Lecture 4: Statistics of Random Polytopes

Spring school at Darmstadt, 25 February-March 1, 2019

Probabilistic Analysis of Geometric Structures

· **Lecture 1**: Probabilistic analysis of Euclidean optimization problems

· **Lecture 2**: Central limit theorems for statistics of geometric structures

· **Lecture 3**: Limit theory for statistics of geometric structures via stabilizing score functions

· **Lecture 4**: Statistics of random polytopes

· **Lecture 5**: Rates of multivariate normal approximation for statistics of geometric structures

- **I Historical Remarks**
- **II Results**
  - Expectation asymptotics
  - Rates of normal approximation
  - Variance asymptotics
- **III Methods**
  - Scaling transform for points in unit ball
  - Scaling transform for Gaussian sample

I Historical remarks

$X_1, ..., X_n$ iid uniform points in $K \subset \mathbb{R}^2$.
K_n: convex hull of $X_1, ..., X_n$.
$f_0(K_n)$: number of vertices in K_n.
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Alikoski, Blaschke, Crofton, Dalla, Efron, Groemer, Herglotz, Larman, Schneider

$d \geq 3$:

$K = \square^3 \quad \mathbb{E}\, f_0(K_5) = \frac{212023}{43200} - \frac{\pi^2}{432}$ (Zinani)

$K = \Delta^3 \quad \mathbb{E}\, f_0(K_n) = \,?$ (Buchta, Reitzner)

Buchta and Reitzner (2001):

Theorem 3. *The expected volume $\mathscr{V}(n)$ of the convex hull of n random points chosen independently and uniformly from a tetrahedron of volume one is given by*

$$
\begin{aligned}
\mathscr{V}(n) = 1 - \frac{2}{n+1} - \frac{3(n-1)n}{4}\left[\frac{1}{(n+1)^3} + \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{(k+3)^3}\right] \\
- \frac{9(n-1)n}{2} \sum_{\substack{j_1+\cdots+j_5=n-2 \\ k_1+k_2+k_3=4 \\ j_1,\ldots,j_5,k_1,k_2,k_3 \geqq 0}} \binom{n-2}{j_1,\ldots,j_4}\binom{4}{k_1,k_2} 2^{k_2} 3^{j_2+j_3} \\
\times B(j_2+2j_3+3j_4+3j_5+k_2+2k_3+1, 3j_1+2j_2+j_3+2k_1+k_2+1) \\
\times B(n+1, j_5+k_3+1) B(2j_1+j_2+k_1+1, j_5+2) \\
\times {}_3F_2(j_5+k_3, n+1, 2j_1+j_2+k_1+1; j_5+k_3+n+2, 2j_1+j_2+j_5+k_1+3; 1) \\
+ 6(n-1)n \sum_{\substack{j_1+\cdots+j_5=n-2 \\ l_1+l_2=2 \\ l_3+l_4=2 \\ j_1,\ldots,j_5,l_1,l_2,l_3,l_4 \geqq 0}} \binom{n-2}{j_1,\ldots,j_4}\binom{2}{l_1}\binom{2}{l_3} 3^{j_2+j_3} \\
\times B(j_2+2j_3+3j_4+3j_5+l_2+l_4+3, 3j_1+2j_2+j_3+l_1+l_3+3) \\
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\times {}_3F_2(j_5+l_4+1, n+1, 2j_1+j_2+l_1+1; j_5+l_4+n+2, 2j_1+j_2+j_5+l_1+4; 1).
\end{aligned}
$$

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$\mathrm{Vol}(K_n) =$ volume of K_n.

Simple questions

1. Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n^K) \le \mathbb{E}\, f_0(K_{n+1}^K)?$$

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2. If $K \subseteq L$ do we have $\mathbb{E}\,\mathrm{Vol} K_n \leq \mathbb{E}\,\mathrm{Vol} L_n$?

   Answer: No (L. Rademacher, 2012).

Monotonicity

Bonnet, Grote, Temesvari, Thaele, Turchi, Vespi (2017) show:

Thm. $\mathbb{E}\, f_{d-1}(K_n) \leq \mathbb{E}\, f_{d-1}(K_{n+1})$ when K_n is the convex hull of n i.i.d. points where the underlying distribution is either the Gaussian distribution on $\mathbb{R}^d$, the uniform distribution on the sphere, or certain heavy-tailed distributions.

Kabluchko and Thaele (2018): the f vector for Gaussian polytope is monotone, i.e.,

$$\mathbb{E}\, f_k(K_n) \leq \mathbb{E}\, f_k(K_{n+1}), \quad k \in \{0, 1, ..., d-1\}$$

when K_n is the convex hull of n i.i.d. random variables with Gaussian distribution.

I Historical remarks: Connections with other fields

1. average complexity of algorithms for computing convex hull (computational geometry)
convex hull used to solve problems in pattern recognition, image processing

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4. approximation of convex sets by random polytopes

II Results: Expectation asymptotics

Difficult to derive explicit formula for statistics of convex hulls on finite number of i.i.d. points.

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The shape of ∂K determines the order of magnitude of $\mathbb{E}\, f_\ell(K_n)$.

II Results: Expectation asymptotics (d = 2)

Rényi and Sulanke (1963-64), X_i i.i.d. in K, ∂K smooth ($d = 2$):

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$$\lim_{n\to\infty} n^{-1/3}\mathbb{E}\, f_0(K_n) = e_{0,d}(\mathrm{Vol}K)^{-1/3}\int_{\partial K}\kappa(x)^{1/3}dx$$

$\kappa(x)$: Gaussian curvature at $x \in \partial K$ (product of principal curvatures)

II Results: Expectation asymptotics ($d \geq 2$)

Dichotomy between smooth K and K which are polytopes. Reitzner (2005):

- ∂K of class C^2, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathbb{E}\, f_\ell(K_n) = e_{\ell,d} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

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- K is a convex polytope, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} (\log n)^{-(d-1)} \mathbb{E}\, f_\ell(K_n) = e'_{\ell,d} \cdot \text{number of flags of K}.$$

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(flag is a maximal chain of faces, each a sub-face of the next in the chain)

II Results: Expectation asymptotics ($d \geq 2$)

- K_n is convex hull of n i.i.d. standard normal r.v. on $\mathbb{R}^d$:

$$\lim_{n\to\infty} (\log n)^{-(d-1)/2} \mathbb{E}\, f_\ell(K_n) = E_{\ell,d}.$$

II Results: Normal approximation

Let K have C^2 boundary or let K be convex polytope.

If the random variable Z_n is either $\mathrm{Vol}(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \le x \right] - \Phi(x) \right| \le c(K)\epsilon(n) = o(1).$$

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Lachièze-Rey, Schulte, Y (2019): if K has C^2 boundary, then $\epsilon(n) = \frac{1}{\sqrt{\mathrm{Var} Z_n}}$.

II Results: Normal approximation

CLT uses Stein method and requires lower bounds on variances.

- What is the precise order of growth of $\mathrm{Var}\mathrm{Vol}(K_n)$?
- What is the precise order of growth of $\mathrm{Var} f_\ell(K_n), \ell \in \{0, ..., d-1\}$?

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These questions arose 25 years ago.

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· Reitzner (2003): Efron-Stein implies sharp upper bounds on $\mathrm{Var} f_\ell(K_n)$.

II Results: Variance asymptotics

Question: But what are the precise variance asymptotics?

As is the case with expectations, the correct scaling depends on the geometry, as shown in the next slide.

II Results: Variance asymptotics

- ∂K of class C^3, $\mathrm{Vol} K = 1$, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathrm{Var} f_\ell(K_n) = \int_{\partial K} \kappa(x)^{1/(d+1)} dx \cdot V_{\ell,d}.$$

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· K is a simple polytope with N vertices, $\mathrm{Vol} K = 1$, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} (\log n)^{-(d-1)} \mathrm{Var} f_\ell(K_n) = N \cdot \nu_{\ell,d}.$$

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· K_n is Gaussian polytope:

$$\lim_{n\to\infty} (2\log n)^{-(d-1)/2} \mathrm{Var} f_\ell(K_n) = v_{\ell,d}.$$

· Calka, Schreiber and Y (2013), Calka and Y (2014,2015,2017)

II Results: Variance asymptotics

How to obtain these results from the general theory of stabilizing functionals?

Can the $f_\ell(K_n), \ell \in \{0, 1, ..., d-1\}$, functional be cast into the form of a sum of stabilizing score functions?

II Results: Scaling limit of convex hull boundaries

What is the scaling limit of the boundary of the convex hull?

Let's start with the case of i.i.d. points in the unit ball.

III Scaling limits of convex hulls in unit ball

· $\mathcal{X}_n$: i.i.d. point set in unit ball of cardinality n.

· Convex geometry: x_0 is extreme in $\mathcal{X}_n$ iff $B_{|x_0|/2}(\frac{x_0}{2})$ is not covered by

$$\bigcup_{x \in \mathcal{X}_n : x \neq x_0} B_{|x|/2}(\frac{x}{2}).$$

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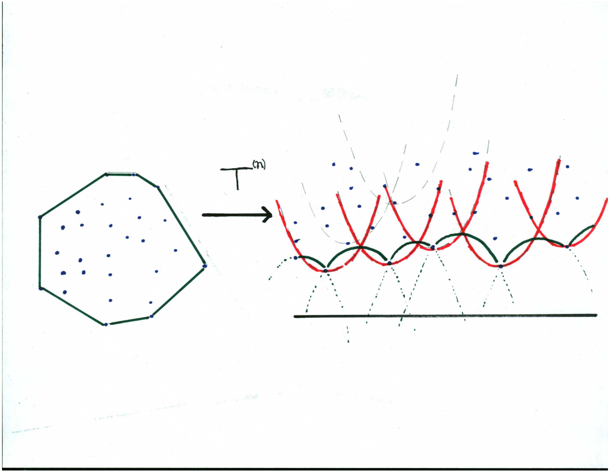
· Scaling limit should preserve this property. Boundary of $B_{|x|/2}(\frac{x}{2})$ is locally parabolic for $|x|$ close to 1: thus any reasonable scaling of the unit ball (into rectangular coordinates) should have the property that its scaling in radial direction should be square of scaling in angular direction.

III Scaling limits of convex hulls in unit ball

We claim that for an i.i.d. point set of size n, that there is a transform $T^{(n)}$ which does the job.

III Scaling limits of convex hulls in unit ball



Def. $w_0 \in T^{(n)}(\mathcal{X}_n)$ is extreme iff the up-paraboloid in $\mathbb{R}^{d-1} \times \mathbb{R}^+$ with apex at w_0 is not covered by the union of the up-paraboloids with apices at $T^{(n)}(\mathcal{X}_n) \setminus w_0$.

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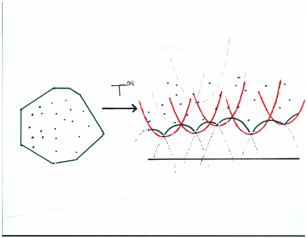
Properties of $T^{(n)}$:

(i) extreme points are preserved: x_0 is extreme in $\mathcal{X}_n$ iff $T^{(n)}(x_0)$ is extreme in $T^{(n)}(\mathcal{X}_n)$



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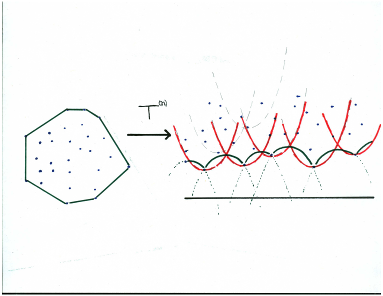

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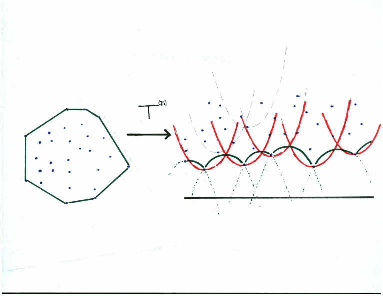

(ii) As $n \to \infty$: $T^{(n)}$ sends uniform samples of size n in B^d to a rate one PPP on $\mathbb{R}^{d-1} \times \mathbb{R}^+$

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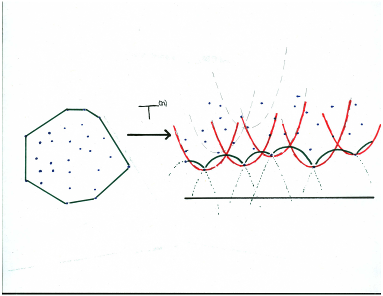

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(iii) $T^{(n)}$ sends extreme pts to extreme pts (blue); $T^{(n)}$ sends the boundary of convex hull into the inverted (green) festoon of paraboloids ($n \to \infty$).

III Scaling limits of convex hulls in unit ball

· Questions related to convex hull of point set $\mathcal{X}_n$ in unit ball are re-interpreted as questions about covering properties of paraboloids with apices at points in $T^{(n)}(\mathcal{X}_n)$.

III Scaling limits of convex hulls in unit ball

· Questions related to convex hull of point set $\mathcal{X}_n$ in unit ball are re-interpreted as questions about covering properties of paraboloids with apices at points in $T^{(n)}(\mathcal{X}_n)$.

· For example, the number of extreme points in convex hull of point set $\mathcal{X}_n$ may be written as a sum of score function on points in $T^{(n)}(\mathcal{X}_n)$:

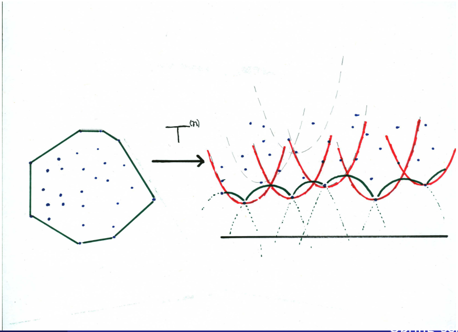
$$\sum_{x \in T^{(n)}(\mathcal{X}_n)} \xi(x, T^{(n)}(\mathcal{X}_n)).$$

· Here $\xi(x, T^{(n)}(\mathcal{X}_n))$ is zero or one according to whether the paraboloid with apex at x is covered by the union of remaining paraboloids.

Advantages to studying re-scaled picture

(i) spatial dependencies are easier to localize in re-scaled picture...

i.e., the parabolic geometry is easier to work with. Whether a paraboloid with apex at $(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ is covered by other paraboloids depends only on the paraboloid geometry inside a space-time cylinder (with axis through v) having a random radius R, but where R has exponentially decaying tails.



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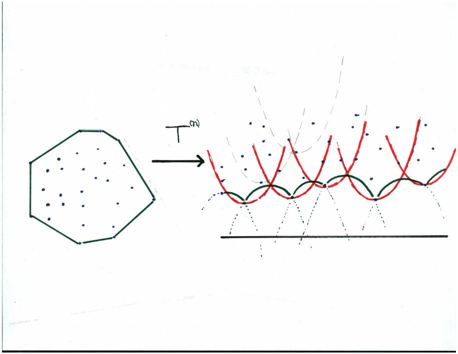

(ii) the space correlations decay exponentially fast wrt spatial distance. This leads to asymptotic independence and CLTs for e.g. the number of extreme points.

III Variance asymptotics in unit ball B^d

$\mathcal{H}$: rate one PPP in upper half-space.

$$\xi(x, \mathcal{H}) := \begin{cases} 1 \text{ if } x \text{ is extreme} \\ 0 \text{ otherwise.} \end{cases}$$

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$$\xi(x,\mathcal{H}) := \begin{cases} 1 \text{ if } x \text{ is extreme} \\ 0 \text{ otherwise.} \end{cases}$$

For all $w_1, w_2 \in \mathbb{R}^d$ put

$$c^{\xi}(w_1,w_2) :=$$

$$\mathbb{E}\,\xi(w_1,\mathcal{H}\cup\{w_2\})\xi(w_2,\mathcal{H}\cup\{w_1\}) - \mathbb{E}\,\xi(w_1,\mathcal{H})\mathbb{E}\,\xi(w_2,\mathcal{H})$$

and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E}\,\xi((\mathbf{0},h),\mathcal{H})dh$$

$$+\int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty} c^{\xi}((\mathbf{0},h),(v,h'))dh'dvdh.$$

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and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E}\, \xi((\mathbf{0}, h), \mathcal{H}) dh$$

$$+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{\xi}((\mathbf{0}, h), (v, h'))dh' dv dh.$$

Thm If K_n is the convex hull of n i.i.d. uniform points in unit ball, then

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathrm{Var} f_0(K_n) = c(d) V_{0,d},$$

where $c(d)$ is explicit constant depending on surface area of unit ball.

III Scaling limits of convex hulls: scaling transform $T^{(n)}$

· What is the transformation $T^{(n)} : B^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}^+$ which does the job? In $d = 2$ we require this transformation:

$$(r, \theta) \mapsto (n^{1/3}\theta, n^{2/3}(1 - r)).$$

· For $d > 2$: T_{u_0}: tangent space to $\mathbb{S}^{d-1}$ at $u_0 = (0, 0, ..., 1)$.

· *Exponential map* $\exp : T_{u_0} \to \mathbb{S}^{d-1}$ maps a vector $v \in T_{u_0}$ to the point $u \in \mathbb{S}^{d-1}$ lying at the end of the geodesic of length $|v|$ starting at u_0 and having direction v.

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· Scaling transform $T^{(n)} : B^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}^+$:

$$T^{(n)}(x) := \left(n^{1/(d+1)} \exp^{-1}(\frac{x}{|x|}), n^{2/(d+1)}(1 - |x|) \right), \; x \in B^d \setminus \{\mathbf{0}\}.$$

· $\exp^{-1}(\cdot)$: inverse exponential map.

Convex hulls for Gaussian input (Gaussian polytopes)

· $X_i, i \geq 1$ are i.i.d. with standard mean zero Gaussian distribution on $\mathbb{R}^d$, i.e., the common density is

$$\phi(x) = (2\pi)^{-d/2} \exp(-\frac{|x|^2}{2}), \; x \in \mathbb{R}^d.$$

III Convex hulls for Gaussian input (Gaussian polytopes)

- $R_n := \sqrt{2\log n - \log(2\cdot(2\pi)^d \cdot \log n)}$.
- Define scaling transform $T^{(n)} : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$

$$T^{(n)}(x) := \left(R_n \exp^{-1} \frac{x}{|x|},\; R_n^2 (1 - \frac{|x|}{R_n}) \right), \quad x \in \mathbb{R}^d.$$

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· K_n : convex hull of n i.i.d. Gaussian points in $\mathbb{R}^d$

· Calka, Y. (2015): $T^{(n)}, n \to \infty$, sends the Gaussian points to Poisson point process $\mathcal{P}$ on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $d\mathcal{P}((v,h)) = e^h dh dv$.

· Here is a picture.

III Convex hulls for Gaussian input (Gaussian polytopes)

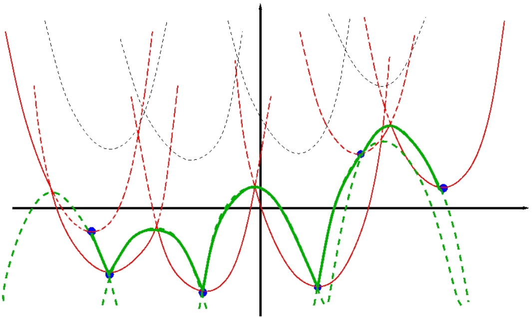

Figure: The blue point process is the image of the extreme points of the convex hull of Gaussian polytope; the green festoon is the scaling limit of the boundary of the convex hull.

· Calka, Y. (2015): The scaling limit of $T^{(n)}(\partial K_n), n \to \infty$, is the green festoon of parabolic surfaces touching points in Poisson point process $\mathcal{P}$ on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $d\mathcal{P}((v,h)) = e^h dh dv$.

III Convex hulls for Gaussian input (Gaussian polytopes)

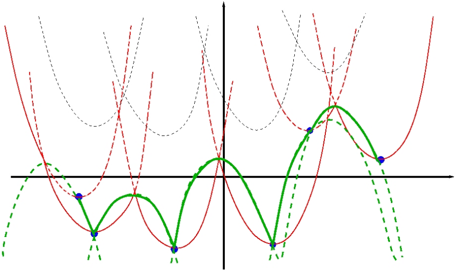

Figure: The blue point process is the image of the extreme points of the convex hull of Gaussian polytope; the red curve is the scaling limit of the germ grain model; the green festoon is the scaling limit of the boundary of the convex hull.

· Calka, Y. (2015): Whether a point in the transformed point set is extreme depends on the 'local' data. In fact the scores stabilize and in this way we prove variance asymptotics, as well as central limit theorems for the k face functional. $f_k(K_n) =$ number of k-faces of K_n, $k \in \{0, ..., d-1\}$.

III Scaling limits of convex hulls: scaling transform $T^{(n)}$

Julian Grote + Christoph Thaele (2017): use the scaling transform $T^{(n)}$ to establish sharp bounds on cumulants of certain statistics of convex hulls of i.i.d. gaussian samples. This leads to exponential estimates for large deviation probabilities of e.g. the number of k-dimensional faces of gaussian samples.

III Convex hulls for uniform input in a polytope

· Let K be a simple polytope (i.e., each vertex is adjacent to d facets).

· Let $\mathcal{X}_n$ denote n i.i.d. uniform random variables on K.

· For each vertex x of K we introduce a scaling transform $T^{(n)}$ and use it to transform the points $\mathcal{X}_n \cap B_r(x)$, i.e., the points in a neighborhood of x. $T^{(n)}$ dilates in the $(d-1)$ spatial directions by a factor of $\log n$.

· The picture looks like this:

III Convex hulls for uniform input in a polytope

· Let K be a simple polytope (i.e., each vertex is adjacent to d facets). WLOG let origin be a vertex.

· $T^{(n)}$ sends $K_n \cap B_r(\mathbf{0})$ to a festoon of inverted cone-like hyper-surfaces pinned to the extreme points of a point process in $\mathbb{R}^{d-1} \times \mathbb{R}$. In the limit as $n \to \infty$ the point process converges in distribution to a Poisson point process with intensity density

$$d\mathcal{P}((x,h)) = \sqrt{d}e^{dh}dhdx, \quad (x,h) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

· The extreme points of $K_n \cap B_r(\mathbf{0})$ converge in distribution to the extreme points of $\mathcal{P}$.

· re-scaled boundary $T^{(n)}((\partial K_n \cap B_r(\mathbf{0}))$ converges in probability to $\partial\Phi(\mathcal{P})$.

· Finally:

$$\lim_{\lambda\to\infty} \frac{\operatorname{Var} f_k(K_\lambda)}{(\log\lambda)^{d-1}} = F_{k,d} \cdot f_0(K).$$

THANK YOU