{\Omega})$ and $c \geq 0$. As $c_0 < K_C$ in $\bar{\Omega}$, by choosing $T \in (0, T_2]$ small enough we have $0 \leq c \leq K_C$ in $[0,T]\times\bar{\Omega}$. Altogether, $\boldsymbol{S} := (c, v, l, \boldsymbol{y}, \kappa)$ is a solution to (7.2.5)-(7.2.7) which satisfies (7.3.4).

Step 3: Uniqueness

We now fix T as chosen in Step 2 and let $\boldsymbol{S}^{(j)} := (c^{(j)}, v^{(j)}, l^{(j)}, \boldsymbol{y}^{(j)}, \kappa^{(j)})$, $j \in \{1,2\}$, denote two solutions to (7.2.5)-(7.2.7) satisfying (7.3.4). As X is continuously embedded into Z and $L^\infty((0,T)\times\Omega)$ due to [A7.22, Lemma II.3.3] and $p > \frac{n+2}{2}$, by (7.3.13)-(7.3.16) and (7.3.18), there exists $C_{11} > 0$ such that

$$\begin{aligned} &\|c^{(j)}\|_{L^\infty((0,T)\times\Omega)} + \|\nabla c^{(j)}\|_{L^{2p}((0,T)\times\Omega)} + \|v^{(j)}\|_{L^\infty((0,T)\times\Omega)} \\ &+ \|\nabla v^{(j)}\|_{L^{2p}((0,T)\times\Omega)} + \|l^{(j)}\|_{L^\infty((0,T)\times\Omega)} + \|\nabla l^{(j)}\|_{L^{2p}((0,T)\times\Omega)} \\ &+ \|\boldsymbol{y}^{(j)}\|_{(L^\infty((0,T)\times\Omega))^2} + \|\kappa^{(j)}\|_{L^\infty((0,T)\times\Omega)} + \|\nabla\kappa^{(j)}\|_{L^{2p}((0,T)\times\Omega)} \leq C_{11} \end{aligned} \tag{7.3.26}$$

is fulfilled for $j \in \{1,2\}$.

Since $z^{(j)} := \frac{1}{v^{(j)}}$ satisfies (7.3.12) with $\tilde{c} = c^{(j)}$, we have

$$\begin{aligned}\left(z^{(1)} - z^{(2)}\right)_t &= \left(-\mu_v + c^{(1)}\left(\delta_v + \eta_2\frac{\mu_v}{K_c}\right)\right)\left(z^{(1)} - z^{(2)}\right)\\ &\quad + \left(\delta_v + \eta_2\frac{\mu_v}{K_c}\right)\left(c^{(1)} - c^{(2)}\right)z^{(2)}\end{aligned}$$

which implies

$$\begin{aligned}\left(z^{(1)} - z^{(2)}\right)(t,\boldsymbol{x}) &= \left(\delta_v + \eta_2\frac{\mu_v}{K_c}\right)\int_0^t \exp\left(\int_s^t \Big(-\mu_v + c^{(1)}(\sigma,\boldsymbol{x})\right.\\ &\quad \left.\cdot\left(\delta_v + \eta_2\frac{\mu_v}{K_c}\right)\Big)d\sigma\right)\left[\left(c^{(1)} - c^{(2)}\right)z^{(2)}\right](s,\boldsymbol{x})\,ds \quad (7.3.27)\end{aligned}$$

for $(t,\boldsymbol{x}) \in (0,T)\times\Omega$.
Hence, we deduce from (7.3.12), (7.3.13) and (7.3.26) that $\frac{1}{K_v} \le z^{(j)} \le \frac{1}{C_2}$,

$$\begin{aligned}\left|\nabla z^{(j)}\right|(t,\boldsymbol{x}) &\le C_{12}\left(\int_0^t \left|\nabla c^{(j)}\right|(\sigma,\boldsymbol{x})\,d\sigma + |\nabla v_0|(\boldsymbol{x})\right),\\ \left|\nabla z^{(1)} - \nabla z^{(2)}\right|(t,\boldsymbol{x}) &\le C_{12}\int_0^t\left(\left|\nabla z^{(2)}\right|(s,\boldsymbol{x}) + \int_0^t\left|\nabla c^{(1)}\right|(\sigma,\boldsymbol{x})\,d\sigma\right)\\ &\quad \cdot\left|c^{(1)} - c^{(2)}\right|(s,\boldsymbol{x})\,ds\\ &\quad + C_{12}\int_0^t\left|\nabla c^{(1)} - \nabla c^{(2)}\right|(s,\boldsymbol{x})\,ds \quad (7.3.28)\end{aligned}$$

are satisfied for $(t,\boldsymbol{x}) \in (0,T)\times\Omega$ and $j \in \{1,2\}$.
Therefore, we have

$$\begin{aligned}\left|v^{(1)} - v^{(2)}\right|(t,\boldsymbol{x}) &= \left|\frac{z^{(2)} - z^{(1)}}{z^{(1)}z^{(2)}}\right|(t,\boldsymbol{x}) \le C_{13}\int_0^t\left|c^{(1)} - c^{(2)}\right|(s,\boldsymbol{x})\,ds,\\ \left|\nabla v^{(1)} - \nabla v^{(2)}\right|(t,\boldsymbol{x}) &= \left|-\frac{\nabla z^{(1)}}{(z^{(1)})^2} + \frac{\nabla z^{(2)}}{(z^{(2)})^2}\right|(t,\boldsymbol{x})\\ &= \left|\frac{\nabla(z^{(2)} - z^{(1)})}{(z^{(2)})^2} + \frac{((z^{(1)})^2 - (z^{(2)})^2)\nabla z^{(1)}}{(z^{(1)})^2(z^{(2)})^2}\right|(t,\boldsymbol{x})\\ &\le C_{13}\left[|\nabla v_0|(\boldsymbol{x}) + \int_0^t\left(\left|\nabla c^{(1)}\right| + \left|\nabla c^{(2)}\right|\right)(\sigma,\boldsymbol{x})\,d\sigma\right]\\ &\quad \cdot\int_0^t\left|c^{(1)} - c^{(2)}\right|(s,\boldsymbol{x})\,ds + C_{13}\int_0^t\left|\nabla c^{(1)} - \nabla c^{(2)}\right|(s,\boldsymbol{x})\,ds \quad (7.3.29)\end{aligned}$$

for $(t,\boldsymbol{x}) \in (0,T)\times\Omega$.

In particular, by Hölder's inequality this implies

$$\begin{aligned}\int_\Omega \left|v^{(1)} - v^{(2)}\right|^2 (t, \boldsymbol{x})\, d\boldsymbol{x} &\leq C_{13}^2 \int_\Omega t \int_0^t \left|c^{(1)} - c^{(2)}\right|^2 (s, \boldsymbol{x})\, ds d\boldsymbol{x} \\ &\leq C_{13}^2 t^2 \sup_{s\in(0,t)} \int_\Omega \left|c^{(1)} - c^{(2)}\right|^2 (s, \boldsymbol{x}) d\boldsymbol{x} \\ &\leq C_{13}^2 T^2 \left\|c^{(1)} - c^{(2)}\right\|^2_{L^\infty(0,t;L^2(\Omega))} \end{aligned} \tag{7.3.30}$$

and

$$\int_0^t \int_\Omega \left|v^{(1)} - v^{(2)}\right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} ds \leq C_{13}^2 T^2 \int_0^t \left\|c^{(1)} - c^{(2)}\right\|^2_{L^\infty(0,s;L^2(\Omega))} ds \tag{7.3.31}$$

for $t \in (0, T)$.

In view of (7.2.5)-(7.2.7), $L := l^{(1)} - l^{(2)}$ satisfies

$$L_t = \alpha \Delta L - \beta L + \delta_l \left(c^{(1)} v^{(1)} - c^{(2)} v^{(2)}\right) \qquad \text{in } (0, T) \times \Omega \tag{7.3.32}$$

together with the homogeneous Neumann boundary condition and $L(0, \boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$. Hence, by [A7.22, Theorem IV.9.1] (and the remark at the end of Section IV.9 concerning the Neumann problem) we obtain

$$\|L\|_{L^2(0,t;W^{2,2}(\Omega))} \leq C_{14} \left\|c^{(1)} v^{(1)} - c^{(2)} v^{(2)}\right\|_{L^2((0,t)\times\Omega)}$$

for all $t \in (0, T)$ and deduce from (7.3.26) and (7.3.31) that

$$\begin{aligned} &\left\|l^{(1)} - l^{(2)}\right\|^2_{L^2((0,t)\times\Omega)} + \left\|\nabla l^{(1)} - \nabla l^{(2)}\right\|^2_{L^2((0,t)\times\Omega)} \\ \leq\ & C_{15} \int_0^t \left\|c^{(1)} - c^{(2)}\right\|^2_{L^\infty(0,s;L^2(\Omega))} ds \end{aligned} \tag{7.3.33}$$

is fulfilled for all $t \in (0, T)$.
Moreover, by using (7.3.32), $L(0, \boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$, (7.3.26), (7.3.30) and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \left|l^{(1)} - l^{(2)}\right|^2 (t, \boldsymbol{x})\, d\boldsymbol{x} &= 2 \int_0^t \int_\Omega L L_t(s, \boldsymbol{x})\, d\boldsymbol{x} ds \\ &= -2\alpha \int_0^t \int_\Omega |\nabla L|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} ds - 2\beta \int_0^t \int_\Omega L^2(s, \boldsymbol{x})\, d\boldsymbol{x} ds \\ &\quad + 2\delta_l \int_0^t \int_\Omega L \left(c^{(1)} v^{(1)} - c^{(2)} v^{(2)}\right) (s, \boldsymbol{x})\, d\boldsymbol{x} ds \\ &\leq \frac{\delta_l^2}{2\beta} \int_0^t \int_\Omega \left(c^{(1)} v^{(1)} - c^{(2)} v^{(2)}\right)^2 (s, \boldsymbol{x})\, d\boldsymbol{x} ds \end{aligned}$$

$$\leq \quad C_{16} \left\| c^{(1)} - c^{(2)} \right\|^2_{L^\infty(0,t;L^2(\Omega))} \tag{7.3.34}$$

for $t \in (0, T)$.

Since $\boldsymbol{y}^{(j)}$ is a solution to a linear ODE, we obtain (similarly as done above for $z^{(j)}$) from (7.3.26), (7.3.29), and the regularity of $\boldsymbol{G}$ that

$$\left| \boldsymbol{y}_i^{(1)} - \boldsymbol{y}_i^{(2)} \right| (t, \boldsymbol{x}) \leq C_{17} \int_0^t \left(\left| c^{(1)} - c^{(2)} \right| + \left| l^{(1)} - l^{(2)} \right| \right) (s, \boldsymbol{x}) \, ds \tag{7.3.35}$$

holds for $t \in (0, T)$ and $i \in \{1, 2\}$. Thus, in a similar manner we have

$$\left| \kappa^{(1)} - \kappa^{(2)} \right| (t, \boldsymbol{x}) \leq C_{18} \int_0^t \left(\left| c^{(1)} - c^{(2)} \right| + \left| l^{(1)} - l^{(2)} \right| \right) (s, \boldsymbol{x}) \, ds \tag{7.3.36}$$

for $t \in (0, T)$ due to (7.3.26), (7.3.35) and the regularity of H.

Next in order to abbreviate notation we define $\varphi_j := \varphi(\kappa^{(j)}, c^{(j)}, v^{(j)})$, $\psi_j := \psi(\kappa^{(j)}, v^{(j)})$ and $f_j := f(c^{(j)}, l^{(j)})$ for $j \in \{1, 2\}$. As $2p \geq 2$ and $c^{(j)}$ is a weak solution to the first equation of (7.2.5) fulfilling (7.2.6), (7.2.7) and (7.3.26), we deduce that

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \left(c^{(1)} - c^{(2)} \right)^2 (t, \boldsymbol{x}) d\boldsymbol{x} \\
= \quad & \frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt} \left(c^{(1)} - c^{(2)} \right)^2 (s, \boldsymbol{x}) \, d\boldsymbol{x} ds \\
= \quad & \int_0^t \int_\Omega \nabla \left(c^{(1)} - c^{(2)} \right) \Big[- \varphi_1 \nabla c^{(1)} + \varphi_2 \nabla c^{(2)} + \psi_1 c^{(1)} \nabla v^{(1)} - \psi_2 c^{(2)} \nabla v^{(2)} \\
& + f_1 c^{(1)} \nabla l^{(1)} - f_2 c^{(2)} \nabla l^{(2)} \Big] (s, \boldsymbol{x}) \, d\boldsymbol{x} ds \\
& + \int_0^t \int_\Omega \mu_c \left(c^{(1)} - c^{(2)} \right) \left[c^{(1)} \left(1 - \frac{c^{(1)}}{K_c} - \eta_1 \frac{v^{(1)}}{K_v} \right) \right. \\
& \left. - c^{(2)} \left(1 - \frac{c^{(2)}}{K_c} - \eta_1 \frac{v^{(2)}}{K_v} \right) \right] (s, \boldsymbol{x}) \, d\boldsymbol{x} ds \\
= \quad & - \int_0^t \int_\Omega \varphi_1 \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x}) \, d\boldsymbol{x} ds \\
& + \int_0^t \int_\Omega \psi_1 c^{(1)} \nabla \left(v^{(1)} - v^{(2)} \right) \nabla \left(c^{(1)} - c^{(2)} \right) (s, \boldsymbol{x}) \, d\boldsymbol{x} ds \\
& + \int_0^t \int_\Omega \nabla \left(c^{(1)} - c^{(2)} \right) \Big[(\varphi_2 - \varphi_1) \nabla c^{(2)} + \left(\psi_1 c^{(1)} - \psi_2 c^{(2)} \right) \nabla v^{(2)} \\
& + \left(f_1 c^{(1)} - f_2 c^{(2)} \right) \nabla l^{(2)} \Big] (s, \boldsymbol{x}) \, d\boldsymbol{x} ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega f_1 c^{(1)}\nabla\left(l^{(1)}-l^{(2)}\right)\nabla\left(c^{(1)}-c^{(2)}\right)(s,\boldsymbol{x})\,d\boldsymbol{x}ds \\
&+\int_0^t\int_\Omega \mu_c\left(c^{(1)}-c^{(2)}\right)\left[c^{(1)}\left(1-\frac{c^{(1)}}{K_c}-\eta_1\frac{v^{(1)}}{K_v}\right)\right. \\
&\left.-c^{(2)}\left(1-\frac{c^{(2)}}{K_c}-\eta_1\frac{v^{(2)}}{K_v}\right)\right](s,\boldsymbol{x})\,d\boldsymbol{x}ds \\
=:\;& -I_1+I_2+I_3+I_4+I_5 \qquad (7.3.37)
\end{aligned}
$$

holds for $t\in(0,T)$.

We further define

$$
\begin{aligned}
g(t,\boldsymbol{x}) &:= \left(\left|c^{(1)}-c^{(2)}\right|+\left|l^{(1)}-l^{(2)}\right|\right)(t,\boldsymbol{x}),\\
h(t,\boldsymbol{x}) &:= \left(\left|\nabla c^{(2)}\right|+\left|\nabla v^{(2)}\right|+\left|\nabla l^{(2)}\right|\right)(t,\boldsymbol{x})
\end{aligned}
\qquad (7.3.38)
$$

for $(t,\boldsymbol{x})\in(0,T)\times\Omega$.
In view of (7.3.26)-(7.3.36), (7.3.16) and (7.3.3) there are positive constants ε and C_{19} such that

$$
\varphi_1\geq\varepsilon \qquad \text{in } (0,T)\times\Omega \qquad (7.3.39)
$$

and

$$
\begin{aligned}
&\left(|\varphi_1-\varphi_2|+\left|\psi_1 c^{(1)}-\psi_2 c^{(2)}\right|+\left|f_1 c^{(1)}-f_2 c^{(2)}\right|\right)(t,\boldsymbol{x})\\
&\quad\leq C_{19}\left(g(t,\boldsymbol{x})+\int_0^t g(s,\boldsymbol{x})\,ds\right), \qquad (7.3.40)\\
&\|h\|_{L^{2p}((0,t)\times\Omega)}\leq C_{19}
\end{aligned}
$$

are fulfilled for $t\in(0,T)$.

Next we fix

$$
r:=\frac{2p}{p-1} \qquad \text{and} \qquad a:=\frac{\frac{1}{2}-\frac{1}{r}}{\frac{1}{n}}=\frac{n}{2p} \qquad (7.3.41)
$$

and remark that $p>\frac{n+2}{2}$ yields $a\in(0,1)$. Therefore, by the inequalities of Gagliardo-Nirenberg and Young, there exist constants $C_{GN}>0$ and $C_\varepsilon>0$ such that

$$
\|u\|_{L^r(\Omega)}\leq C_{GN}\left(\|\nabla u\|^a_{L^2(\Omega)}\|u\|^{1-a}_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\right)\leq\varepsilon\|\nabla u\|_{L^2(\Omega)}+C_\varepsilon\|u\|_{L^2(\Omega)} \quad (7.3.42)
$$

is satisfied for all $u\in W^{1,2}(\Omega)$.

Furthermore, by (7.3.2) and (7.3.26) there is $C_{20}>0$ such that $w(t,\boldsymbol{x}):=h(t,\boldsymbol{x})+|\nabla v_0|(\boldsymbol{x})+\int_0^t(|\nabla c^{(1)}|+|\nabla c^{(2)}|)(\sigma,\boldsymbol{x})\,d\sigma$ satisfies

$$
\|w\|_{L^{2p}((0,t)\times\Omega)}\leq C_{20} \qquad (7.3.43)
$$

for all $t \in (0,T)$.
Thus, (7.3.29), (7.3.33), (7.3.34) and (7.3.40)-(7.3.43) along with $a \in (0,1)$ and $p(1-a) = \frac{2p-n}{2} > 1$ and the inequalities of Hölder and Young yield

$$
\begin{aligned}
& I_2 + I_3 \\
\leq\ & \int_0^t \int_\Omega \psi_1 c^{(1)} \nabla \left(v^{(1)} - v^{(2)} \right) \nabla \left(c^{(1)} - c^{(2)} \right) (s, \boldsymbol{x}) d\boldsymbol{x} ds \\
& + C_{19} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (s, \boldsymbol{x}) \cdot h(s, \boldsymbol{x}) \left(g(s, \boldsymbol{x}) + \int_0^s g(\sigma, \boldsymbol{x})\, d\sigma \right) d\boldsymbol{x} ds \\
\leq\ & C_{21} \int_\Omega \int_0^t \int_0^s \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (\sigma, \boldsymbol{x}) d\sigma \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (s, \boldsymbol{x}) ds d\boldsymbol{x} \\
& + C_{21} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (s, \boldsymbol{x}) \cdot w(s, \boldsymbol{x}) \int_0^s g(\sigma, \boldsymbol{x})\, d\sigma d\boldsymbol{x} ds \\
& + C_{19} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (s, \boldsymbol{x}) \cdot h(s, \boldsymbol{x}) \cdot g(s, \boldsymbol{x})\, d\boldsymbol{x} ds \\
\leq\ & C_{21} \int_\Omega \int_0^t \frac{1}{2} \frac{d}{ds} \left(\int_0^s \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (\sigma, \boldsymbol{x}) d\sigma \right)^2 ds d\boldsymbol{x} \\
& + C_{21} \int_0^t \int_0^s \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |w|^{2p} (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \\
& \cdot \left(\int_\Omega |g|^{\frac{2p}{p-1}} (\sigma, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{p-1}{2p}} d\sigma ds + C_{19} \int_0^t \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \\
& \cdot \left(\int_\Omega |h|^{2p} (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \cdot \left(\int_\Omega |g|^{\frac{2p}{p-1}} (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{p-1}{2p}} ds \\
\leq\ & C_{21} \int_\Omega \frac{1}{2} \left(\int_0^t \left| \nabla c^{(1)} - \nabla c^{(2)} \right| (s, \boldsymbol{x}) ds \right)^2 d\boldsymbol{x} \\
& + C_{21} \int_0^t \int_0^s \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |w|^{2p} (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \\
& \cdot \left[\varepsilon \left(\int_\Omega |\nabla g|^2 (\sigma, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} + C_\varepsilon \left(\int_\Omega g^2 (\sigma, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \right] d\sigma ds \\
& + C_{19} C_{GN} \int_0^t \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |h|^{2p} (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \\
& \cdot \left[\left(\int_\Omega |\nabla g|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{a}{2}} \cdot \left(\int_\Omega g^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1-a}{2}} + \left(\int_\Omega g^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \right] ds \\
\leq\ & C_{21} \frac{t}{2} \int_\Omega \int_0^t \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 ds d\boldsymbol{x} \\
& + \varepsilon C_{21} t^{\frac{1}{2}} \int_0^t \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s, \boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \left(\int_\Omega |w|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \cdot \left(\int_0^s \int_\Omega |\nabla g|^2(\sigma,\boldsymbol{x})\, d\boldsymbol{x} d\sigma \right)^{\frac{1}{2}} ds \\
& + C_\varepsilon C_{21} T^{\frac{1}{2}} \int_0^t \left(\int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2}} \\
& \cdot \left(\int_\Omega |w|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{2p}} \cdot \left(\int_0^s \int_\Omega g^2(\sigma,\boldsymbol{x})\, d\boldsymbol{x} d\sigma \right)^{\frac{1}{2}} ds \\
& + \frac{\varepsilon}{36} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} ds + \frac{\varepsilon}{72} \int_0^t \int_\Omega |\nabla g|^2(s,\boldsymbol{x})\, d\boldsymbol{x} ds \\
& + C_{22} \int_0^t \left(\int_\Omega |h|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{p(1-a)}} \int_\Omega g^2(s,\boldsymbol{x})\, d\boldsymbol{x} ds \\
& + C_{22} \int_0^t \left(\int_\Omega |h|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{p}} \int_\Omega g^2(s,\boldsymbol{x})\, d\boldsymbol{x} ds \\
\leq \; & C_{21} \frac{t}{2} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& + \varepsilon C_{21} t^{\frac{1}{2}} \Bigg[\left(\int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (\sigma,\boldsymbol{x})\, d\boldsymbol{x} d\sigma \right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_\Omega \left| \nabla l^{(1)} - \nabla l^{(2)} \right|^2 (\sigma,\boldsymbol{x})\, d\boldsymbol{x} d\sigma \right)^{\frac{1}{2}} \Bigg] \\
& \cdot \left(\int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} ds \right)^{\frac{1}{2}} \cdot \left(\int_0^t \left(\int_\Omega |w|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{p}} ds \right)^{\frac{1}{2}} \\
& + C_\varepsilon C_{21} T^{\frac{1}{2}} \left(\int_0^t \int_\Omega g^2(\sigma,\boldsymbol{x})\, d\boldsymbol{x} d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} ds \right)^{\frac{1}{2}} \\
& \cdot \left(\int_0^t \left(\int_\Omega |w|^{2p}(s,\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{p}} ds \right)^{\frac{1}{2}} \\
& + \frac{\varepsilon}{18} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} ds \\
& + \frac{\varepsilon}{36} \int_0^t \int_\Omega \left| \nabla l^{(1)} - \nabla l^{(2)} \right|^2 (s,\boldsymbol{x})\, d\boldsymbol{x} ds \\
& + C_{22} \left(\sup_{s \in (0,t)} \int_\Omega g^2(s,\boldsymbol{x})\, d\boldsymbol{x} \right) \cdot \left(C_{19}^{\frac{2}{1-a}} t^{\frac{p(1-a)-1}{p(1-a)}} + C_{19}^2 t^{\frac{p-1}{p}} \right) \\
\leq \; & \left(C_{21} \frac{t}{2} + \varepsilon C_{21} t^{\frac{1}{2}} \cdot C_{20} t^{\frac{p-1}{2p}} \right) \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& + \frac{\varepsilon}{18} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{9\varepsilon C_{21}^2 T}{2} \cdot C_{20}^2 T^{\frac{p-1}{p}} \cdot C_{15} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& +\frac{\varepsilon}{18} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& +C_\varepsilon^2 C_{21}^2 T \cdot \frac{9}{2\varepsilon} \cdot C_{20}^2 T^{\frac{p-1}{p}} \cdot (2C_{15}+2) \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& +\frac{\varepsilon}{18} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& +\frac{\varepsilon}{36} \cdot C_{15} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& +C_{22} \left(C_{19}^{\frac{4p}{2p-n}} t^{\frac{2p-n-2}{2p-n}} + C_{19}^2 t^{\frac{p-1}{p}} \right) \cdot (2C_{16}+2) \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,t;L^2(\Omega))}^2 \\
\leq\ & \left(C_{21} \frac{t}{2} + \varepsilon C_{21} C_{20} t^{\frac{2p-1}{2p}} + \frac{\varepsilon}{6} \right) \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& +C_{23} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& +C_{24} \left(t^{\frac{2p-n-2}{2p-n}} + t^{\frac{p-1}{p}} \right) \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,t;L^2(\Omega))}^2 \qquad (7.3.44)
\end{aligned}
$$

for $t \in (0, T)$.

Thus, fixing $t_0 := \min\{\frac{\varepsilon}{3C_{21}}, (6C_{20}C_{21})^{-\frac{2p}{2p-1}}, (8C_{24})^{-\frac{2p-n}{2p-n-2}}, (8C_{24})^{-\frac{p}{p-1}}, T\}$, inserting (7.3.39) and (7.3.44) into (7.3.37) and using Young's inequality along with (7.3.26), (7.3.29) and (7.3.33), we conclude that

$$
\begin{aligned}
& \frac{1}{2} \int_\Omega \left(c^{(1)} - c^{(2)} \right)^2 (t, \boldsymbol{x}) d\boldsymbol{x} \\
& \quad \leq\ -\varepsilon \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds + \frac{\varepsilon}{2} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds \\
& \qquad +C_{23} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds + \frac{1}{4} \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,t;L^2(\Omega))}^2 \\
& \qquad +\frac{\varepsilon}{2} \int_0^t \int_\Omega \left| \nabla c^{(1)} - \nabla c^{(2)} \right|^2 d\boldsymbol{x} ds + \frac{C_{25}}{\varepsilon} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& \qquad +C_{26} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds \\
& \quad \leq\ C_{27} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,s;L^2(\Omega))}^2 ds + \frac{1}{4} \left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,t;L^2(\Omega))}^2
\end{aligned}
$$

holds for all $t \in (0, t_0)$. As the right-hand side of the last inequality is nondecreasing for $t \in (0, t_0)$, we obtain

$$
\left\| c^{(1)} - c^{(2)} \right\|_{L^\infty(0,t;L^2(\Omega))}^2 = \sup_{s \in (0,t)} \int_\Omega \left(c^{(1)} - c^{(2)} \right)^2 (s, \boldsymbol{x}) d\boldsymbol{x}
$$

$$\leq \quad 4C_{27} \int_0^t \left\| c^{(1)} - c^{(2)} \right\|^2_{L^\infty(0,s;L^2(\Omega))} ds$$

for $t \in (0, t_0)$. In view of $c^{(1)}(0, \boldsymbol{x}) = c^{(2)}(0, \boldsymbol{x}) = c_0(\boldsymbol{x})$, Gronwall's lemma implies that $c^{(1)} = c^{(2)}$ in $[0, t_0] \times \Omega$.

As all the constants depend on T but not on t_0, by repeating this argument we have $c^{(1)} = c^{(2)}$ in $[mt_0, \min\{(m+1)t_0, T\}] \times \Omega$ for all $m \in \mathbb{N}$ such that $mt_0 \leq T$. Hence, $c^{(1)} = c^{(2)}$ in $[0, T] \times \Omega$. In view of (7.3.29)-(7.3.36), we further deduce that the solutions $(c^{(j)}, v^{(j)}, l^{(j)}, \boldsymbol{y}^{(j)}, \kappa^{(j)})$, $j \in \{1, 2\}$, to (7.2.5)-(7.2.7) coincide. Thus, the proof of the theorem is completed. ∎

7.3.2 The case with constant delay

Now we consider the case with a constant delay $\tau > 0$ in equation (7.2.4) for the cell contractivity. We prove the local existence by using the method of steps which is well-known in the context of delay differential equations (see e.g., [A7.3, A7.4] and the references therein).

Theorem 7.3.2 *Suppose that* $\tau > 0$ *and* $p \in (\frac{n+2}{2}, \infty)$ *and let* (7.3.2), (7.3.3) *and* (7.2.3) *be fulfilled. Then there exists* $T > 0$ *such that* (7.2.5)-(7.2.7) *has a unique solution satisfying* (7.3.4).

Proof. We take $T > 0$ as defined in Theorem 7.3.1 and set $t_m := \min\{m\tau, T\}$ for $m \in \mathbb{N}_0$ and $m_0 := \max\{m \in \mathbb{N}_0 : t_m < T\}$. Then, in view of (7.2.7), $\tilde{\boldsymbol{y}}(t, \boldsymbol{x}) := \boldsymbol{y}(t - \tau, \boldsymbol{x})$ satisfies $\tilde{\boldsymbol{y}}(t, \boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x})$ for $(t, \boldsymbol{x}) \in [0, t_1] \times \bar{\Omega}$ and therefore fulfills $\tilde{\boldsymbol{y}} \in Z^2 \cap V^2$ and $\tilde{\boldsymbol{y}} \in Y$ in $[0, t_1] \times \bar{\Omega}$ due to (7.3.2).

Hence, (7.2.4) is a linear ODE in $(0, t_1] \times \Omega$ so that the existence of a unique solution $\boldsymbol{S}^{(1)} := (c^{(1)}, v^{(1)}, l^{(1)}, \boldsymbol{y}^{(1)}, \kappa^{(1)})$ to (7.2.5)-(7.2.7) in $[0, t_1] \times \Omega$ satisfying (7.3.4) is proved in exactly the same way as in Theorem 7.3.1 (in fact, even statements (7.3.16), (7.3.24) and (7.3.36) concerning κ remain unchanged).

Now assume that we have a solution $\boldsymbol{S}^{(m)}$ to (7.2.5)-(7.2.7) in $[0, t_m] \times \Omega$ satisfying (7.3.4) for some $m \leq m_0$. Then $\tilde{\boldsymbol{y}} \in Z^2 \cap V^2$ and $\tilde{\boldsymbol{y}} \in Y$ in $[0, t_{m+1}] \times \bar{\Omega}$ hold due to (7.3.4). Hence, by Theorem 7.3.1 there exists a unique solution $\boldsymbol{S}^{(m+1)}$ to (7.2.5)-(7.2.7) in $[0, t_{m+1}] \times \Omega$ satisfying (7.3.4). In view of the uniqueness, we have $\boldsymbol{S}^{(m+1)} = \boldsymbol{S}^{(m)}$ in $[0, t_m] \times \Omega$. By mathematical induction we obtain a unique solution to (7.2.5)-(7.2.7) in $[0, T) \times \Omega$ which fulfills (7.3.4). ∎

7.4 Nondimensionalization

Before performing our numerical simulations, we write system (7.2.5) in terms of dimensionless variables. To this end we rescale

$$\begin{aligned} &\tilde{c} := \frac{c}{K_c}, \quad \tilde{v} := \frac{v}{K_v}, \quad \tilde{l} := \frac{l}{\lambda}, \quad \tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \\ &\tilde{t} := \frac{t}{T}, \quad \tilde{y}_1 = \frac{y_1}{R_0}, \quad \tilde{y}_2 = \frac{y_2}{R_0}, \quad \tilde{\theta} = \frac{t}{\chi T}, \end{aligned} \tag{7.4.1}$$

where L is the reference length scale, T is the reference time unit, λ is the reference concentration of proteolytic rests. Since the processes on the subcellular level are much faster than the ones on the macrolevel we set $\tilde{t} = \chi\tilde{\theta}$ where $\chi \in (0,1)$.
After using (7.2.8) and the transformations (7.4.1), we obtain the nondimensionalized system for (7.2.5) as

$$\begin{cases} \tilde{c}_{\tilde{t}} = \nabla \cdot \left(\tilde{D}_c \dfrac{\kappa}{1+\tilde{c}\tilde{v}} \nabla \tilde{c} \right) - \nabla \cdot \left(\dfrac{\tilde{D}_H \kappa \tilde{v}}{1+\tilde{v}} \tilde{c} \nabla \tilde{v} \right) - \nabla \cdot \left(\dfrac{\tilde{D}_k}{1+\tilde{c}\tilde{l}} \tilde{c} \nabla \tilde{l} \right) \\ \qquad + \tilde{\mu}_c \tilde{c} \left(1 - \tilde{c} - \eta_1 \tilde{v}\right), \\ \tilde{v}_{\tilde{t}} = -\tilde{\delta}_v \tilde{c}\tilde{v} + \tilde{\mu}_v \tilde{v} \left(1 - \eta_2 \tilde{c} - \tilde{v}\right), \\ \tilde{l}_{\tilde{t}} = \tilde{\alpha} \Delta \tilde{l} + \tilde{\delta}_l \tilde{c}\tilde{v} - \tilde{\beta}\tilde{l}, \\ \dot{\tilde{y}}_1 = \tilde{k}_1 (1 - \tilde{y}_1 - \tilde{y}_2)\tilde{v} - \tilde{k}_{-1}\tilde{y}_1, \\ \dot{\tilde{y}}_2 = \tilde{k}_2 (1 - \tilde{y}_1 - \tilde{y}_2)\tilde{l} - \tilde{k}_{-2}\tilde{y}_2 \\ \dot{\kappa} = -\tilde{q}\kappa + \dfrac{\tilde{M}\tilde{y}_1(\tilde{\theta} - \tilde{\tau})}{1 + \tilde{y}_2(\tilde{\theta} - \tilde{\tau})} \end{cases} \tag{7.4.2}$$

with 'upper dot' denoting the time derivative with respect to $\tilde{\theta}$ and the dimensionless parameters

$$\begin{aligned} &\tilde{D}_c = \frac{D_c T}{L^2}, && \tilde{D}_H = \frac{D_H T K_v}{L^2}, && \tilde{D}_k = \frac{D_k T \lambda}{L^2}, && \tilde{\mu}_c = \mu_c T, \\ &\tilde{\delta}_v = \delta_v K_c T, && \tilde{\mu}_v = \mu_v T, && && \\ &\tilde{\alpha} = \frac{\alpha T}{L^2}, && \tilde{\delta}_l = \frac{K_c K_v T \delta_l}{\lambda}, && \tilde{\beta} = \beta T, && \\ &\tilde{k}_1 = K_v k_1 \chi T, && \tilde{k}_{-1} = k_{-1}\chi T, && \tilde{k}_2 = \lambda k_2 \chi T, && \tilde{k}_{-2} = k_{-2}\chi T, \\ &\tilde{q} = q\chi T, && \tilde{M} = M\chi T, && \tilde{\tau} = \frac{\tau}{\chi T}. && \end{aligned}$$

For the ease of notation we omit the tildes and continue with system (7.4.2).

7.5 Numerical Results

In this section we investigate the qualitative behavior of the model via numerical simulations in 1-D. To this end we consider (7.4.2) with the initial conditions

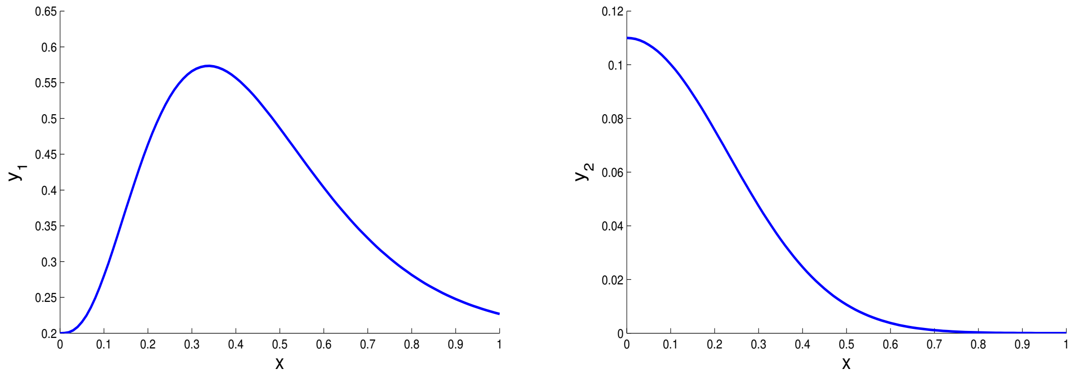
$$\begin{aligned} &c(0,x) = \exp\left(\frac{-x^2}{\varepsilon}\right), \quad x \in [0,1] \text{ and } \varepsilon > 0, \\ &v(0,x) = 1 - \exp\left(\frac{-x^2}{\varepsilon}\right), \quad x \in [0,1] \text{ and } \varepsilon > 0, \\ &l(0,x) = \zeta c(0,x), \quad x \in [0,1] \text{ and } \zeta \in [0,1), \end{aligned} \tag{7.5.1}$$

for the cancer cell density, ECM density, and concentration of proteolytic residuals, respectively. We assume that initially the space is mainly occupied by the ECM, while there is a cluster of cancer cells which have already penetrated a short distance into the tissue. The initial density of proteolytic rests is proportional to the initial cancer cell density. Throughout our numerical simulations we take $\varepsilon = 0.01$, $\zeta = 0.3$, and impose homogeneous Neumann boundary conditions as in equation (7.2.6), hence we assume that there is no flux of tumor cells, ECM fibers and proteolytic residuals across the boundary of the domain $\Omega = (0,1)$.
On the subcellular level we expect the concentration y_1 of the integrins binding to ECM fibers to increase on the left side of the domain (due to the high concentration of fibers) and to decrease on the rest of the domain (as cancer cells have not reached that portion yet). On the other hand, the initial concentration y_2 of integrins binding to the soluble ligand originating from proteolysis depends on the initial densities of c and l and thus should decrease throughout the spatial domain. Hence, we choose a gamma probability density function for y_1, whereas for the initial y_2 we take a function with a decaying exponential profile (see Figure 7.1). Moreover, since contractivity is mainly the outcome of biochemical processes initiated by the binding of integrins to the ECM fibers, we consider κ_0 to be proportional to $y_1(0)$.
For the discretization of the model we use the finite difference method (FDM). We divide the space interval $[0,1]$ into k parts with $k+1$ nodes, with the space step Δx (in our computations $\Delta x = 0.01$). We start solving system (7.4.2) with the equation for the ECM density v. We use forward differences for the time derivatives in our system which after the discretization of the ECM equation leads to

$$v_i^{n+1} = \frac{1}{1+\delta_v c_i^n \overline{\Delta t}}[v_i^n + \overline{\Delta t}\mu_v v_i^n(1-\eta_2 c_i^n - v_i^n)], \quad i = 0,1,2,...,k, \tag{7.5.2}$$

with n denoting the time level. We choose the time increment for the macrolevel as $\overline{\Delta t} = \chi \Delta t$, where $\Delta t = 0.01$ is the time step for the events on the microscale. In our computations we use $\chi = 0.01$.
In order to discretize the diffusion term on the right-hand side of the equation for proteolytic residuals, we use the central difference and obtain

$$\frac{l_i^{n+1} - l_i^n}{\overline{\Delta t}} = \alpha \cdot \frac{l_{i-1}^{n+1} - 2l_i^{n+1} + l_{i+1}^{n+1}}{(\Delta x)^2} + \delta_l c_i^n v_i^{n+1} - \beta l_i^{n+1}, \quad i = 0,1,...,k. \tag{7.5.3}$$



(a) (b)

Figure 7.1: Initial condition for the vector $\boldsymbol{y}$ of bound integrins

leading to the $(k+1)\times(k+1)$ linear system of equations

$$\mathbf{A}_l \mathbf{l}^{n+1} = \mathbf{l}^n + \tilde{\boldsymbol{\vartheta}}_l^{\tilde{n}}, \tag{7.5.4}$$

where $\mathbf{l}^{n+1}$ is the vector containing the values of l for the $k+1$ space nodes at (n+1)-th time level, $\mathbf{A}_l$ is the tridiagonal matrix coming from the FDM discretization and $\tilde{\boldsymbol{\vartheta}}_l^{\tilde{n}}$ is the vector with the entries $\overline{\Delta t}\delta_l c_i^n v_i^{n+1}$ for $i = 0, 1, 2, ..., k$ where we make use of the updated values v_i^{n+1} found by solving (7.5.2).
Before solving the equation for the evolution of cancer cell density, we solve the ODEs on the microlevel in order to update the values for the contractivity κ. The corresponding system of delay differential equations is discretized by using the semi-implicit Euler method:

$$\begin{aligned}
(y_1^{n+1})_i &= \frac{1}{1 + k_{-1}\Delta t + k_1 v_i^{n+1}\Delta t}[(y_1^n)_i + k_1 \Delta t(1-(y_2^n)_i)v_i^{n+1}], \\
(y_2^{n+1})_i &= \frac{1}{1 + k_{-2}\Delta t + k_2 l_i^{n+1}\Delta t}[(y_2^n)_i + k_2 \Delta t(1-(y_1^{n+1})_i)l_i^{n+1}], \\
\kappa_i^{n+1} &= \frac{1}{1+q\Delta t}\left[\kappa_i^n + \frac{\Delta t M(\widehat{y}_1)_i}{1+(\widehat{y}_2)_i}\right],
\end{aligned} \tag{7.5.5}$$

where $(\widehat{y}_m)_i$, is the vector containing the values of y_m $(m = 1, 2)$ at the space node i $(i = 0, 1, 2, ...k)$ and at time $(n+1)\Delta t - \tau$, with τ denoting the delay.
For the discretization of the PDE for cancer cells we use a nonstandard finite difference scheme [A7.27, A7.10, A7.26] which handles the diffusion part explicitly and the reaction terms implicitly. While adapting the method into the first three terms on the right-hand side of the first equation in (7.4.2) we handle the diffusion, haptotaxis, and chemotaxis

coefficients explicitly (w.r.t c) and the rest implicitly:

$$\begin{aligned}
\nabla(\varphi(\kappa,c,v)\nabla c)|_{x_i} &= \frac{1}{2(\Delta x)^2}\sum_{k\in N_i}(\varphi(\kappa_k^{n+1},c_k^n,v_k^{n+1})+\varphi(\kappa_i^{n+1},c_i^n,v_i^{n+1})) \\
&\quad \cdot(c_k^{n+1}-c_i^{n+1}), \\
\nabla(\psi(\kappa,v)c\nabla v)|_{x_i} &= \frac{1}{2(\Delta x)^2}\sum_{k\in N_i}(\psi(\kappa_k^{n+1},v_k^{n+1})c_k^{n+1}+\psi(\kappa_i^{n+1},v_i^{n+1})c_i^{n+1}) \\
&\quad \cdot(v_k^{n+1}-v_i^{n+1}), \\
\nabla(f(c,l)c\nabla l)|_{x_i} &= \frac{1}{2(\Delta x)^2}\sum_{k\in N_i}(f(c_k^n,l_k^{n+1})c_k^{n+1}+f(c_i^n,l_i^{n+1})c_i^{n+1}) \\
&\quad \cdot(l_k^{n+1}-l_i^{n+1}), \qquad (7.5.6)
\end{aligned}$$

where $N_i=\{i-1,i+1\}$ is the index set pointing at the direct neighbors of the node x_i. After employing (7.5.6) for the discretization of the equation for cancer cells we get

$$\mathbf{A}_c\mathbf{c}^{n+1}=\mathbf{c}^n+\tilde{\boldsymbol{\vartheta}}_c^{\tilde{n}}, \qquad (7.5.7)$$

with the $(k+1)\times(k+1)$ tridiagonal matrix $\mathbf{A}_c$ and the vector $\tilde{\boldsymbol{\vartheta}}_c^{\tilde{n}}$ of length $k+1$ which has entries $\overline{\Delta t}\mu_c c_i^n(1-c_i^n-\eta_1 v_i^{n+1})$ for $i=0,1,2,...,k$.
In our simulations we fixed the following parameters:

$$\begin{array}{llllll}
D_c=10^{-3}, & D_H=1, & D_k=0.5, & \mu_c=1, & \eta_1=0.05, & \\
\delta_v=10, & \mu_v=0.3, & \eta_2=0.9, & \alpha=1, & \delta_l=0.05, & \beta=0.15, \\
k_1=2, & k_{-1}=0.06, & k_2=0.31, & k_{-2}=0.048, & q=3, & M=1,
\end{array}$$

which are chosen from the parameter ranges given in Table 7.1.

We illustrate the variations of the cancer cell density, ECM density, concentration of proteolytic rests, and contractivity function in space. As mentioned in Section 7.2, the cell contractivity is the outcome of a sequence of biochemical processes and thus we introduce a delay (τ) in our system characterizing the time elapsed between integrin binding and the reorganization of the cell's shape by contractivity. In order to see the effect of the delay we draw the set of plots in Figure 7.2. We show the evolution of cancer cells, ECM fibers, protelytic rests, and contractivity function at different times with a time lag $\tau=4$ and respectively without delay. As expected, the invasion of cancer cells is faster in the case without delay.
Still in the case with a time lag of $\tau=4$, we are now interested in the effects of including subcellular dynamics. To this aim we compare the pure macroscopic setting (hence $\kappa=1$) with our multiscale model (7.4.2). The simulations are shown for a sequence of time steps in Figure 7.3. Observe that accounting for the subcellular dynamics slows down the invasion of tumor cells into the tissue, but leads at later times to higher peaks of the aggregates at the invasion front. This is what one would expect from a qualitative point of

Parameters	**Range**	**Source**
D_c (Diffusion coefficient for c)	$10^{-5} - 10^{-3}$	[A7.9]
D_H (Haptotaxis coefficient)	$10^{-3} - 1$	consistent with [A7.9]
D_k (Chemotaxis coefficient)	$10^{-3} - 1$	consistent with [A7.9]
μ_c (Proliferation of cancer cells)	$0.05 - 2$	[A7.9]
δ_v (Rate of degradation of ECM)	1-20	[A7.9]
μ_v (Proliferation of ECM)	0.15-2.5	[A7.9]
α (Diffusion coefficient for l)	$0.001 - 1$	[A7.9]
β (Decay of l)	$0.13 - 0.95$	[A7.9]
δ_l (Production rate of l)	$0.05 - 1$	[A7.9]
k_2 (association rate constant for y_2)	$3 \times 10^{-1} - 1$	consistent with [A7.11]
k_{-2} (dissociation rate constant for y_2)	$4 \times 10^{-2} - 10^{-1}$	consistent with [A7.11]

Table 7.1: Parameter ranges in the model

view, too. The genuinely macroscopic setting also seems to exacerbate the tissue degradation, while the two settings do not appear to make any difference to the concentration of proteolytic enzymes. Furthermore, notice that ignoring the microscale predicts a decrease in the original tumor, which is actually not expected in practice.

7.6 Discussion

In this work we proposed and analyzed a mathematical model for tumor cell migration through tissue networks, influenced both by haptotaxis and chemotaxis. Our multiscale setting is connecting the macroscopic level of cell population, fiber density, and chemoattractant concentration with the microscopic one of integrin binding dynamics. The coupling is realized with the aid of a contractivity function involved in the diffusion and haptotaxis coefficients of the cancer cell equation written on the macroscale. The time lag between integrin binding and translation of the initiated signal into motile behavior of the cell population is accounted for via a delay term in the equation for the contractivity function. The multiscality and the coupling between different types of equations increase the complexity of the resulting system, for which we proved the (local) existence of a unique solution. Due to the lack of a priori bounds for the cancer cell density, the global existence result is still out of reach unless generous assumptions are made on the problem's data, which, however, are usually not satisfied in the framework of a concrete biological problem inferring a large variety of fluctuations.
But including microscale dynamics is not only interesting from a mathematical point of view; it can help gaining a deeper insight into the processes involved in and influenced by tumor cell migration. Hence, the cell-ECM interaction modeled by integrin dynamics as in (7.2.4) has been found to play a crucial role in explaining fingering patterns for glioma [A7.12] in a micro-meso setting, while in the context of bacterial motion the intracellu-

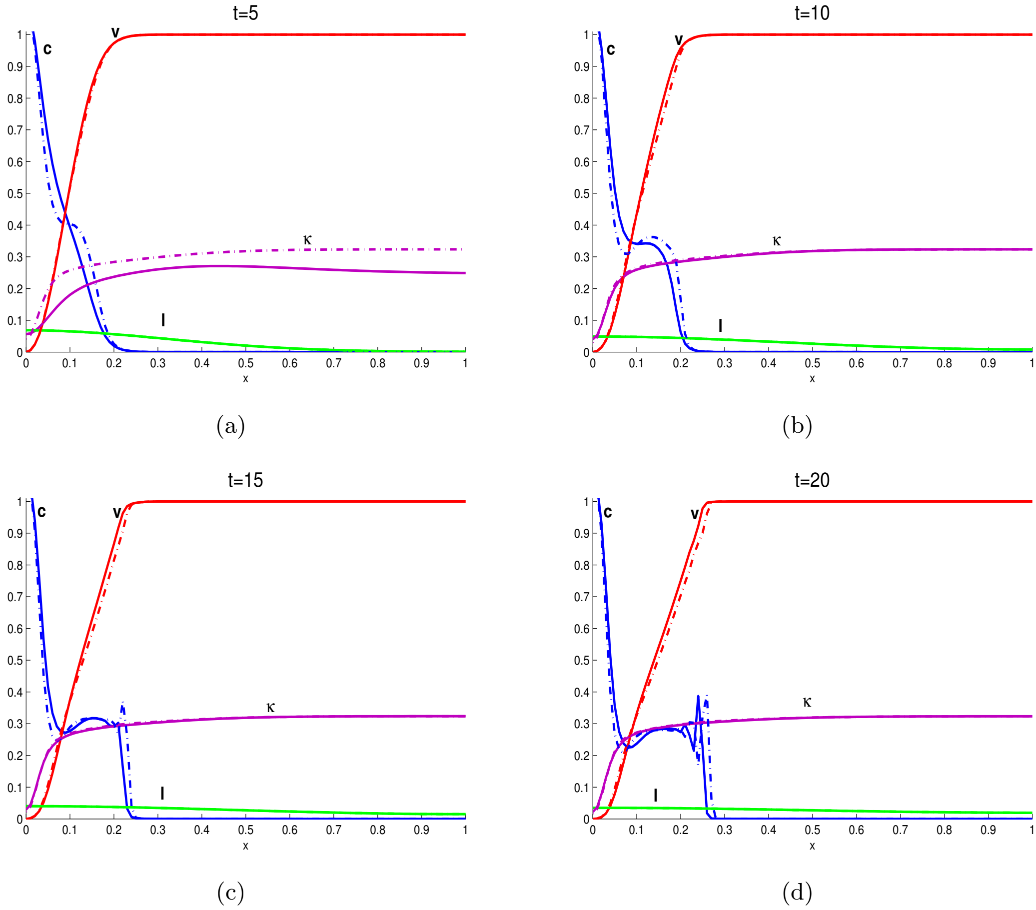


Figure 7.2: Evolution of tumor cell density (blue), ECM fiber density (red), concentration of proteolytic rests (green), and contractivity function (purple) in the cases with $\tau = 0$ (dash-dot line) and with $\tau = 4$ (solid line).

lar excitation-adaptation mechanism was shown to influence the motile, aggregation or tactic behavior of the corresponding cell population, see, e.g., [A7.34, A7.32]. The model proposed in [A7.26] in order to assess the effects of heat shock proteins (HSP) on tumor invasion also aligns to the micro-macro approach proposed in this work, however, it provides a much simplified, rather phenomenological description of the events on the subcellular level connected to HSP dynamics.

The numerical simulations and the comparisons performed in Section 7.5 illustrate the effects of introducing the microscopic dynamics: the 'classical', purely macroscopic diffusion-haptotaxis-chemotaxis model overestimates the effective distance invaded in the tissue by the cancer cells and underestimates the peaks of their aggregates at the front of the invasion for later times. Furthermore, that setting predicts a decrease in the original tumor, which seems unrealistic from a biological point of view.

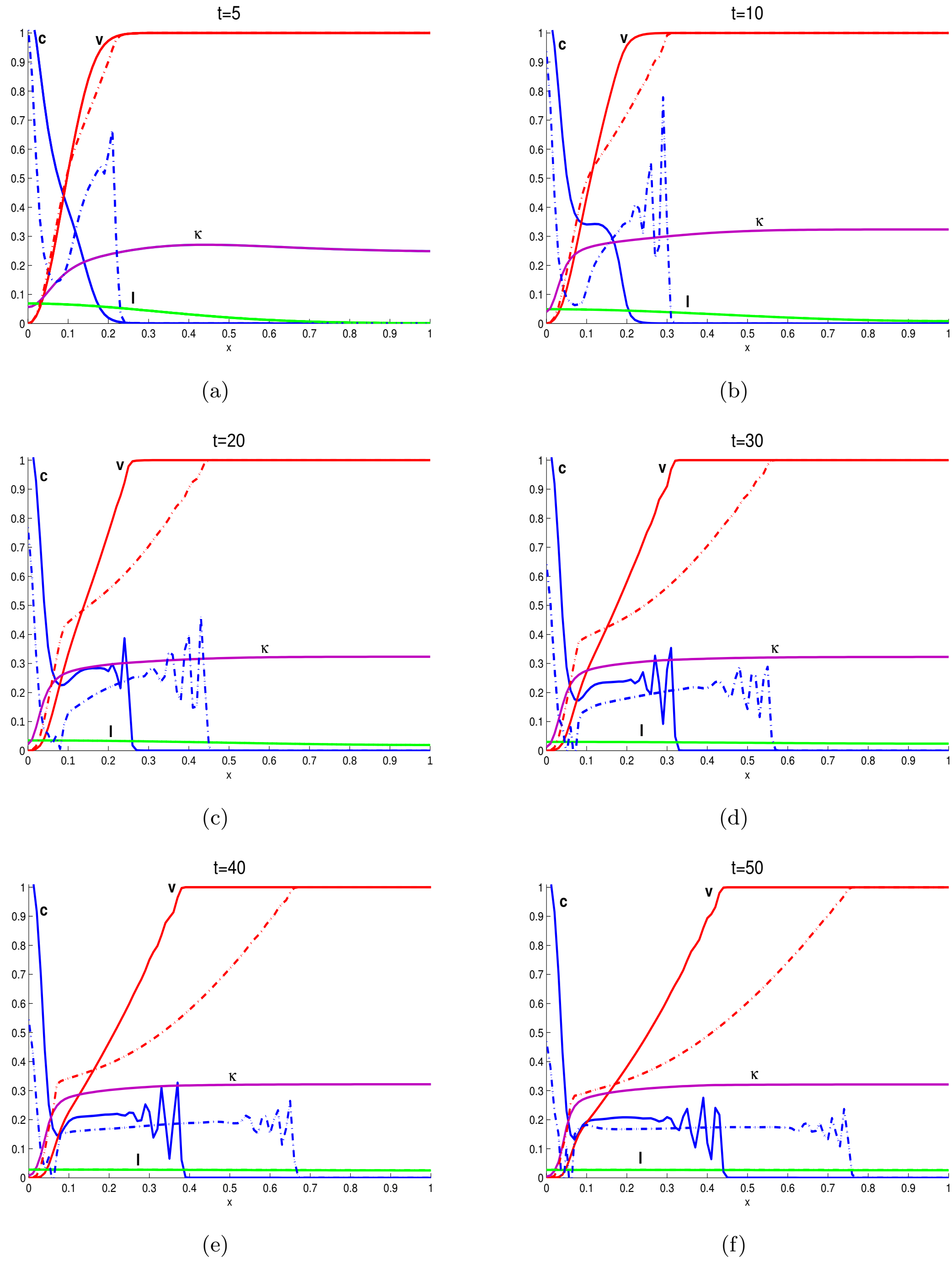


Figure 7.3: Evolution of tumor cell density (blue), ECM fiber density (red), concentration of proteolytic rests (green), and contractivity function (purple) in the pure macroscopic setting (dash-dot line) and with κ (solid line) satisfying the last equation in (7.4.2).

Finally, we would like to stress out that the model presented here is merely a paradigm for further multiscale settings, in which enhanced attention can be paid to a more detailed description of subcellular events and hence to their effects on the population spread. The validation of the model predictions would be desirable, which calls for the availability of adequate medical data.

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Article 8:
A multiscale model for pH-tactic invasion with time-varying carrying capacities

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http://imamat.oxfordjournals.org/cgi/content/abstract/hxu055?ijkey=MWQLeeZskAvoKmZ&keytype=ref)

Abstract

We propose a model for acid-mediated tumor invasion involving two different scales: the microscopic one, for the dynamics of intracellular protons and their exchange with their extracellular counterparts, and the macroscopic scale of interactions between tumor cell and normal cell populations, along with the evolution of extracellular protons. We also account for the tactic behavior of cancer cells, the latter being assumed to biase their motion according to a gradient of extracellular protons (following [A8.2, A8.31] we call this pH taxis). A time dependent (and also time delayed) carrying capacity for the tumor cells in response to the effects of acidity is considered as well. The global well posedness of the resulting multiscale model is proved with a regularization and fixed point argument. Numerical simulations are performed in order to illustrate the behavior of the model.

Key words: tumor cell migration, multiscale models, pH-taxis, time-delayed carrying capacities, reaction-diffusion-taxis equations
MSC 2010: 35Q92, 92C17, 35K57, 35B40

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8.1 Introduction

A recent approach to cancer invasion is based on the role of the peritumoral environment in determining cancer malignancy. Gatenby & Gillies [A8.12] suggested that biochemical events therein may drive the selection of the cancerous phenotype, and such informations can be used to conceive new therapies. Hypoxia and acidity, for instance, are factors that can trigger the progression from benign to malignant growth [A8.10, A8.42]. Cancer cells upregulate certain mechanisms which allow for extrusion of excessive protons and hence acidify their surroundings. This triggers apoptosis of normal cells and thus allows the neoplastic tissue to extend into the space becoming available. Moreover, the pH was found to directly influence the metastatic potential of tumor cells [A8.1, A8.28]. The mathematical modeling of acid-mediated tumor invasion seems to have begun some decades ago with the work by Gatenby & Gawlinski [A8.9], who proposed a model involving reaction-diffusion equations to describe the interaction between the density of normal cells, tumor cells, and the concentration of H^+ ions produced by the latter. The well posedness of that model was investigated in [A8.26], thereby also explicitly allowing for crowding effects (due to competition with cancer cells) in the growth of normal cells. Still in the framework of [A8.9], traveling waves were used to characterize the aggressive action of cancer cells on their surroundings [A8.7]. Further developments of Gatenby & Gawlinski's model involve both vascular and avascular growth of multicellular tumor spheroids, assuming rotational symmetry, for which existence and qualitative properties of the solutions were investigated [A8.35].

The mentioned models all have a monoscale character and describe the interaction of cancer and normal cell populations, coupled with the evolution of extracellular H^+ concentration and possibly also with the concentration of matrix degrading enzymes [A8.27]. However, this macroscale dynamics is regulated by and influences the intracellular proton dynamics [A8.22, A8.37, A8.42]. Webb et al. [A8.43, A8.44] proposed some mathematical settings for the interdependence between the activity of several membrane ion transport systems and the changes in the peritumoral space. The models involve even more biological details, like intracellular proton buffering, effects on the expression/activation of matrix metalloproteinases (MMPs) and proton removal by vasculature. Webb, Sherratt, and Fish [A8.43] also account for the influence of alkaline intracellular pH on the growth of tumor cells, hence their model can be seen as a first step towards multiscale settings. However, the spatial dependence is essential for assessing the actual invasive behavior. This leads to more complex models, coupling the subcellular level with the macroscopic scale of populations. Some models of this new class involving both the subcellular and the population levels have been recently proposed and analyzed [A8.29, A8.30, A8.36, A8.39]. A multiscale setting addressing acid-mediated tumor invasion has been presented in [A8.17]; it also accounts for stochasticity, which is a relevant feature inherent to many biological processes occurring on all modeling levels and in particular, it seems to greatly influence subcellular dynamics and individual cell behavior [A8.8, A8.14, A8.38]. Further multiscale settings concerning tumor cell migration -so far, however, not necessarily in connection

with acidosis- take into account more modeling levels and allow for a relatively detailed description of processes taking place on the mesoscale, i.e., on the rank of individual cells and their interactions with their environment [A8.3, A8.6, A8.19, A8.24].
Here we propose a multiscale model for acid-mediated cancer invasion, to be developed in Section 8.2 and analyzed w.r.t. global well posedness in Section 8.3. We present some numerical simulations in Section 8.4 to illustrate its performance and eventually give in Section 8.5 a short discussion of the results.

8.2 The model

We denote by $c(x,t)$ the density of cancer cells, by $n(x,t)$ the density of normal cells, and by $h(x,t)$ and $y(x,t)$ the concentrations of extracellular and intracellular protons, respectively.

8.2.1 Subcellular dynamics

Glycolysis is a metabolic pathway for rather inefficient energy production and normally used by cells under hypoxic conditions. Nevertheless, cancer cells consistently rely on the glucose metabolism even in normoxic conditions. The high glycolytic rate of neoplastic tissues is clinically used to diagnose and assess (via positron emission tomography, shortly PET) tumor responses to treatment [A8.11]. Cancer cells seem to use this aerobic glycolytic phenotype for invasion and metastasis, as -unlike normal cells- they are able to develop resistance against acid-induced toxicity. Environmental acidosis has been found to be directly related to enhanced tumor proliferation [A8.25] and regulating angiogenesis [A8.18]. The proton dynamics inside and outside tumor cells is controlled by a plethora of processes. Relying on the facts in [A8.4, A8.44] and following [A8.17], we describe the intracellular proton dynamics with the aid of the equation

$$\frac{dy}{dt} = -R(y,h) - \alpha y + g(c), \tag{8.2.1}$$

where $R(y,h)$ denotes the decay term for intracellular H^+ due to membrane transporters (e.g., NDCBE, NHE, and AE) [13], production by aerobic glycolysis (possibly depending on microenvironmental vascularization), and buffering by organelles. It describes a (saturated) growth with respect to the concentration y of intracellular protons and decay with respect to the extracellular proton concentration h and takes in the nondimensionalized model the form given in (8.2.6) below. The coefficient α in (8.2.1) denotes some decay constant, and $g(c)$ represents a source term due to the production (with saturation) by cancer cells. Observe that (8.2.1) is an ordinary differential equation (ODE), the variable x denoting the position of a cell having the intracellular proton concentration $y(x,t)$ and being seen as a parameter in that ODE.

[13] NDCBE (Na^+ dependent Cl^--HCO_3^- exchanger), NHE (Na^+ and H^+ exchanger) and AE (Cl^--HCO_3^- or *a*nion *e*xchanger) are specific transporters present on the cell membrane.

8.2.2 Extracellular proton concentration

In order to maintain an advantageous intracellular pH, cancer cells upregulate proton extrusion through membrane transporters, leading to acidosis of the tumor environment. The concentration of extracellular protons h is a macroscopic quantity explicitly depending on time and position. It is produced the same way the intracellular protons decay by transport through the cell membrane and it diffuses through the extracellular space with a diffusion constant D_h:

$$\partial_t h = D_h \Delta h + R(y, h) \tag{8.2.2}$$

8.2.3 Cell dynamics on the macroscale

On the population level we are interested in the dynamics of tumor cells in interdependence with the normal cells and under the influence of proton concentrations. The evolution of cancer cell density is characterized by nonlinear diffusion, with the diffusion coefficient depending on the solution, more precisely inversely proportional to the interactions between cancer and normal cells, as these are slowing down the diffusivity. Furthermore, we assume the tumor cells to biase their motion in response to a gradient of extracellular protons and call this behavior pH-taxis. The notion has been firstly proposed in the context of bacteria avoiding acidic regions (hence the protons playing the role of a chemorepellent) [A8.20] and more recently also in reference to motility of cancer cells, the latter being enhanced in the direction of extracellular pH gradient [A8.2, A8.31]. The pH-tactic sensitivity function $f(h, c)$ (a concrete choice of which is proposed in (8.2.6) below) is nonlinearly depending on the tumor cell density and the interaction of cancer cells with extracellular protons. The tumor cell proliferation is modeled with a logistic growth with crowding. Thereby, the carrying capacity K_c of cancer cells depends on the extracellular proton concentration to stress out that the cancer cells are allowed to infer an enhanced growth due to more space becoming available through normal cell killing by acidity. However, at the same time the tumor cell density cannot exceed a certain threshold, according to the acidity level: a too acidic environment is baneful even for cancer cells. The time delay expresses the fact that their adaptation of the carrying capacity to the acidosis in the peritumoral region is not instantaneous. [14] The proliferation rate $\mu_c(y)$ depends on the intracellular proton concentration. Indeed, malignant cells whose intracellular medium gets alkalinized were found to exhibit enhanced proliferation [A8.5, A8.16, A8.46], which motivates the choice of μ_c in (8.2.6).

The normal cell dynamics is much easier to describe: normal tissue is not diffusing, it is only degraded by the environmental acidity (i.e., by the action of extracellular protons) and normal cell proliferation is supposed to be well enough described by logistic growth with crowding, also against tumor cells. The carrying capacity of the latter is still depending on the acidity, but now no longer needs to infer a time lag, as the focus is on the proliferation of normal cells.

[14] The idea of introducing time-lagged carrying capacities was previously used by Yukalov et al. [A8.47] in a simpler ODE framework to model punctuated evolution in a population under several regimes.

Altogether, our multiscale model for acid-mediated tumor invasion takes the following form:

$$\begin{cases} \partial_t c = \nabla \cdot (\varphi(c,n)\nabla c) - \nabla \cdot (f(h,c)\nabla h) \\ \qquad +\mu_c(y)c\left(1 - \frac{c}{K_c(h(\cdot,t-\tau))} - \eta_1 \frac{n}{K_n}\right) & \text{in } \Omega \times (0,T), \\ \partial_t n = -\delta_n hn + \mu_n n\left(1 - \eta_2 \frac{c}{K_c(h(\cdot,t))} - \frac{n}{K_n}\right) & \text{in } \Omega \times (0,T), \\ \partial_t h = D_h \Delta h + R(y,h) & \text{in } \Omega \times (0,T), \\ \partial_t y = -R(y,h) - \alpha y + g(c) & \text{in } \Omega \times (0,T), \end{cases} \tag{8.2.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \leq 3$. Denoting by ν the outward unit normal on $\partial\Omega$, we further endow (8.2.3) with the boundary conditions

$$\partial_\nu c = \partial_\nu h = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{8.2.4}$$

and initial conditions

$$\begin{array}{ll} c(x,0) = c_0(x), \quad n(x,0) = n_0(x), \quad y(x,0) = y_0(x) & \text{for } x \in \Omega, \\ \qquad h(x,t) = h_0(x,t) & \text{for } x \in \Omega,\ t \in [-\tau, 0]. \end{array} \tag{8.2.5}$$

For the coefficient functions involved in (8.2.3), we propose

$$\begin{array}{lll} \varphi(c,n) := \frac{D_c}{1+\frac{cn}{C_0 K_n}}, & f(h,c) := \frac{Mc}{1+\frac{ch}{C_0 H_0}}, & \mu_c(y) := \frac{\kappa_1}{1+\frac{y}{Y_0}}, \\ K_c(h) := \frac{C_0+bh}{1+dh^2}, & g(c) := \frac{\rho c}{1+\frac{c}{C_0}}, & R(y,h) := \frac{\gamma_h y}{1+\frac{y^2}{Y_0^2}+\alpha_h h^2} - \frac{\beta_h h}{1+\frac{y^2}{Y_0^2}} \end{array} \tag{8.2.6}$$

and assume that all the constants in (8.2.3) and (8.2.6) are positive along with $\eta_1, \eta_2 \in (0,1)$.

8.3 Global existence

In order to prove the existence of a global weak solution to (8.2.3), let

$$\begin{array}{c} \varphi, f, R \in C^1([0,\infty)^2), \quad \mu_c, K_c, g \in C^1([0,\infty)) \text{ such that } g \in L^\infty((0,\infty)) \\ \text{and} \quad g \geq 0,\ \mu_c > 0,\ K_c > 0 \quad \text{on } [0,\infty). \end{array} \tag{8.3.1}$$

Moreover, we assume that there exist $H_0, Y_0 \in (0,\infty)$ such that

$$\begin{array}{c} R(y,0) \geq 0,\ R(y,H_0) \leq 0 \quad \text{for all } y \in [0,Y_0], \quad R(0,h) \leq 0 \quad \text{for all } h \in [0,H_0], \\ -R(Y_0,h) - \alpha Y_0 + \|g\|_{L^\infty((0,\infty))} \leq 0 \quad \text{for all } h \in [0,H_0]. \end{array} \tag{8.3.2}$$

H_0 and Y_0 are upper bounds for the concentrations of the extra- and intracellular protons, respectively. As R describes the effect of the proton exchange between the interior of the cancer cell and its environment, e.g. the first two conditions in (8.3.2) mean that there is no proton transport into the tumor cell if there are no extracellular protons, while protons cannot leave the cell if the extracellular proton concentration is at its maximal value. With H_0 and Y_0 as defined above, we further assume that there exist positive constants C_1 and C_2 such that

$$0 \le f(h,c) \le C_1(1+c), \quad \frac{C_2}{1+c} \le \varphi(c,n) \le C_1 \;\; \forall (c,n,h) \in [0,\infty) \times [0,K_n] \times [0,H_0], \tag{8.3.3}$$

and that for any $a \in (0, H_0)$ there is $C_a > 0$ such that

$$f(h,c) \le C_a \quad \text{for all } (c,h) \in [0,\infty) \times [a, H_0]. \tag{8.3.4}$$

Observe that the functions given in (8.2.6) satisfy (8.3.1)-(8.3.4). Concerning the initial data suppose that

$$\begin{gathered} c_0, n_0, y_0 \in C^0(\bar{\Omega}), \quad h_0 \in C^0([-\tau,0]; W^{1,q}(\Omega)), \\ c_0 \ge 0, \quad 0 \le n_0 \le K_n, \quad 0 \le y_0 \le Y_0 \quad \text{in } \bar{\Omega}, \quad \delta \le h_0 \le H_0 \quad \text{in } \bar{\Omega} \times [-\tau, 0] \end{gathered} \tag{8.3.5}$$

with some $q \in (N+2, \infty)$ and $\delta > 0$. The following solution concept will be appropriate.

Definition 8.3.1 *Let $T \in (0,\infty)$. A weak solution to* (8.2.3)-(8.2.5) *consists of nonnegative functions*

$$\begin{aligned} &c \in L^\infty(\Omega \times (0,T)) \cap L^2((0,T); W^{1,2}(\Omega)), \quad n, y \in L^\infty(\Omega \times (0,T)), \\ &h \in L^\infty(\Omega \times (-\tau,T)) \cap L^2((0,T); W^{1,2}(\Omega)) \end{aligned}$$

which satisfy for all $\psi \in C_0^\infty(\bar{\Omega} \times [0,T))$ the equations

$$\begin{aligned} -\int_0^T \int_\Omega c\partial_t \psi - \int_\Omega c_0 \psi(\cdot,0) = &-\int_0^T \int_\Omega \varphi(c,n) \nabla c \cdot \nabla \psi + \int_0^T \int_\Omega f(h,c) \nabla h \cdot \nabla \psi \\ &+ \int_0^T \int_\Omega \mu_c(y) c \left(1 - \frac{c}{K_c(h(\cdot,t-\tau))} - \eta_1 \frac{n}{K_n}\right) \psi, \end{aligned} \tag{8.3.6}$$

$$-\int_0^T \int_\Omega n\partial_t \psi - \int_\Omega n_0 \psi(\cdot,0) = \int_0^T \int_\Omega \left(-\delta_n h n + \mu_n n \left(1 - \frac{\eta_2 c}{K_c(h)} - \frac{n}{K_n}\right)\right) \psi, \tag{8.3.7}$$

$$-\int_0^T \int_\Omega h\partial_t \psi - \int_\Omega h_0 \psi(\cdot,0) = -D_h \int_0^T \int_\Omega \nabla h \cdot \nabla \psi + \int_0^T \int_\Omega R(y,h) \psi, \tag{8.3.8}$$

$$-\int_0^T \int_\Omega y\partial_t \psi - \int_\Omega y_0 \psi(\cdot,0) = \int_0^T \int_\Omega \left(-R(y,h) - \alpha y + g(c)\right) \psi. \tag{8.3.9}$$

If (c,n,h,y) is a weak solution to (8.2.3)-(8.2.5) *for all $T \in (0,\infty)$, then we call it a global weak solution to* (8.2.3)-(8.2.5).

Now we state the main result of this section which establishes the existence of a global weak solution.

Theorem 8.3.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $N \in \mathbb{N}$ and assume that* (8.3.1)-(8.3.5) *are fulfilled. Then there exists a global weak solution to* (8.2.3)-(8.2.5) *in the sense of Definition 8.3.1 satisfying*

$$\begin{gathered} c \in L^\infty_{loc}(\bar\Omega \times [0,\infty)), \quad 0 \le n \le K_n \quad \text{and} \quad 0 \le y \le Y_0 \quad \text{in } \Omega \times (0,\infty), \\ h \in L^\infty((0,\infty); W^{1,q}(\Omega)), \quad 0 \le h \le H_0 \quad \text{in } \Omega \times (-\tau,\infty). \end{gathered} \tag{8.3.10}$$

If in addition $c_0 \in C^\beta(\bar\Omega)$ is satisfied with some $\beta \in (\frac{1}{N+2}, 1)$, then there is a unique global weak solution within the class of functions satisfying the conditions of Definition 8.3.1 and $h \in L^r_{loc}([0,\infty); W^{1,r}(\Omega))$ for some $r > N+2$.

In order to prove this result, we use the following regularized problems for $\varepsilon \in (0,1)$:

$$\begin{cases} \partial_t c_\varepsilon = \nabla\cdot(\varphi_\varepsilon(c_\varepsilon,n_\varepsilon)\nabla c_\varepsilon) - \nabla\cdot(f_\varepsilon(h_\varepsilon,c_\varepsilon)\nabla h_\varepsilon) \\ \qquad +\mu_c(y_\varepsilon)c_\varepsilon\left(1-\frac{c_\varepsilon}{K_c(h_\varepsilon(\cdot,t-\tau))}-\eta_1\frac{n_\varepsilon}{K_n}\right) \quad \text{in } \Omega\times(0,T_\varepsilon), \\ \partial_t n_\varepsilon = -\delta_n h_\varepsilon n_\varepsilon + \mu_n n_\varepsilon\left(1-\eta_2\frac{c_\varepsilon}{K_{c\varepsilon}(h_\varepsilon(\cdot,t))}-\frac{n_\varepsilon}{K_n}\right) \quad \text{in } \Omega\times(0,T_\varepsilon), \\ \partial_t h_\varepsilon = D_h\Delta h_\varepsilon + R(y_\varepsilon,h_\varepsilon) \quad \text{in } \Omega\times(0,T_\varepsilon), \\ \partial_t y_\varepsilon = -R(y_\varepsilon,h_\varepsilon) - \alpha y_\varepsilon + g(c_\varepsilon) \quad \text{in } \Omega\times(0,T_\varepsilon), \\ \partial_\nu c_\varepsilon = \partial_\nu h_\varepsilon = 0 \qquad \text{on } \partial\Omega\times(0,T_\varepsilon), \\ c_\varepsilon(x,0)=c_{0\varepsilon}(x), \quad n_\varepsilon(x,0)=n_{0\varepsilon}(x), \quad y_\varepsilon(x,0)=y_{0\varepsilon}(x) \quad \text{for } x\in\Omega, \\ h_\varepsilon(x,t)=h_{0\varepsilon}(x,t) \quad \text{for } x\in\Omega,\ t\in[-\tau,0]. \end{cases} \tag{8.3.11}$$

Here, we choose families of functions $c_{0\varepsilon}$, $n_{0\varepsilon}$, $h_{0\varepsilon}$, $y_{0\varepsilon}$, φ_ε and f_ε, $\varepsilon \in (0,1)$, satisfying

$$\begin{gathered} c_{0\varepsilon}, n_{0\varepsilon}, y_{0\varepsilon} \in C^3(\bar\Omega), \quad h_{0\varepsilon} \in C^3(\bar\Omega\times[-\tau,0]), \quad \tfrac{\delta}{2} \le h_{0\varepsilon} \le H_0 \quad \text{in } \bar\Omega\times[-\tau,0], \\ c_{0\varepsilon} \ge 0, \quad 0 \le n_{0\varepsilon} \le K_n, \quad 0 \le y_{0\varepsilon} \le Y_0 \quad \text{in } \bar\Omega, \\ \partial_\nu c_{0\varepsilon} = \partial_\nu h_{0\varepsilon} = 0 \qquad \text{on } \partial\Omega, \\ \varphi_\varepsilon \in C^3([0,\infty)^2) \cap W^{2,\infty}([0,\infty)\times[0,K_n]), \quad \max\{\varepsilon, \tfrac{\tilde C_2}{1+c}\} \le \varphi_\varepsilon(c,n) \le \tilde C_1 \\ f_\varepsilon \in C^3([0,\infty)^2) \cap W^{2,\infty}([0,H_0]\times[0,\infty)), \quad 0 \le f_\varepsilon(h,c) \le \tilde C_1(1+c), \\ K_{c\varepsilon} \in C^3([0,\infty)), \quad 0 < a_2 \le K_{c\varepsilon}(h) \le a_1 \end{gathered} \tag{8.3.12}$$

with positive constants $\tilde C_1$, $\tilde C_2$, a_1, a_2 for all $(c,n,h) \in [0,\infty)\times[0,K_n]\times[0,H_0]$ and any $\varepsilon \in (0,1)$ as well as

$$\begin{gathered} c_{0\varepsilon} \to c_0 \quad \text{and} \quad n_{0\varepsilon} \to n_0 \quad \text{and} \quad y_{0\varepsilon} \to y_0 \quad \text{in } C^0(\bar\Omega) \\ h_{0\varepsilon} \to h_0 \quad \text{in } C^0([-\tau,0];W^{1,q}(\Omega)), \quad \varphi_\varepsilon \to \varphi \quad \text{in } C^1([0,r_0]\times[0,K_n]), \\ f_\varepsilon \to f \quad \text{in } C^1([0,H_0]\times[0,r_0]), \quad K_{c\varepsilon} \to K_c \quad \text{in } C^1([0,H_0]) \end{gathered} \tag{8.3.13}$$

as $\varepsilon \searrow 0$ for all $r_0 > 0$. Furthermore, we assume that for any $a \in (0,H_0)$ there is $\tilde C_a > 0$ such that

$$f_\varepsilon(h,c) \le \tilde C_a \quad \text{for all } (c,h) \in [0,\infty)\times[a,H_0] \tag{8.3.14}$$

and all $\varepsilon \in (0,1)$.

8.3.1 Global existence for the regularized problems

We first state the local existence of classical solutions for (8.3.11) along with an extensibility criterion and prove this result like in [A8.36, Lemma 3.1].

Lemma 8.3.3 *Let* $\varepsilon \in (0,1)$ *and assume that* (8.3.1), (8.3.2) *and* (8.3.12) *are fulfilled. Then there exist a maximal existence time* $T_\varepsilon \in (0,\infty]$ *and functions* $c_\varepsilon, n_\varepsilon, h_\varepsilon \in C^{2,1}(\bar{\Omega} \times [0,T_\varepsilon))$ *and* $y_\varepsilon \in C^1([0,T_\varepsilon); C^0(\bar{\Omega}))$ *which solve* (8.3.11) *in the classical sense and satisfy*

$$\begin{aligned} c_\varepsilon \geq 0, \quad 0 \leq n_\varepsilon \leq K_n, \quad 0 \leq y_\varepsilon \leq Y_0 \quad \text{in } \bar{\Omega} \times [0,T_\varepsilon), \\ 0 \leq h_\varepsilon \leq H_0 \quad \text{in } \bar{\Omega} \times [-\tau, T_\varepsilon). \end{aligned} \tag{8.3.15}$$

If moreover $T_\varepsilon < \infty$ *holds, then*

$$\limsup_{t \nearrow T_\varepsilon} \Big(\|c_\varepsilon(\cdot,t)\|_{C^0(\bar{\Omega})} + \|h_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \Big) = \infty \tag{8.3.16}$$

is fulfilled, where $q \in (N+2, \infty)$ *is defined in* (8.3.5).

Proof. We fix $\beta \in (0,1)$, $T := 1$ and set

$$A := \|c_{0\varepsilon}\|_{C^{2+\beta}(\bar{\Omega})} + \|n_{0\varepsilon}\|_{C^{2+\beta}(\bar{\Omega})} + \|h_{0\varepsilon}\|_{C^{2+\beta}(\bar{\Omega})} + \|h_{0\varepsilon}\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[-\tau,0])} + \|y_{0\varepsilon}\|_{C^{\beta}(\bar{\Omega})} < \infty.$$

Moreover, let $c_{0\varepsilon t}(x)$, $n_{0\varepsilon t}(x)$ and $h_{0\varepsilon t}(x)$ denote the right-hand side of the first, second and third equation of (8.3.11) evaluated at $(x,t) = (x,0)$, respectively, so that

$$\begin{aligned} B \;:=\; & \|c_{0\varepsilon}\|_{C^{\beta}(\bar{\Omega})} + \|c_{0\varepsilon t}\|_{C^0(\bar{\Omega})} + \|h_{0\varepsilon}\|_{C^{\beta}(\bar{\Omega})} + \|h_{0\varepsilon t}\|_{C^0(\bar{\Omega})} \\ & + \|n_{0\varepsilon}\|_{C^{1+\beta}(\bar{\Omega})} + \|n_{0\varepsilon t}\|_{C^1(\bar{\Omega})} \qquad \leq C_3(A) < \infty \end{aligned} \tag{8.3.17}$$

holds with some constant $C_3(A)$ depending on A. Then we define

$$\begin{aligned} X \;:=\; & \Big\{ (c_\varepsilon, h_\varepsilon, n_\varepsilon) \in (C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T]))^2 \times C^{1+\beta,\frac{1+\beta}{2}}(\bar{\Omega}\times[0,T]) \; : \\ & c_\varepsilon \geq 0, \; 0 \leq h_\varepsilon \leq H_0, \; 0 \leq n_\varepsilon \leq K_n, \\ & \|c_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} + \|h_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} + \|n_\varepsilon\|_{C^{1+\beta,\frac{1+\beta}{2}}(\bar{\Omega}\times[0,T])} \leq B+3 \Big\}. \end{aligned}$$

Given fixed $(c_\varepsilon, h_\varepsilon, n_\varepsilon) \in X$, by (8.3.1), (8.3.2), (8.3.12), the theory of ODEs (see, e.g., [A8.33]) and the comparison principle, there is a solution $y_\varepsilon \in C^1((0,T); C^0(\bar{\Omega}))$ to the fourth equation of (8.3.11) with initial data $y_{0\varepsilon}$ such that

$$0 \leq y_\varepsilon \leq Y_0 \quad \text{in } \bar{\Omega}\times[0,T], \qquad \|y_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} + \|\partial_t y_\varepsilon\|_{C^0(\bar{\Omega}\times[0,T])} \leq C_4(A), \tag{8.3.18}$$

where the latter Hölder estimate with repect to x follows from the regularity properties (8.3.1) and an application of Gronwall's inequality to $y_\varepsilon(x_1,t) - y_\varepsilon(x_2,t)$. Next, (8.3.1), (8.3.2), (8.3.12), (8.3.18) along with [A8.21, Theorems V.7.4 and IV.5.3] and the parabolic comparison principle imply the existence of a solution $\tilde{h}_\varepsilon \in C^{2+\beta,1+\frac{\beta}{2}}(\bar{\Omega}\times[0,T])$ to the third equation of (8.3.11) with the homogeneous Neumann boundary condition and initial data $h_{0\varepsilon}$, satisfying

$$\begin{gathered} 0 \le \tilde{h}_\varepsilon \le H_0 \quad \text{in } \bar{\Omega} \times [-\tau, T], \\ \|\tilde{h}_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[-\tau,T])} + \|\tilde{h}_\varepsilon\|_{C^{2+\beta,1+\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} \le C_5(A) \end{gathered} \tag{8.3.19}$$

with some constant $C_5(A) > 0$. Using next (8.3.1), (8.3.12), (8.3.18) and (8.3.19), by the comparison principle, [A8.21, Theorem III.5.1] and [A8.23, Theorem 1.1] there exists a weak solution $\tilde{c}_\varepsilon \in C^{1+\beta_1,\frac{1+\beta_1}{2}}(\bar{\Omega} \times [0,T]) \cap W_2^{1,\frac{1}{2}}(\bar{\Omega} \times [0,T])$ to the first equation of (8.3.11) (with $\tilde{h}_\varepsilon$ instead of h_ε) satisfying the respective boundary and initial conditions, where $\beta_1 \in (0,\beta]$. By the last reference and [A8.21, Theorem IV.5.3], we further obtain that $\tilde{c}_\varepsilon \in C^{2+\beta_1,1+\frac{\beta_1}{2}}(\bar{\Omega} \times [0,T])$ is a classical solution satisfying

$$\tilde{c}_\varepsilon \ge 0 \quad \text{in } \bar{\Omega} \times [0,T], \qquad \|\tilde{c}_\varepsilon\|_{C^{2+\beta_1,1+\frac{\beta_1}{2}}(\bar{\Omega}\times[0,T])} \le C_6(A) \tag{8.3.20}$$

with a positive constant $C_6(A)$. Combining this with (8.3.12) and (8.3.19), we apply the theory of ODEs (see e.g. Theorem 2 in [A8.33, Section 2.3]) and the comparison principle to get a solution $\tilde{n}_\varepsilon \in C^{2+\beta_1,1+\frac{\beta_1}{2}}(\bar{\Omega} \times [0,T])$ to the second equation of (8.3.11) (with $\tilde{c}_\varepsilon$, $\tilde{h}_\varepsilon$ instead of c_ε, h_ε) with initial data $n_{0\varepsilon}$, such that

$$\begin{gathered} 0 \le \tilde{n}_\varepsilon \le K_n \quad \text{in } \bar{\Omega} \times [0,T], \\ \|\tilde{n}_\varepsilon\|_{C^{2+\beta_1,1+\frac{\beta_1}{2}}(\bar{\Omega}\times[0,T])} + \|\partial_t \tilde{n}_\varepsilon\|_{C^{1+\beta_1,\frac{1+\beta_1}{2}}(\bar{\Omega}\times[0,T])} \le C_7(A), \end{gathered} \tag{8.3.21}$$

with some positive constant $C_7(A)$, where the Hölder estimate with respect to x is done as described above for y_ε. In particular, recalling the definitions of $c_{0\varepsilon t}$, $h_{0\varepsilon t}$ and $n_{0\varepsilon t}$ before (8.3.17), from (8.3.19)-(8.3.21) we obtain $c_{0\varepsilon t}(x) = \partial_t \tilde{c}_\varepsilon(x,0)$, $h_{0\varepsilon t}(x) = \partial_t \tilde{h}_\varepsilon(x,0)$ and $n_{0\varepsilon t}(x) = \partial_t \tilde{n}_\varepsilon(x,0)$ for $x \in \bar{\Omega}$ so that there is $T_0 \in (0,T]$ only depending on A such that

$$\|\tilde{c}_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} + \|\tilde{h}_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])} + \|n_\varepsilon\|_{C^{1+\beta,\frac{1+\beta}{2}}(\bar{\Omega}\times[0,T])} \le B + 3. \tag{8.3.22}$$

Here we used that $\|\psi\|_{C^{\frac{\beta}{2}}([0,T_0])} \le \|\psi\|_{C^1([0,T_0])}$ holds for $\psi \in C^1([0,T_0])$ due to $T_0 \le 1$. Hence, setting $T := T_0$, the map $F : X \to X$, $F(c_\varepsilon, h_\varepsilon, n_\varepsilon) := (\tilde{c}_\varepsilon, \tilde{h}_\varepsilon, \tilde{n}_\varepsilon)$ is well-defined and compact due to (8.3.19)-(8.3.22) so that F has a fixed point $(c_\varepsilon, h_\varepsilon, n_\varepsilon)$ by Schauder's fixed point theorem. The above reasoning thus ensures the existence of a classical solution to (8.3.11) in $\Omega \times (0,T)$ which has the claimed regularity properties and satisfies (8.3.15).
Next, let $T_\varepsilon < \infty$ and assume for contradiction that (8.3.16) does not hold. Then there is $C_8 > 0$ such that

$$\|c_\varepsilon\|_{L^\infty(\Omega\times(0,T_\varepsilon))} + \|h_\varepsilon\|_{L^\infty((0,T_\varepsilon);W^{1,q}(\Omega))} \le C_8. \tag{8.3.23}$$

Combining this estimate with (8.3.1), (8.3.12), (8.3.15) and (8.3.11), we have

$$\partial_t c_\varepsilon = \nabla \cdot (a_\varepsilon(x,t,\nabla c_\varepsilon)) + b_\varepsilon(x,t) \qquad \text{in } \Omega \times (0,T_\varepsilon),$$

where

$$a_\varepsilon(x,t,\xi)\cdot\xi \geq \frac{\tilde{C}_2}{2(1+C_8)}|\xi|^2 - \psi_0(x,t), \quad |a_\varepsilon(x,t,\xi)| \leq \tilde{C}_1|\xi| + \psi_1(x,t)$$

holds for all $(x,t,\xi) \in \Omega\times(0,T_\varepsilon)\times\mathbb{R}^N$ with $\psi_0, \psi_1^2 \in L^\infty((0,T_\varepsilon); L^{\frac{q}{2}}(\Omega))$ and $b_\varepsilon \in L^\infty(\Omega\times(0,T_\varepsilon))$. Hence, in view of $\frac{q}{2} > \frac{N}{2}$, [A8.34, Theorem 1.3 and Remark 1.4] and (8.3.1) imply that

$$\|c_\varepsilon\|_{C^{\beta_2,\frac{\beta_2}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \leq C_9 \tag{8.3.24}$$

with some $C_9 > 0$ and $\beta_2 \in (0,1)$. By using the same results and possibly diminishing β_2, we also have

$$\|h_\varepsilon\|_{C^{\beta_2,\frac{\beta_2}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \leq C_{10},$$

where $C_{10} > 0$ and $\beta_2 \in (0,1)$. Hence, (8.3.18) and (8.3.19) are valid with $T = T_\varepsilon$ and $\beta = \beta_2$ so that a combination with (8.3.24) yields

$$\|n_\varepsilon\|_{C^{\beta_2,\frac{\beta_2}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \leq C_{11}$$

with some $C_{11} > 0$. Thus, we are able to apply [A8.23, Theorem 1.1] to obtain

$$\|c_\varepsilon\|_{C^{1+\beta_3,\frac{1+\beta_3}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \leq C_{12}$$

with constants $C_{12} > 0$ and $\beta_3 \in (0,\beta_2]$. Now this implies

$$\|n_\varepsilon\|_{C^{1+\beta_3,\frac{1+\beta_3}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \leq C_{13}$$

with some $C_{13} > 0$ due to (8.3.19) and [A8.33, Theorem 2 in Section 2.3]. Finally, as A with $\beta = \beta_3$ is finite, (8.3.18)-(8.3.21) yield

$$\begin{aligned} A_1 \;:=\; & \|c_\varepsilon\|_{C^{2+\beta_4,1+\frac{\beta_4}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} + \|n_\varepsilon\|_{C^{2+\beta_4,1+\frac{\beta_4}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} + \|h_\varepsilon\|_{C^{2+\beta_4,1+\frac{\beta_4}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} \\ & + \|h_\varepsilon\|_{C^{\beta_4,\frac{\beta_4}{2}}(\bar{\Omega}\times[-\tau,T_\varepsilon])} + \|y_{0\varepsilon}\|_{C^{\beta_4,\frac{\beta_4}{2}}(\bar{\Omega}\times[0,T_\varepsilon])} < \infty \end{aligned}$$

with some $\beta_4 \in (0,\beta_3]$. Therefore, by the first part of this proof the solution can be extended to a classical solution of (8.3.11) in $\Omega\times(0,T_\varepsilon+\frac{T_0}{2})$ with some $T_0 = T_0(A_1) > 0$ which contradicts the maximality of T_ε and proves (8.3.16). ■

In order to prove the global existence for the solution to (8.3.11), we will show appropriate bounds on c_ε and ∇h_ε which are independent of ε. To this end, we remark that (8.3.1), (8.3.12) and (8.3.13) imply

$$0 < a_2 \leq K_c(h) \leq a_1, \quad 0 < b_2 \leq \mu_c(y) \leq b_1, \quad |R(y,h)| \leq M_R \\ \text{for all } h \in [0,H_0],\ y \in [0,Y_0] \tag{8.3.25}$$

with positive constants b_1, b_2, M_R. Moreover, we denote by $(e^{t\Delta})_{t\geq 0}$ the heat semigroup in Ω with homogeneous Neumann boundary conditions and define $\lambda_1 > 0$ to be the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. It is well-known (see e.g. [A8.45, Lemma 1.3]) that there exists $C_3 > 0$ such that

$$\begin{aligned} &\|\nabla e^{t\Delta} v\|_{L^\rho(\Omega)} \leq C_3 \left(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{r}-\frac{1}{\rho})}\right) e^{-\lambda_1 t}\|v\|_{L^r(\Omega)} && \text{for all } t > 0 \\ &\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \leq C_3 e^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)} && \text{for all } t > 0 \end{aligned} \tag{8.3.26}$$

holds for any $v \in L^r(\Omega)$, $w \in W^{1,p}(\Omega)$, $1 \leq r \leq \rho \leq \infty$ and $p \in [2, \infty)$. Using these estimates, we prove the following elementary bounds for c_ε and h_ε.

Lemma 8.3.4 *Let $q \in (N + 2, \infty)$ be as defined in* (8.3.5)*. There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ we have*

$$\int_\Omega c_\varepsilon(x,t)\,dx \leq m := \max\left\{\sup_{\varepsilon\in(0,1)} \int_\Omega c_{0\varepsilon}\,dx, \frac{a_1 b_1|\Omega|}{b_2}\right\} < \infty \quad \textit{for all } t \in (0, T_\varepsilon), \tag{8.3.27}$$

$$\|h_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C \qquad \textit{for all } t \in (0, T_\varepsilon). \tag{8.3.28}$$

Proof. Integrating the first equation of (8.3.11) and using (8.3.15), (8.3.25) along with the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt}\int_\Omega c_\varepsilon \leq b_1 \int_\Omega c_\varepsilon - \frac{b_2}{a_1}\int_\Omega c_\varepsilon^2 \leq b_1 \int_\Omega c_\varepsilon - \frac{b_2}{a_1|\Omega|}\left(\int_\Omega c_\varepsilon\right)^2 \quad \text{for all } t \in (0, T_\varepsilon),$$

so that (8.3.27) follows by an ODE comparison and (8.3.13).
Next, we use the Neumann heat semigroup and Lemma 8.3.3 to obtain from (8.3.11)

$$h_\varepsilon(\cdot,t) = e^{tD_h\Delta}h_{0\varepsilon}(\cdot,0) + \int_0^t e^{(t-s)D_h\Delta}R(y_\varepsilon(\cdot,s), h_\varepsilon(\cdot,s))\,ds, \quad t \in (0, T_\varepsilon).$$

In view of $q \geq 2$, (8.3.15), (8.3.25) and (8.3.26), this implies

$$\begin{aligned} &\|\nabla h_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \\ &\quad\leq \|\nabla e^{tD_h\Delta}h_{0\varepsilon}(\cdot,0)\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)D_h\Delta}R(y_\varepsilon(\cdot,s), h_\varepsilon(\cdot,s))\|_{L^q(\Omega)}\,ds \\ &\quad\leq C_3\|h_{0\varepsilon}(\cdot,0)\|_{W^{1,q}(\Omega)} \\ &\qquad + C_3\int_0^t \left(1 + (D_h(t-s))^{-\frac{1}{2}}\right) e^{-\lambda_1 D_h(t-s)}\|R(y_\varepsilon(\cdot,s), h_\varepsilon(\cdot,s))\|_{L^q(\Omega)}\,ds \\ &\quad\leq C_3 \sup_{\varepsilon\in(0,1)} \|h_{0\varepsilon}(\cdot,0)\|_{W^{1,q}(\Omega)} + C_3 M_R|\Omega|^{\frac{1}{q}}\int_0^\infty \left(1 + (D_h\sigma)^{-\frac{1}{2}}\right) e^{-\lambda_1 D_h\sigma}\,d\sigma \end{aligned}$$

for all $t \in (0, T_\varepsilon)$, which proves (8.3.28) due to (8.3.13) and (8.3.15). ∎

The next lemma is the main step toward the global existence. It uses Lemma 8.3.4 as a starting point to obtain bounds in $L^\infty((0,T); L^p(\Omega))$ for any finite p. We adapt ideas from [A8.40, Lemma 3.3] for its proof.

Lemma 8.3.5 *Let $T \in (0,\infty)$ such that $T \le T_\varepsilon$. Then there are $C(T) > 0$ and $\tilde{C}(T) > 0$ such that*

$$h_\varepsilon(x,t) \ge C(T) \quad \text{for all } (x,t) \in \bar{\Omega} \times [0,T), \tag{8.3.29}$$

$$\int_0^T \int_\Omega |\nabla c_\varepsilon|^2(x,t)\,dxdt \le \tilde{C}(T) \tag{8.3.30}$$

are fulfilled for every $\varepsilon \in (0,1)$. Moreover, for any $p \in [1,\infty)$ there exists $C_p(T) > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C_p(T) \qquad \text{for all } t \in (0,T). \tag{8.3.31}$$

Proof. In view of (8.3.1), (8.3.2) and (8.3.15), there exists $C_R > 0$ such that the third equation in (8.3.11) implies

$$\partial_t h_\varepsilon \ge D_h \Delta h_\varepsilon + R(y_\varepsilon, 0) - C_R h_\varepsilon \ge D_h \Delta h_\varepsilon - C_R h_\varepsilon \quad \text{in } \Omega \times (0, T_\varepsilon)$$

for all $\varepsilon \in (0,1)$. Hence, by (8.3.12) and the comparison principle, we have

$$h_\varepsilon(x,t) \ge \frac{\delta}{2} e^{-C_R t} \qquad \text{for all } (x,t) \in \bar{\Omega} \times [0, T_\varepsilon) \tag{8.3.32}$$

and all $\varepsilon \in (0,1)$, which proves (8.3.29).

Next, we fix $p \in [2,\infty)$ and $T \in (0,\infty)$ such that $T \le T_\varepsilon$. Defining $a := C(T) > 0$ with $C(T)$ from (8.3.29), testing the first equation in (8.3.11) by $(1+c_\varepsilon)^{p-1}$ and using (8.3.12), (8.3.14), (8.3.15), (8.3.25) as well as the inequalities of Young and Hölder, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (1+c_\varepsilon)^p &= \int_\Omega (1+c_\varepsilon)^{p-1}\partial_t c_\varepsilon \\
&\le -(p-1)\int_\Omega \varphi_\varepsilon(c_\varepsilon, n_\varepsilon)(1+c_\varepsilon)^{p-2}|\nabla c_\varepsilon|^2 \\
&\quad +(p-1)\int_\Omega f_\varepsilon(h_\varepsilon, c_\varepsilon)(1+c_\varepsilon)^{p-2}\nabla c_\varepsilon \cdot \nabla h_\varepsilon \\
&\quad +b_1 \int_\Omega c_\varepsilon(1+c_\varepsilon)^{p-1} - \frac{b_2}{a_1}\int_\Omega c_\varepsilon^2(1+c_\varepsilon)^{p-1} \\
&\le -\frac{\tilde{C}_2(p-1)}{2}\int_\Omega (1+c_\varepsilon)^{p-3}|\nabla c_\varepsilon|^2 + \frac{\tilde{C}_a^2(p-1)}{2\tilde{C}_2}\int_\Omega (1+c_\varepsilon)^{p-1}|\nabla h_\varepsilon|^2 \\
&\quad + \left(b_1 + \frac{2b_2}{a_1}\right)\int_\Omega (1+c_\varepsilon)^p - \frac{b_2}{a_1}\int_\Omega (1+c_\varepsilon)^{p+1} \\
&\le -\frac{2\tilde{C}_2}{p-1}\int_\Omega |\nabla (1+c_\varepsilon)^{\frac{p-1}{2}}|^2 \\
&\quad + \frac{\tilde{C}_a^2(p-1)}{2\tilde{C}_2}\left(\int_\Omega (1+c_\varepsilon)^{\frac{(p-1)q}{q-2}}\right)^{\frac{q-2}{q}}\left(\int_\Omega |\nabla h_\varepsilon|^q\right)^{\frac{2}{q}} \\
&\quad + \left(b_1 + \frac{2b_2}{a_1}\right)\int_\Omega (1+c_\varepsilon)^p - \frac{b_2}{a_1|\Omega|^{\frac{1}{p}}}\left(\int_\Omega (1+c_\varepsilon)^p\right)^{\frac{p+1}{p}}
\end{aligned} \tag{8.3.33}$$

for all $t \in (0,T)$ in view of $q > 2$. Abbreviating $\theta := \frac{q}{q-2} \in (1, \frac{N+2}{N})$, the Gagliardo-Nirenberg inequality and (8.3.27) yield

$$\begin{aligned}
&\left(\int_\Omega (1+c_\varepsilon)^{(p-1)\theta}\right)^{\frac{1}{\theta}} \\
&= \left\|(1+c_\varepsilon)^{\frac{p-1}{2}}\right\|^2_{L^{2\theta}(\Omega)} \\
&\le C_{GN}\left\|\nabla(1+c_\varepsilon)^{\frac{p-1}{2}}\right\|^{2d}_{L^2(\Omega)}\left\|(1+c_\varepsilon)^{\frac{p-1}{2}}\right\|^{2(1-d)}_{L^{\frac{2}{p-1}}(\Omega)} + C_{GN}\left\|(1+c_\varepsilon)^{\frac{p-1}{2}}\right\|^2_{L^{\frac{2}{p-1}}(\Omega)} \\
&\le C_{GN}\Big((m+|\Omega|)^{(p-1)(1-d)}\left\|\nabla(1+c_\varepsilon)^{\frac{p-1}{2}}\right\|^{2d}_{L^2(\Omega)} + (m+|\Omega|)^{p-1}\Big) \qquad (8.3.34)
\end{aligned}$$

for all $t \in (0, T_\varepsilon)$, since

$$d := \frac{\frac{p-1}{2} - \frac{1}{2\theta}}{\frac{1}{N} - \frac{1}{2} + \frac{p-1}{2}} \in (0,1)$$

is satisfied due to $\theta \in (1, \frac{N+2}{N})$ and $p \ge 2$. In view of (8.3.28) and $d < 1$, by inserting (8.3.34) into (8.3.33) and applying Young's inequality we arrive at

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (1+c_\varepsilon)^p + \frac{\tilde{C}_2}{p-1}\int_\Omega |\nabla(1+c_\varepsilon)^{\frac{p-1}{2}}|^2 \\
\le\ &\left(b_1 + \frac{2b_2}{a_1}\right)\int_\Omega (1+c_\varepsilon)^p - \frac{b_2}{a_1|\Omega|^{\frac{1}{p}}}\left(\int_\Omega (1+c_\varepsilon)^p\right)^{\frac{p+1}{p}} + C_4(a,p) \qquad (8.3.35)
\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ with some positive constant $C_4(a,p)$. This proves (8.3.31) upon an ODE comparison due to (8.3.13) and (8.3.27). Then, integrating (8.3.35) for $p = 3$ with respect to $t \in (0,T)$ and using (8.3.31), we conclude that (8.3.30) is valid. ■

Now we are in the position to obtain the global existence for the regularized problem (8.3.11) by using a result from [A8.40].

Lemma 8.3.6 *For any $\varepsilon \in (0,1)$, the solution to* (8.3.11) *obtained in Lemma 8.3.3 exists globally in time, which means that $T_\varepsilon = \infty$. Furthermore, for any $T \in (0,\infty)$ there exists $C_\infty(T) > 0$ such that*

$$0 \le c_\varepsilon \le C_\infty(T) \quad \text{in } \bar{\Omega} \times [0,T] \qquad (8.3.36)$$

holds for any $\varepsilon \in (0,1)$.

Proof. We fix $T \in (0,\infty)$ with $T \le T_\varepsilon$. Keeping the notation from [A8.40, Appendix A], by (8.3.11), (8.3.12), (8.3.14) and Lemmas 8.3.3-8.3.5, we have

$$\partial_t c_\varepsilon \le \nabla\cdot(D_\varepsilon(x,t,c_\varepsilon)\nabla c_\varepsilon) + \nabla\cdot(F_\varepsilon(x,t)) + G_\varepsilon(x,t), \quad (x,t) \in \Omega \times (0,T),$$

where c_ε and F_ε satisfy the homogeneous Neumann boundary condition, D_ε and F_ε are C^1-functions and G_ε is continuous such that $F_\varepsilon \in L^\infty((0,T); L^{q_1}(\Omega))$, $G_\varepsilon \in L^\infty((0,T); L^{q_2}(\Omega))$,

$c_\varepsilon \in L^\infty((0,T);L^{p_0}(\Omega))$ and $D_\varepsilon(x,t,c_\varepsilon) \geq \tilde{C}_2(1+c_\varepsilon)^{\tilde{m}-1}$ for $\tilde{m}=0$, $q_1 = q > N+2$ and all $p_0 \in (1,\infty)$ and $q_2 \in (\frac{N+2}{2},\infty]$. In view of $\tilde{m}=0$, we may apply [A8.40, Lemma A.1] with some $p_0 > \max\{1, \frac{N}{2}\}$ and obtain a constant $C_\infty(T) > 0$ such that

$$\|c_\varepsilon\|_{L^\infty(\Omega\times(0,T))} \leq C_\infty(T).$$

As $C_\infty(T)$ depends on $\tilde{C}_2$, $\sup_{\varepsilon\in(0,1)} \|c_{0\varepsilon}\|_{L^\infty(\Omega)}$ and the norms of c_ε, F_ε and G_ε in the spaces mentioned above, we conclude that $C_\infty(T)$ does not depend on $\varepsilon \in (0,1)$ and just depends on T via (8.3.29) and (8.3.31). Hence, in view of (8.3.28), the criterion (8.3.16) proves the lemma. ∎

Let us finalize this subsection with the following remark.

Remark 8.3.7 *The conditions imposed on φ and f in* (8.3.3) *and* (8.3.4) *are motivated by biological considerations and are in particular satisfied for the example given in* (8.2.6). *If we assume instead*

$$0 \leq f(h,c) \leq C_1, \quad \frac{C_2}{1+c} \leq \varphi(c,n) \leq C_1 \quad \forall (c,n,h) \in [0,\infty)\times[0,K_n]\times[0,H_0]$$

and corresponding estimates for φ_ε and f_ε, then $\|c_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \leq C_\infty$ holds for some $C_\infty > 0$ which does not depend on $\varepsilon \in (0,1)$, as the constants C_p and C_∞ do not depend on T any more. This result remains true if we only assume the nonnegativity of h_0 and $h_{0\varepsilon}$ instead of their strict positivity as we do not need (8.3.29) *in this setting.*
With appropriately adapted proofs of Lemmas 8.3.5 and 8.3.6, the result $\|c_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \leq C_\infty$ holds for some $C_\infty > 0$ which does not depend on $\varepsilon \in (0,1)$, if we assume

$$0 \leq f(h,c) \leq C_1(1+c)^{m_1}, \quad C_2(1+c)^{-m_2} \leq \varphi(c,n) \leq C_1(1+c)^{m_3}$$

for all $(c,n,h) \in [0,\infty)\times[0,K_n]\times[0,H_0]$ with some real numbers m_j, $j=1,2,3$, satisfying $2m_1 + m_2 < 3$ as well as $h_0 \in C^0([-\tau,0];W^{1,\infty}(\Omega))$ and the nonnegativity of h_0.

8.3.2 Existence of a global weak solution to the original problem

In order to obtain a global weak solution to (8.2.3), we next prove appropriate precompactness properties of the solutions to (8.3.11) which are based on the results of the preceding subsection.

Lemma 8.3.8 *Let $T \in (0,\infty)$ be arbitrary. For the global solutions to* (8.3.11) *from Lemma 8.3.3 we have that $(c_\varepsilon)_{\varepsilon\in(0,1)}$, $(n_\varepsilon)_{\varepsilon\in(0,1)}$, $(h_\varepsilon)_{\varepsilon\in(0,1)}$ and $(y_\varepsilon)_{\varepsilon\in(0,1)}$ are strongly precompact in $L^2(\Omega\times(0,T))$.*

Proof. Throughout this proof we will frequently make use of (8.3.12), (8.3.15), (8.3.25), (8.3.28), (8.3.30) and (8.3.36) without explicitly mentioning this every time. Using these properties, there exists a constant $C_4(T) > 0$ such that for all $\psi \in C_0^\infty(\Omega)$ and all $\varepsilon \in (0,1)$ we obtain from (8.3.11) and the Hölder inequality that

$$\int_\Omega \partial_t c_\varepsilon \psi = -\int_\Omega \varphi_\varepsilon(c_\varepsilon,n_\varepsilon)\nabla c_\varepsilon \cdot \nabla\psi + \int_\Omega f_\varepsilon(h_\varepsilon,c_\varepsilon)\nabla h_\varepsilon \cdot \nabla\psi$$

$$
\begin{aligned}
&+\int_\Omega \mu_c(y_\varepsilon)c_\varepsilon\left(1-\frac{c_\varepsilon}{K_c(h_\varepsilon(\cdot,t-\tau))}-\eta_1\frac{n_\varepsilon}{K_n}\right)\psi\\
\leq\quad &\left[C_4(T)+\left(\int_\Omega|\nabla c_\varepsilon|^2\right)^{\frac{1}{2}}\right]\|\psi\|_{W_0^{1,2}(\Omega)}\qquad\text{for all } t\in(0,T).
\end{aligned}
$$

Hence, $(\partial_t c_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2((0,T);(W_0^{1,2})^*)$ by (8.3.30). As furthermore $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2((0,T);W^{1,2}(\Omega))$, $W^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega)$ and $L^2(\Omega)\subset(W_0^{1,2})^*$, the Aubin-Lions lemma (see [A8.41, Theorem 2.1 in Chapter III]) implies the strong precompactness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2((0,T);L^2(\Omega))$.

Similarly, $(\partial_t h_\varepsilon)_{\varepsilon\in(0,1)}$ and $(h_\varepsilon)_{\varepsilon\in(0,1)}$ are uniformly bounded in $L^2((0,T);(W_0^{1,2})^*)$ and $L^2((0,T);W^{1,2}(\Omega))$, respectively, in view of $q\geq 2$, so that $(h_\varepsilon)_{\varepsilon\in(0,1)}$ is strongly precompact in $L^2((0,T);L^2(\Omega))$.

In order to prove the claimed results for $(n_\varepsilon)_{\varepsilon\in(0,1)}$, we recall that by Kolmogorov-Riesz for a bounded domain $\mathcal{D}\subset\mathbb{R}^s$ with $s\in\mathbb{N}$ a set $\mathcal{M}\subset L^2(\mathcal{D})$ is strongly precompact in $L^2(\mathcal{D})$ if and only if

$$
\sup_{F\in\mathcal{M}}\|F\|_{L^2(\mathcal{D})}<\infty\quad\text{and}\quad\lim_{z\to 0}\left(\sup_{F\in\mathcal{M}}\|F^z-F\|_{L^2(\mathcal{D})}\right)=0,
$$

where $z\in\mathbb{R}^s$ and $F^z(\zeta):=F(\zeta+z)$ for $\zeta\in\mathcal{D}$ such that $\zeta+z\in\mathcal{D}$ and $F^z(\zeta)=0$ if $\zeta+z\notin\mathcal{D}$. Setting $\mathcal{D}:=\Omega\times(0,T)$ and $\zeta:=(x,t)\in\mathcal{D}$, for $z\in\mathbb{R}^{N+1}$ we obtain from an integration of the second equation of (8.3.11) and the regularity and boundedness properties of its right-hand side that

$$
\begin{aligned}
&\int_\Omega (n_\varepsilon^z-n_\varepsilon)^2(\cdot,t)\,dx\\
&\leq\int_\Omega(n_{0\varepsilon}^z-n_{0\varepsilon})^2\,dx+C_5(T)\int_0^t\int_\Omega\left(|n_\varepsilon^z-n_\varepsilon|+|c_\varepsilon^z-c_\varepsilon|+|h_\varepsilon^z-h_\varepsilon|\right)|n_\varepsilon^z-n_\varepsilon|\,dxds\\
&\leq\int_\Omega(n_{0\varepsilon}^z-n_{0\varepsilon})^2\,dx\\
&\quad+C_6(T)\int_0^t\int_\Omega\left((n_\varepsilon^z-n_\varepsilon)^2+(c_\varepsilon^z-c_\varepsilon)^2+(h_\varepsilon^z-h_\varepsilon)^2\right)\,dxds
\end{aligned}
\tag{8.3.37}
$$

with some positive constants $C_5(T)$ and $C_6(T)$ for all $t\in(0,T)$ and all $\varepsilon\in(0,1)$. Hence, by Gronwall's inequality there exists $C_7(T)>0$ such that

$$
\begin{aligned}
\sup_{\varepsilon\in(0,1)}\|n_\varepsilon^z-n_\varepsilon\|_{L^2(\Omega\times(0,T))}\quad\leq\quad &C_7(T)\sup_{\varepsilon\in(0,1)}\Big(\|n_{0\varepsilon}^z-n_{0\varepsilon}\|_{L^2(\Omega)}\\
&+\|c_\varepsilon^z-c_\varepsilon\|_{L^2(\Omega\times(0,T))}+\|h_\varepsilon^z-h_\varepsilon\|_{L^2(\Omega\times(0,T))}\Big).
\end{aligned}
\tag{8.3.38}
$$

As $(c_\varepsilon)_{\varepsilon\in(0,1)}$, $(h_\varepsilon)_{\varepsilon\in(0,1)}$ are strongly precompact in $L^2(\Omega\times(0,T))$ and $(n_{0_\varepsilon})_{\varepsilon\in(0,1)}$ is strongly precompact in $L^2(\Omega)$ due to (8.3.13), the right-hand side of (8.3.38) converges to zero as $z\to 0$ by Kolmogorov-Riesz. Since furthermore $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded

in $L^2(\Omega \times (0,T))$, the mentioned criterion yields the strong precompactness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2(\Omega \times (0,T))$.
Similar arguments also show that $(y_\varepsilon)_{\varepsilon\in(0,1)}$ is strongly precompact in $L^2(\Omega \times (0,T))$. ■

With these compactness properties at hand, we are able to prove the existence of a global weak solution to the original problem (8.2.3).

Proof of Theorem 8.3.2.
By Lemma 3.8, (8.3.13), (8.3.15), (8.3.28), (8.3.30), (8.3.36) as well as a standard extraction argument involving diagonal sequences, there are a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions

$$c \in L^\infty_{loc}(\bar\Omega \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \quad n, y \in L^\infty(\Omega \times (0,\infty)),$$
$$h \in L^\infty(\Omega \times (-\tau,\infty)) \cap L^\infty((0,\infty); W^{1,q}(\Omega))$$

satisfying (8.3.10) such that

$$c_\varepsilon \to c, \quad n_\varepsilon \to n, \quad y_\varepsilon \to y \quad \text{strongly in } L^2_{loc}([0,\infty); L^2(\Omega)) \text{ and a.e. in } \Omega \times (0,\infty),$$
$$h_\varepsilon \to h \quad \text{strongly in } L^2_{loc}([-\tau,\infty); L^2(\Omega)) \text{ and a.e. in } \Omega \times (-\tau,\infty),$$
$$\nabla c_\varepsilon \rightharpoonup \nabla c, \quad \nabla h_\varepsilon \rightharpoonup \nabla h \quad \text{weakly in } L^2_{loc}([0,\infty); L^2(\Omega))$$

as $\varepsilon = \varepsilon_j \searrow 0$. Combining these properties with (8.3.1), (8.3.12) and (8.3.13), for any fixed $T \in (0,\infty)$ we may then pass to the limit as $\varepsilon_j \searrow 0$ in the weak formulation of (8.3.11) corresponding to (8.3.6)-(8.3.9) and use the dominated convergence theorem to conclude that (c,n,h,y) is a global weak solution to (8.2.3)-(8.2.5).
In order to prove the uniqueness claim, let in addition $c_0 \in C^\beta(\bar\Omega)$ be fulfilled with some $\beta \in (\frac{1}{N+2}, 1)$ and assume that (c_j, n_j, h_j, y_j), $j = 1,2$, are global weak solutions to (8.2.3)-(8.2.5) such that for all $T \in (0,\infty)$ there is $C_4(T) > 0$ with

$$\begin{aligned} &\|c_j\|_{L^\infty(\Omega\times(0,T))} + \|\nabla c_j\|_{L^2(\Omega\times(0,T))} + \|n_j\|_{L^\infty(\Omega\times(0,T))} \\ &+\|h_j\|_{L^\infty(\Omega\times(0,T))} + \|\nabla h_j\|_{L^r(\Omega\times(0,T))} + \|y_j\|_{L^\infty(\Omega\times(0,T))} \le C_4(T) \end{aligned} \tag{8.3.39}$$

for $j = 1,2$ and some $r > N+2$ which is independent of $T \in (0,\infty)$ and satisfies $1 - \frac{N+1}{r} < \beta$. Observe that the global weak solution constructed above satisfies (8.3.39) in view of $q > N+2$.
We next fix an arbitrary $T \in (0,\infty)$. Then, similarly to (8.3.37) and (8.3.38), we obtain from (8.3.7), (8.3.9), (8.3.1), (8.3.39) and Gronwall's inequality that there is $C_5(T) > 0$ such that

$$|n_1 - n_2|(x,t) \le C_5(T) \int_0^t (|c_1 - c_2| + |h_1 - h_2|)(x,s)\, ds, \tag{8.3.40}$$
$$|y_1 - y_2|(x,t) \le C_5(T) \int_0^t (|c_1 - c_2| + |h_1 - h_2|)(x,s)\, ds \tag{8.3.41}$$

are fulfilled for a.e. $(x,t) \in \Omega \times (0,T)$. Furthermore, (8.3.8), (8.3.1) and (8.3.39) imply

$$\int_\Omega |h_1 - h_2|^2(x,t)\, dx \le -D_h \int_0^t \int_\Omega |\nabla(h_1 - h_2)|^2\, dxds$$

$$+C_6(T)\int_0^t (|h_1-h_2|^2+|y_1-y_2|^2)\,dxds$$

for a.e. $t\in(0,T)$ with some $C_6(T)>0$. Hence, using (8.3.41) along with Gronwall's inequality we conclude that there is $C_7(T)>0$ such that

$$\int_\Omega |h_1-h_2|^2(x,t)\,dx \;\le\; C_7(T)\int_0^t |c_1-c_2|^2(x,s)\,dxds, \qquad (8.3.42)$$

$$\int_0^t\int_\Omega |\nabla(h_1-h_2)|^2\,dxds \;\le\; C_7(T)\int_0^t |c_1-c_2|^2(x,s)\,dxds \qquad (8.3.43)$$

for a.e. $t\in(0,T)$. Next, in view of (8.3.1), (8.3.39) and $c_0\in C^\beta(\bar\Omega)$, we may apply [A8.34, Theorem 1.3 and Remark 1.4] to obtain that $c_j,h_j\in C^{\gamma,\frac{\gamma}{2}}(\bar\Omega\times[0,T])$ for some $\gamma\in(0,\beta)$ (like in the proof of (8.3.24)) which also implies $n_j,y_j\in C^0(\bar\Omega\times[0,T])$, $j=1,2$, due to (8.3.40), (8.3.41). Therefore, the closed operator $\mathcal{B}_j(t)$ with $\mathcal{B}_j(t)u:=-\nabla\cdot(\varphi(c_j(\cdot,t),n_j(\cdot,t))\nabla u)$ for $u\in W^{1,r}(\Omega)$, defines a continuous map $\mathcal{B}_j:[0,T]\to\mathcal{L}(W^{1,r}(\Omega),W^{-1,r}(\Omega))$ by [A8.15, (6.3)]. As furthermore $-\mathcal{B}_j$ is uniformly elliptic on $[0,T]$, c_j solves

$$\partial_t u(t)+\mathcal{B}_j(t)u(t)=f_j(t),\quad t\in[0,T],\qquad u(0)=c_0$$

with $f_j\in L^r((0,T);W^{-1,r}(\Omega))$ and $c_0\in W^{1-\frac{1}{r},r}(\Omega)$ due to (8.3.1), (8.3.6), (8.3.39), $\beta>1-\frac{N+1}{r}$ and the smoothness of $\partial\Omega$, we may apply the result of maximal parabolic regularity from [A8.15, Theorem 5.4, Remark 5.5, Proposition 6.1] to conclude that $c_j\in W^{1,r}((0,T);W^{-1,r}(\Omega))\cap L^r((0,T);W^{1,r}(\Omega))$. Here, the regularity of c_0 and [A8.15, Theorem 6.14] shows that we can apply the maximal regularity result also to $u(0)=c_0\not\equiv 0$. Hence,

$$\|\nabla c_j\|_{L^r(\Omega\times(0,T))}\le C_8(T) \qquad (8.3.44)$$

is satisfied for $j=1,2$ with some $C_8(T)>0$. Finally, using (8.3.39)-(8.3.44), we can apply the method from the uniqueness proof in [A8.29, Theorem 3.1] (starting at [A8.29, (3.37)] and setting $\psi\equiv 0$, $l:=h$ and $2p:=r>N+2$) to conclude that there is $t_0\in(0,T)$ sufficiently small (just depending on T) such that

$$\|c_1-c_2\|^2_{L^\infty((0,t);L^2(\Omega))}\le C_9(T)\int_0^t \|c_1-c_2\|^2_{L^\infty((0,s);L^2(\Omega))}\,ds$$

holds for all $t\in(0,t_0)$ with some $C_9(T)>0$. Hence, an application of Gronwall's inequality yields $c_1=c_2$ a.e. in $\Omega\times(0,t_0)$ so that (8.3.40)-(8.3.42) and an iteration of this argument show that the two global weak solutions coincide a.e. in $\Omega\times(0,T)$. ■

8.4 Numerical simulations

For the numerical simulations we first introduce the dimensionless variables

$$\tilde c=\frac{c}{C_0},\quad \tilde n=\frac{n}{K_n},\quad \tilde h=\frac{h}{H_0},\quad \tilde y=\frac{y}{Y_0},\quad \tilde x=\frac{x}{L},\quad \tilde t=\frac{t}{T},\quad \theta=\frac{t}{\chi T},$$

where K_n is the carrying capacity of the normal cells, C_0 is the reference carrying capacity of the cancer cells, H_0 and Y_0 are the reference concentrations of extracellular and intracellular protons, L is the reference length scale and T is the reference time unit. As the processes on the subcellular scale are much faster than those on the macroscale, $\theta = \frac{\tilde{t}}{\chi}$ with some $\chi \in (0,1)$ represents the time on the microscale.
Using these variables and (8.2.6), we transform (8.2.3) to the nondimensionalized system

$$\begin{cases} \partial_t c = \nabla \cdot \left(\frac{D_c}{1+cn} \nabla c\right) - \nabla \cdot \left(\frac{Mc}{1+ch} \nabla h\right) + \frac{\kappa_1}{1+y} c \left(1 - \frac{c}{K_c(h(\cdot,t-\tau))} - \eta_1 n\right), \\ \partial_t n = -\delta_n h n + \mu_n n \left(1 - \eta_2 \frac{c}{K_c(h(\cdot,t))} - n\right), \\ \partial_t h = D_h \Delta h + \frac{\gamma_h y}{1+y^2+\alpha_h h^2} - \frac{\beta_h h}{1+y^2}, \\ \partial_\theta y = -\frac{\gamma_y y}{1+y^2+\alpha_h h^2} + \frac{\beta_y h}{1+y^2} - \alpha y + \frac{\rho c}{1+c}, \end{cases} \tag{8.4.1}$$

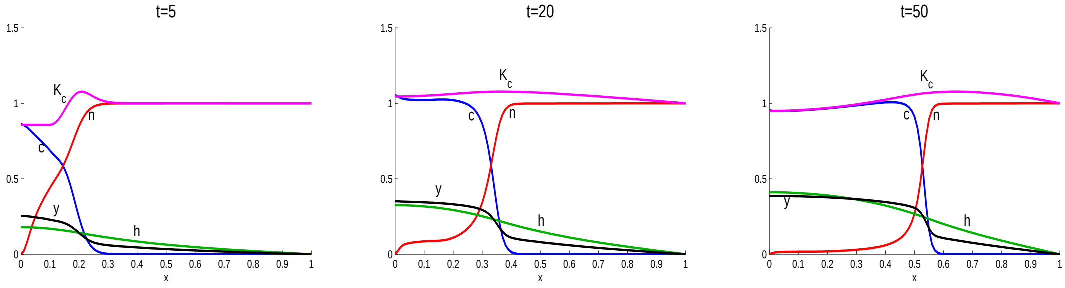
in $\Omega \times (0,T)$ with $\Omega := (0,1) \subset \mathbb{R}$, where we omit the tildes in variables and constants for the ease of notation.
We endow (8.4.1) with the boundary and initial conditions (8.2.4)-(8.2.5) and set

$$c_0(x) := \exp\left(-\frac{x^2}{\varepsilon}\right), \quad n_0(x) := 1 - \exp\left(-\frac{x^2}{\varepsilon}\right), \quad y_0(x) := \xi_y c_0(x),$$
$$h_0(x,t) := \xi_h c_0((x - x_0)_+)$$

for $x \in [0,1]$ and $t \in [-\tau, 0]$, where $(s)_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$. Here the choice of h_0 accounts for an already elevated intratumoral acidosis, which decays towards the tumor border. This is done to include the fact that a too acidic environment is harmful also for cancer cells.
We perform numerical simulations by using an implicit-explicit finite difference scheme as in [A8.29] and choosing the parameters $\tau = 8$, $\chi = 0.01$, $D_c = 2 \cdot 10^{-6}$, $M = 10^{-3}$, $D_h = 0.1$, $\eta_1 = 0.35$, $\eta_2 = 0.05$, $\xi_y = 0.3$, $\xi_h = 0.5$ and $x_0 = 0.1$.
We provide some time snapshots for two different choices of the carrying capacity K_c of the cancer cells. First, we choose $K_c(h) = \frac{1+bh}{1+dh^2}$ with $d > 0$ so that a too acidic environment of the tumor causes a smaller carrying capacity, which leads to a decrease in the original tumor (see Figure 8.1). In contrast to this, for $K_c(h) = 1 + bh$ the original tumor infers enhanced growth, but does not seem to affect the invasion speed (see Figure 8.2). The classical choice of a constant carrying capacity (see Figure 8.3) does not allow the tumor to adapt its growth to the environmental acidity.

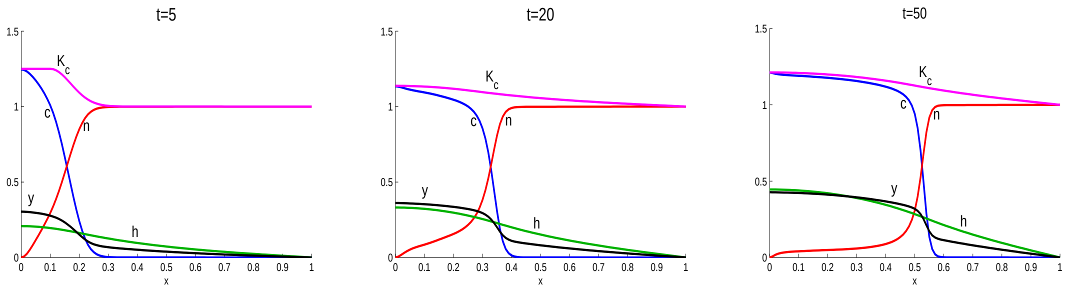
8.5 Discussion

In this work we proposed a multiscale model for acid-mediated tumor invasion which assigns more importance to the intracellular proton dynamics on the microscale leading to the behavior of the interacting normal and cancer cell populations on the macroscale. The coupling between the two scales is realized through the spatio-temporal evolution of the extracellular protons, which are controlled by and influence the dynamics of the processes



(a) (b) (c)

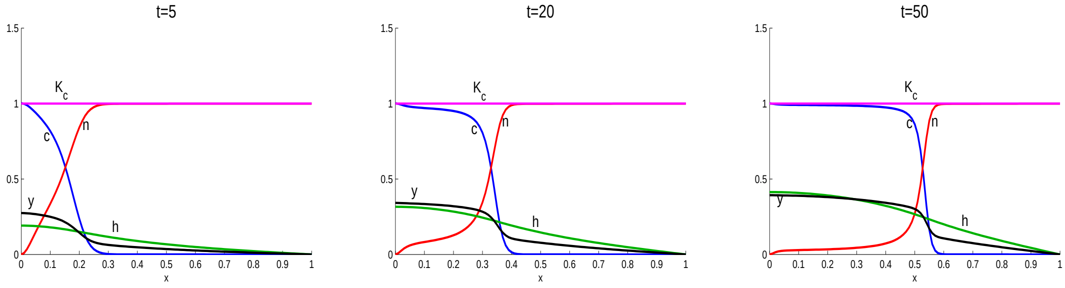
Figure 8.1: Evolution of tumor cell (blue) and normal cell densities (red), concentrations of extracellular (green) and intracellular protons (black), and carrying capacity $K_c(h(t-\tau))$ of cancer cells (purple) with $K_c(h) = \frac{1+h}{1+3h^2}$ for model (8.4.1).



(a) (b) (c)

Figure 8.2: Evolution of tumor cell (blue) and normal cell densities (red), concentrations of extracellular (green) and intracellular protons (black) and carrying capacity $K_c(h(t-\tau))$ of cancer cells (purple) with $K_c(h) = 1 + \frac{h}{2}$ for model (8.4.1).

taking place both on the microscale and on the macroscale. Moreover, we account for pH-taxis, which characterizes the tactic behavior of tumor cells in response to an extracellular pH gradient, a feature which was only lately proposed in the context of cancer invasion [A8.2, A8.31]. The model was shown to be well posed in the weak sense; the result holds even for a more general case than the concrete situation described in Section 8.2. The global boundedness of the solution remains, however, open (unless for supplementary assumptions, see Remark 8.3.7). Numerical simulations illustrate the behavior of the solution as predicted by our model. Thereby, the choice of the carrying capacity as a (delayed) function of the extracellular proton concentration was shown to be relevant: as expected, a larger carrying capacity will enhance tumor growth (though it seems to hardly have an effect on the invasion speed, but proving this mathematically is still an open issue). This would endorse the therapeutical approach aiming to reduce the acidity in the tumor environment to control the neoplasm development [A8.13, A8.32].



(a) (b) (c)

Figure 8.3: Evolution of tumor cell (blue) and normal cell densities (red), concentrations of extracellular (green) and intracellular protons (black) and carrying capacity K_c of cancer cells (purple) with $K_c = 1$ for model (8.4.1).

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