Lecture 2: Central limit theorems for statistics of geometric structures

Joe Yukich

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Probabilistic Analysis of Geometric Structures

· **Lecture 1**: Probabilistic analysis of Euclidean optimization problems

· **Lecture 2**: Central limit theorems for statistics of geometric structures

· **Lecture 3**: Limit theory for statistics of geometric structures via stabilizing score functions

· **Lecture 4**: Statistics of random polytopes

· **Lecture 5**: Rates of multivariate normal approximation for statistics of geometric structures

Lecture 2: Central limit theorems for statistics of geometric structures

- **I Introduction**
- **II Models and Results**

  Random packing models, Growth models, Gilbert Graph

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- **I Introduction**
- **II Models and Results**

  Random packing models, Growth models, Gilbert Graph
- **III Key Idea**

  Stability of difference operator
- **IV Central limit theorems**

  Statement

  Proof
- **V Large Deviation Principles**

I Introduction

· Most models of physical systems involve particles which interact ‘locally’, inducing long-range interactions.

· We take our particles to be points, usually the realization of an i.i.d. collection of r.v. $X_i, i \geq 1$ or a homogeneous Poisson point process $\mathcal{P}_1$.

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· We take our particles to be points, usually the realization of an i.i.d. collection of r.v. $X_i, i \geq 1$ or a homogeneous Poisson point process $\mathcal{P}_1$.

· For ease of exposition, we consider $Q_n := [-\frac{1}{2}n, \frac{1}{2}n]^d$ and let $U_i, i \leq n$ be i.i.d. uniform on Q_n.

· We let H be a generic functional defined on finite point sets.

· We are interested in the behavior of the **Poisson** functional $H(\mathcal{P}_1 \cap Q_n)$ and the **binomial** functional $H(U_1, ..., U_n)$.

I Introduction

Natural questions:

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2. (CLT) When do we have

$$\frac{H(U_1,...,U_n) - \mathbb{E}\, H(U_1,...,U_n)}{\sqrt{\mathrm{Var} H(U_1,...,U_n)}} \xrightarrow{\mathcal{D}} N(0,1)?$$

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3. (Probability bounds) Seek good bounds for

$$\mathbb{P}(H(U_1,...,U_n) \geq t).$$

a. **Packing Model**. Unit volume balls $B_1, ..., B_k$ arrive sequentially and uniformly at random in the cube $Q_n := [-\frac{1}{2}n, \frac{1}{2}n]^d$. Packing rules:

· Pack ball B_1.

II Models and Results

a. **Packing Model**. Unit volume balls $B_1, ..., B_k$ arrive sequentially and uniformly at random in the cube $Q_n := [-\frac{1}{2}n, \frac{1}{2}n]^d$. Packing rules:

· Pack ball B_1.

· Pack ball $B_i, \ i > 1$, if B_i does not overlap any ball in $B_1, B_2, ..., B_{i-1}$ which has already been packed.

· Picture for $d = 1$ looks like this:

· **a. Packing Model (contd)**. Fix $k \in \{1, 2, ..., \infty\}$. If balls $B_1, ..., B_k$ have centers at points $U_1, ..., U_k \in Q_n$ with respective arrival times $\tau_1, ..., \tau_k$, then let

$$H_n(U_1, ..., U_k)$$

denote the number of packed (accepted) balls on the substrate Q_n.

· The random variable $H_n(U_1, ..., U_k)$ is the number of accepted particles in the random sequential adsorption (RSA) model. $H_n(U_1, ..., U_\infty)$ is 'packing number'.

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Rényi's Thm: $d = 1 \Rightarrow$

$$\lim_{n\to\infty} \frac{\mathbb{E}\, H_n(U_1, ..., U_\infty)}{n} = \int_0^\infty \exp(-2 \int_0^t \frac{1 - e^{-u}}{u} du) dt \sim 0.748.$$

Dvoretsky + Robbins CLT: $d = 1 \Rightarrow$

$$\frac{H_n(U_1, ..., U_\infty) - \mathbb{E}\, H_n(U_1, ..., U_\infty)}{\sqrt{\mathrm{Var} H_n(U_1, ..., U_\infty)}} \xrightarrow{\mathcal{D}} N(0, 1).$$

· **a. Packing Model (contd)** What about the case $d \geq 2$? Hurdles: lack of subadditivity, lack of independence over subsets of cube Q_n.

LLN: $d \geq 2, \alpha \in (0, \infty)$:

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b. Geometric Graph (Gilbert Graph)

Def. Given a finite point set $\mathcal{X}$, $r \in (0, \infty)$, put

$$C_B(\mathcal{X}, r) := \bigcup_{x \in \mathcal{X}} B_r(x).$$

When $\mathcal{X}$ is PPP we get the Boolean model. It gives rise to the geometric graph $GG_r(\mathcal{X})$: join two points x and y with an edge iff $B_{r/2}(x) \cap B_{r/2}(y) \neq \emptyset$.

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Def. Let $H(\mathcal{X})$ be the number of edges in geometric graph $GG_r(\mathcal{X})$.

CLT: $d \geq 2, r > 0$:

$$\frac{H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)}{\sqrt{\mathrm{Var} H(\mathcal{P}_1 \cap Q_n)}} \xrightarrow{\mathcal{D}} N(0, 1).$$

c. Spatial Birth Growth Models

· The model: cells form at random locations $U_1, ..., U_k \in Q_n$ at times $\tau_1, ..., \tau_k$, respectively.

· Initially the new cell around U_i takes the form of a ball of radius $R_i \geq 0$ centered at U_i; then the cell grows radially in all directions with constant speed v.

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· New cells form only in the *uncovered space* in $\mathbb{R}^d$.

· This models crystal growth, cavitation.

· $H(U_1, ..., U_n)$ is the volume of the region covered by the first n cells.

c. Spatial Birth Growth Models (contd)

· $H(U_1, ..., U_n)$ is the volume of the region covered by the first n cells.

CLT: $\frac{H(U_1,...,U_n) - \mathbb{E}\, H(U_1,...,U_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$

Chiu, Quine,...

III Key Ideas

Recall:

H: a function defined on point set of $\mathbb{R}^d$.

$Q_n := [-\frac{1}{2}n, \frac{1}{2}n]^d$

$U_i, 1 \leq i \leq n$, i.i.d. uniform on Q_n.

Goal: Seek conditions on H yielding

$$\frac{H(U_1, ..., U_n)}{n} \xrightarrow{P} \text{constant} \quad (LLN)$$

and

$$\frac{H(U_1, ..., U_n) - \mathbb{E}\, H(U_1, ..., U_n)}{\sqrt{\mathrm{Var} H(U_1, ..., U_n)}} \xrightarrow{\mathcal{D}} N(0,1) \quad (CLT)$$

III Key Ideas

- Write $Q_n := \cup_{i=1}^n Q_{n,i}$, where $Q_{n,i}$ are disjoint sub-cubes of volume 1.
- Abbreviate $\{U_1, ..., U_n\}$ by $\mathcal{U}_n$.
- In general

$$H(\mathcal{U}_n) \neq \sum_{i=1}^{n} H(\mathcal{U}_n \cap Q_{n,i}),$$

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i.e., H is NOT additive.

· If H were additive, then we could deduce LLN and CLT for $H(\mathcal{U}_n)$ from the classical limit theorems.

III Key Ideas

· Restriction of H to disjoint sets does not give independence

· $H(\mathcal{U}_n \cap A)$ and $H(\mathcal{U}_n \cap B)$ are dependent in general!

III Key Ideas

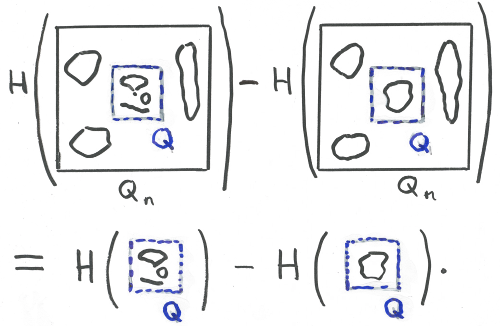
A key idea is to compare the functional $H(\mathcal{U}_n)$ with $H(\mathcal{U}_n \cup \{\mathbf{0}\})$.

Let's start by comparing $H(\mathcal{P}_1 \cap Q_n)$ with $H((\mathcal{P}_1 \cap Q_n) \cup \{\mathbf{0}\})$.

In other words, what happens to H when we insert an extra point at the origin into homogenous rate 1 Poisson input $\mathcal{P}_1$? How does H change?

III Key Ideas

Consider this situation:



This picture says that for the fixed deterministic square Q we have

$$H((\mathcal{P}_1 \cap Q_n) \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q_n)$$

$$= H((\mathcal{P}_1 \cap Q) \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q).$$

III Key Ideas: Stability of Difference Operators

The above phenomena rarely happens. Fortunately, a slightly weaker one does. Let $\mathcal{P}_1$ be unit intensity PPP on $\mathbb{R}^d$.

Def. Say that H **stabilizes** on $\mathcal{P}_1$ if there is a cube Q, $\mathrm{diam}(Q) < \infty$ a.s., such that

$$\lim_{n \to \infty} H((\mathcal{P}_1 \cap Q_n) \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q_n)$$

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$$= H((\mathcal{P}_1 \cap Q) \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q).$$

This condition says that the 'add-one cost' does not propagate far; it is confined to Q.

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Def. $D_{\mathbf{0}}H(\mathcal{P}_1) := H(\mathcal{P}_1 \cup \{\mathbf{0}\}) - H(\mathcal{P}_1)$.

$D_{\mathbf{0}}$ is called the **first order difference operator**.

Stabilization says that the first order difference operator has a behavior which is determined by local data.

III Key Ideas: Stability of Difference Operators

Which functionals H stabilize in the above sense?

Consider the nearest neighbors graph (put an edge between every point and its nearest neighbor). Let $H(\mathcal{X})$ be the total edge length of the nearest neighbor graph on $\mathcal{X}$.

IV General CLT and Variance Asymptotics

Theorem. Let H be a functional on points in $\mathbb{R}^d$ which satisfies:

(i) translation invariance, i.e., $H(\mathcal{X} + y) = H(\mathcal{X}),\ y \in \mathbb{R}^d$,

(ii) $(2+\epsilon)$ bounded increments:

$$\sup_{Q \subset \mathbb{R}^d,\ Q \text{ a cube}} \mathbb{E}\,|H((\mathcal{P}_1 \cap Q) \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q)|^{2+\epsilon} < \infty,$$

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Then

$$\frac{H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\,H(\mathcal{P}_1 \cap Q_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

$$\frac{\mathrm{Var} H(\mathcal{P}_1 \cap Q_n)}{n} \to \sigma^2.$$

Applications

We can show that the following functionals stabilize and satisfy the CLT and variance asymptotics:

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a. The number of balls accepted in the RSA packing model,

b. The volume of the occupied region in spatial birth-growth models,

c. The number of edges in the random geometric graph with parameter r,

d. Total edge length of nearest neighbors graph.

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Limitations:

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· we are unable to show that L_{TSP} stabilizes. Same for L_{MM}, L_{SMST}.

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$$\frac{\mathrm{Var} H(\mathcal{P}_1 \cap Q_n)}{n} \to \sigma^2.$$

If the add-one cost $D_{\mathbf{0}} H(\mathcal{P}_1) := H(\mathcal{P}_1 \cup \{\mathbf{0}\}) - H(\mathcal{P}_1)$ is non-degenerate, then $\sigma^2 > 0$.

IV General CLT and Variance Asymptotics

Proof of CLT. The proof involves these ingredients:

1. A stability lemma for stabilizing functionals,

2. Expressing H as a sum of martingale differences,

3. McLeish CLT (this result says that to prove a CLT it is enough to prove an L^1 WLLN).

IV General CLT and Variance Asymptotics

If H is stabilizing and trans. invariant, then there is a r.v. Δ_0 such that

$$\lim_{n\to\infty}[H((\mathcal{P}_1 \cap Q_n) \cup \mathbf{0}) - H(P_1 \cap Q_n)] = \mathbf{\Delta_0} \quad a.s.$$

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Thus, if H is stabilizing and trans. invariant, then $\forall x, y \in \mathbb{R}^d$

$$H((\mathcal{P}_1 \cap Q_n) \cup x) - H(\mathcal{P}_1 \cap Q_n) \to \Delta_x \quad a.s.$$

$$H((\mathcal{P}_1 \cap Q_n) \cup y) - H(\mathcal{P}_1 \cap Q_n) \to \Delta_y \quad a.s.$$

Consequently,

$$H((\mathcal{P}_1 \cap Q_n) \cup x) - H((\mathcal{P}_1 \cap Q_n) \cup y) \to \Delta_x - \Delta_y \quad a.s.$$

i.e, replacing x by y produces a change in H which converges a.s.

Stability lemma for stabilizing functionals.

Def. $\forall x \in \mathbb{Z}^d$ let $C(x)$ be unit volume cube with center x.

Def. $\mathcal{P}_1'$ an independent copy of $\mathcal{P}_1$. $\forall x \in \mathbb{Z}^d$ let

$$\mathcal{P}_{C(x)} := \mathcal{P}_1 \setminus (\mathcal{P}_1 \cap C(x)) \cup (\mathcal{P}_1' \cap C(x)).$$

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Stability Lemma. Let H be stabilizing and trans. invariant. Then $\forall x \in \mathbb{Z}^d$ there is a r.v. $\Delta_{C(x)}$ such that as $n \to \infty$

$$H(\mathcal{P}_1 \cap Q_n) - H(\mathcal{P}_{C(x)} \cap Q_n) \to \Delta_{C(x)}.$$

Martingale Differences

· $X_i, 1 \leq i \leq n$, independent r.v.

· $\mathcal{F}_i := \sigma(X_1, ..., X_i)$.

· Let $f : \mathbb{R}^n \to \mathbb{R}$.

· Then

$$f(X_1, ..., X_n) - \mathbb{E}\, f(X_1, ..., X_n) = \sum_{i=1}^{n} D_i,$$

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$$\begin{aligned} D_i &:= \mathbb{E}\,(f(X_1, ..., X_n)|\mathcal{F}_i) - \mathbb{E}\,(f(X_1, ..., X_n)|\mathcal{F}_{i-1}) \\ &= \mathbb{E}\,(f(X_1, ..., X_n)|\mathcal{F}_i) - \mathbb{E}\,(f(X_1, ..., X_i', ..., X_n)|\mathcal{F}_i). \end{aligned}$$

Here X_i' is a copy of X_i.

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Here X_i' is a copy of X_i.

Seek an analogous decomposition for $H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)$ in terms of martingale differences. But first:

IV General CLT and Variance Asymptotics

Thm (McLeish): Let $D_i, i \geq 1$, be martingale differences such that:

· $\sup_n \mathbb{E} \max_{i \leq n}(\frac{D_i^2}{n}) < \infty$,

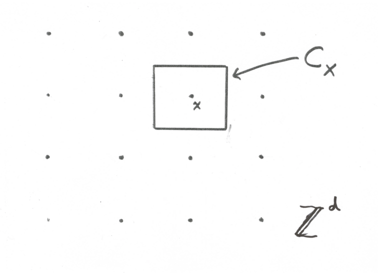
· $\frac{1}{\sqrt{n}} \max_{i \leq n} |D_i| \xrightarrow{P} 0$,

· $\frac{1}{n} \sum_{i=1}^n D_i^2 \xrightarrow{L^1} \sigma^2$.

Then as $n \to \infty$ we have

$$\frac{\sum_{i=1}^n D_i}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

Martingale differences for $H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)$



· $\mathbb{Z}^d$ ordered with dictionary ordering $\prec$.

· $\forall x \in \mathbb{Z}^d$ let $\mathcal{F}_x := \sigma(\mathcal{P}_1 \cap \cup_{y \preceq x} C_y)$.

Martingale differences for $H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)$:

· $Q_n := [-\frac{1}{2}n, \frac{1}{2}n]^d$

· Write $\mathbb{Z}^d \cap Q_n = \{x_1, x_2, ..., x_n\}, \quad x_{i-1} \prec x_i$.

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· Write $H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n) = \sum_{i=1}^n D_i$, with

$$\begin{aligned} D_i &:= \mathbb{E}\,(H(\mathcal{P}_1 \cap Q_n)|\mathcal{F}_{x_i}) - \mathbb{E}\,(H(\mathcal{P}_1 \cap Q_n)|\mathcal{F}_{x_{i-1}}) \\ &= \mathbb{E}\,(H(\mathcal{P}_1 \cap Q_n)|\mathcal{F}_{x_i}) - \mathbb{E}\,(H(\mathcal{P}_{C(x_i)} \cap Q_n)|\mathcal{F}_{x_i}) \\ &= \mathbb{E}\,([H(\mathcal{P}_1 \cap Q_n) - H(\mathcal{P}_{C(x_i)} \cap Q_n)] \mid \mathcal{F}_{x_i}) \end{aligned}$$

IV General CLT and Variance Asymptotics

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To apply the McLeish CLT we need to show WLLN in L^1 sense:

$$\frac{1}{n}\sum_{i=1}^n D_i^2 \xrightarrow{L^1} \sigma^2.$$

Proof that $\frac{1}{n}\sum_{i=1}^n D_i^2 \xrightarrow{L^1} \sigma^2.$ **(**)**
(i) $\forall x \in \mathbb{Z}^d \cap Q_n$, stability lemma implies for all $1 \leq i \leq n$:

$$[H(\mathcal{P}_1 \cap Q_n) - H(\mathcal{P}_{C(x_i)} \cap Q_n)] \to \Delta_{C(x_i)} \quad a.s.$$

$(2+\epsilon)$-bounded increments condition implies uniformly in $x_i \in \mathbb{Z}^d \cap Q_n$

$$D_{x_i}^2 := (\mathbb{E}([\ldots\ldots]|\mathcal{F}_{x_i}))^2 \xrightarrow{L^1} \mathbb{E}(\Delta_{C(x_i)}|\mathcal{F}_{x_i})^2.$$

Proof that $\frac{1}{n}\sum_{i=1}^n D_i^2 \xrightarrow{L^1} \sigma^2$. **(**)**
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(ii) Ergodic thm implies

$$\frac{1}{n}\sum_{x_i \in Q_n} \left(\mathbb{E}\left(\Delta_{C(x_i)}|\mathcal{F}_{x_i}\right)\right)^2 \xrightarrow{L^1} \sigma^2 := \mathbb{E}\left(\mathbb{E}\left(\Delta_{C(0)}|\mathcal{F}_0\right)\right)^2.$$

(iii) Combining (i) and (ii) we deduce the desired convergence (**).

Martingale differences for $H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n :)$

Thm (McLeish): Let $D_i, i \geq 1$, be martingale differences such that:

$\cdot$ $\sup_n \mathbb{E} \max_{i \leq n} (\frac{D_i^2}{n}) < \infty$, $\frac{1}{\sqrt{n}} \max_{i \leq n} |D_i| \xrightarrow{P} 0$, $\frac{1}{n} \sum_{i=1}^n D_i^2 \xrightarrow{L^1} \sigma^2$.

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Thm (CLT). Let H be a functional on points in $\mathbb{R}^d$ which satisfies:

(i) translation invariance, i.e., $H(\mathcal{X} + y) = H(\mathcal{X}),\ y \in \mathbb{R}^d$,

(ii) bounded increments: $\sup_Q \mathbb{E}\, |H(\mathcal{P}_1 \cap Q \cup \{\mathbf{0}\}) - H(\mathcal{P}_1 \cap Q)|^{2+\epsilon} < \infty$,

(iii) stability of 'add-one cost".

Then

$$\frac{H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

General CLT for $H(\mathcal{P}_1 \cap Q_n)$.

Under certain moment conditions on H we have

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{H(\mathcal{P}_1 \cap Q_n) - \mathbb{E}\, H(\mathcal{P}_1 \cap Q_n)}{\sqrt{\mathrm{Var} H(\mathcal{P}_1 \cap Q_n)}} \leq x \right) - \mathbb{P}(N(0,1) \leq x) \right| \leq \sum_{i=1}^{5} \tau_i.$$

The τ_i are integrals of sums and products of first order difference operators for H:

$$D_x H(\mathcal{P}_1) := H(\mathcal{P}_1 \cup x) - H(\mathcal{P}_1)$$

as well as second order difference operators

$$D^{(2)}_{x_1,x_2} H(\mathcal{P}_1) := D_{x_1} D_{x_2}(H(\mathcal{P}_1))$$

$$= H(\mathcal{P}_1 \cup \{x_1, x_2\}) - H(\mathcal{P}_1 \cup \{x_2\}) - H(\mathcal{P}_1 \cup x_1) + H(\mathcal{P}_1).$$

The results are an improvement because they provide rates of convergence. The unwieldy terms τ_i can only be simplified when the functional H is a stabilizing functional. We discuss this in the next lecture.

Example. $X_i, i \geq 1$, i.i.d. $N(0,1)$. $S(n) = \sum_{i=1}^{n} X_i$; $\frac{S(n)}{\sqrt{n}} \stackrel{\mathcal{D}}{=} N(0,1)$.

Thus for all Borel $A \subset \mathbb{R}$ we have

$$\mathbb{P}(\frac{S(n)}{\sqrt{n}} \in A) = \frac{1}{\sqrt{2\pi}} \int_A \exp(-\frac{x^2}{2}) dx.$$

So

$$\mathbb{P}(|\frac{S(n)}{n}| \geq t) = \frac{2}{\sqrt{2\pi}} \int_{t\sqrt{n}} \exp(-\frac{x^2}{2}) dx \sim \sqrt{\frac{2}{\pi}} \frac{1}{t\sqrt{n}} \exp(-\frac{t^2 n}{2}),$$

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which gives as $n \to \infty$

$$\frac{\log \mathbb{P}(|\frac{S(n)}{n}| \geq t)}{n} \to \frac{-t^2}{2}.$$

· With small probability, $|\frac{S(n)}{n}|$ takes on relatively large values.

· $\frac{-t^2}{2}$ is the rate function.

V Large Deviation Principles

Cramer's LDP

· $X_i, i \geq 1$, i.i.d. $S(n) = \sum_{i=1}^{n} X_i$; $\mathbb{E}\, X_1 = \mu$.

· log m.g.f.

$$\Lambda(\lambda) := \log \mathbb{E}\, \exp(\lambda X_1) \geq \lambda\mu.$$

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· log m.g.f.

$$\Lambda(\lambda) := \log \mathbb{E}\, \exp(\lambda X_1) \geq \lambda\mu.$$

· $\Lambda(\lambda)$ is convex. Fenchel-Legendre transform (convex dual) of $\Lambda(\lambda)$ is

$$\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda(\lambda)) \geq 0.$$

· The convex dual is quadratic if X_1 is $N(\mu, \sigma^2)$.

Cramer's LDP

· Assume $\Lambda(\lambda) < \infty$. Then

(a) For all closed $F \subseteq \mathbb{R}$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{S(n)}{n} \in F) \leq - \inf_{x \in F} \Lambda^*(x).$$

(b) For all open $O \subseteq \mathbb{R}$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{S(n)}{n} \in O) \geq - \inf_{x \in O} \Lambda^*(x).$$

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The sequence $\frac{S(n)}{n}$ satisfies the LDP with rate function Λ^*.

· Can one show that statistics of geometric structures also satisfy a LDP?

LDP for the TSP

· Notation: $X(n) := L_{TSP}(\mathcal{P}_1 \cap Q_n)$.

· The limit of the log m.g.f. for the sequence $X(n), n \geq 1$, is

$$\Lambda(t) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \exp(tX(n)).$$

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- It is a fact that the limit exists.
- Convex dual of $\Lambda(t)$ is

$$\Lambda^*(x) := \sup_{t\in\mathbb{R}} (tx - \Lambda(t)).$$

- Now we can state an LDP for $X(n) := L_{TSP}(\mathcal{P}_1 \cap Q_n)$.

Donsker-Varadhan LDP for the TSP

· $X(n) := L_{TSP}(\mathcal{P}_1 \cap Q_n)$ and $\frac{X(n)}{n} \to \gamma_{TSP}$ $a.s.$ (Lecture 1)

· $\Lambda(t) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp(tX(n)) < \infty$ and

(a) For all closed $F \subseteq \mathbb{R}$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{X(n)}{n} \in F) \leq -\inf_{x\in F} \Lambda^*(x).$$

(b) For all open $O \subseteq \mathbb{R}$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{X(n)}{n} \in O) \geq -\inf_{x\in O} \Lambda^*(x).$$

The sequence $\frac{X(n)}{n}$ satisfies the LDP with rate function Λ^* and

$$\Lambda^*(x) = 0 \Leftrightarrow x = \gamma_{TSP} \text{ and } \Lambda^*(x) \leq C(x - \gamma_{TSP})^2.$$

See Seppalainen + Y.

LDP for the TSP
Remark.

$\frac{X(n)}{n}$ converges exponentially to γ_{TSP}, i.e., $\forall \epsilon > 0$ there is $M := M(\epsilon) > 0$ such that $\forall n$

$$\mathbb{P}(|\frac{X(n)}{n} - \gamma_{TSP}| > \epsilon) \leq \exp(-Mn).$$

Final thoughts: LDP for the TSP

How does one prove an LDP for TSP and other statistics of large geometric structures?

Key. The sequence $X(n), n \geq 1$, satisfies near additivity in the sense that $\forall \epsilon > 0$, $\forall C > 0$, there is n_0 such that for $n \geq n_0$ and for all integers m we have

$$\mathbb{P}\left(|X(nm^d) - \sum_{i=1}^{m^d} X_i(n)| \geq \epsilon nm^d \right) \leq \exp(-Cnm^d)$$

where $X_i(n)$ are i.i.d. copies of $X(n)$.

Near additivity says that except on a set with exponentially small probability, $X(nm^d)$ can be expressed as a sum of i.i.d. random variables with a suitably small error.

THANK YOU