

# ASYMPTOTIC GEOMETRY OF RANDOM POLYTOPES

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ABSTRACT. This set of notes is based on the educational lecture given at the spring school on ‘Selected topics in stochastic geometry’ at TU Darmstadt. We give a very brief introduction to random polytopes and their geometric aspects within the framework of asymptotic geometric analysis. At the same time, we present some of the central tools in their study. The model we are looking at is as follows: we sample random points  $X_1, \dots, X_N$  independently and uniformly inside an isotropic convex body  $K \subseteq \mathbb{R}^n$  ( $N \geq n$ ) and consider their absolute convex hull  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$ . The latter is a random convex set in  $\mathbb{R}^n$ , called a random polytope, and the goal is to understand its typical asymptotic shape. Given the time constraints, we will focus on the expected mean width of a random polytope only and present optimal bounds up to absolute constants. In the regime  $N \geq n^{1+\delta}$  ( $\delta > 0$ ) these beautiful results are due to Dafnis, Giannopoulos, and Tsolomitis [14] (actually they considered the whole sequence of Minkowski quermassintegrals and obtained results with high-probability as well). When the number of points is linear in the dimension, that is, when  $N \approx n$ , the optimal bound was proved by Alonso-Gutiérrez and Prochno [5] (but we will not discuss this case here).

## 1. INTRODUCTION

In asymptotic geometric analysis the hyperplane conjecture is one of the outstanding open problems that first appeared explicitly in a work of Jean Bourgain [10] on high-dimensional maximal functions from 1986. It asks about the existence of an absolute constant  $c \in (0, \infty)$  such that every convex body of unit volume has a hyperplane section of volume bounded from below by  $c$ , independently of the space dimension. An equivalent formulation, following from a result of Doug Hensley [19], is that the isotropic constant of every convex body is bounded above by an absolute constant (a formal definition is provided below). Although the hyperplane conjecture has an affirmative answer for several classes of convex bodies such as unconditional convex bodies [10, 32], zonoids and duals of zonoids [8], bodies with a bounded outer volume ratio [32], or unit balls of Schatten  $p$ -classes [27], the general case still remains one of the central open problems in this area. The best general upper bound for the isotropic constant known up to now is due to Bo’az Klartag [24] and gives an upper bound of order  $\sqrt[4]{n}$  with  $n$  being the space dimension. This improves by a logarithmic factor the previous bound of Jean Bourgain [11].

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Let us now motivate why, from the point of view of asymptotic geometric analysis, random polytopes are interesting<sup>1</sup>. In 1988, Vitali Milman and Alain Pajor [32] discovered an interesting connection between the hyperplane conjecture and geometric properties associated with random polytopes. More precisely, they proved that the second moment of the volume of a random simplex in an isotropic convex body is closely related to the value of its isotropic constant (see also [12, Theorem 3.5.7]). Furthermore, since the pioneering work of Efim Gluskin [17], random polytopes are a major source for extremizers in high-dimensional geometric analysis. More precisely, Gluskin used random symmetric polytopes to show the existence of two  $n$ -dimensional normed spaces with Banach-Mazur distance greater than or equal to  $cn$ ,  $c \in (0, 1)$  some absolute constant<sup>2</sup> - we present a few more details in the appendix. So far it had only been known that, as a consequence of Fritz John's theorem, the Banach-Mazur distance is less than  $n$ , but no example of two spaces actually exhibiting this "maximal" behavior was available<sup>3</sup>. But Gluskin's probabilistic approach via symmetric random polytopes showed that if we close our eyes, and pick two spaces at random, then these spaces have, up to constant, maximal Banach-Mazur distance. As a consequence, they are natural candidates for a potential counterexample to the hyperplane or isotropic constant conjecture stated above. Following this philosophy, the isotropic constant has been studied for several classes of random polytopes in the last decade. More exactly, it has been shown in [2, 13, 20, 21, 25, 37] that the isotropic constant of the random polytopes is bounded by an absolute constant with probability tending to one, as the space dimension tends to infinity. The models studied so far are Gaussian random polytopes [25], random convex hulls of points from the Euclidean unit sphere [2], random polytopes that arise from uniform random points chosen in the interior of an isotropic convex body [13], random polyhedra associated with a parametric class of Poisson hyperplane tessellations [20], random polytopes spanned by points that are chosen from an  $\ell_p$ -sphere with respect to the cone probability measure [21], and a generalization of the latter to points chosen with respect to the cone measure from the boundary of an unconditional isotropic convex body [37]. Following along the lines of [25], it has recently been proved independently in [4] and [16] that if  $K_N$  is the symmetric convex hull of  $N \geq n$  independent random vectors uniformly distributed in the interior of an  $n$ -dimensional isotropic convex body  $K$ , then the isotropic constant  $L_{K_N}$  of  $K_N$  is bounded by a constant multiple of  $\sqrt{\log(2N/n)}$  with overwhelming probability (see also [3, 6, 22] for earlier results).

Having briefly discussed the importance of random polytopes in asymptotic geometric analysis, it is more than natural to ask the following question, which will at the same time be our point of departure:

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<sup>1</sup>There are various other ways to motivate the interest in random polytopes, for instance, coming from theoretical computer science. However, we will stay within the asymptotic geometric analysis framework.

<sup>2</sup>Note that each symmetric convex bodies  $K$  in  $\mathbb{R}^n$  is the unit ball of a normed space where the norm is given by  $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ . So each realization of a random symmetric polytope corresponds to an  $n$ -dimensional normed space.

<sup>3</sup>Note that even  $\ell_1^n$  and  $\ell_\infty^n$  are only at distance  $\sqrt{n}$  from one another.

“What is the typical asymptotic shape of a random polytope in an isotropic convex body?”

Given the time constraints and also not wanting to overload these notes with notions and machinery from convex geometry such as mixed volumes and Minkowski’s quermassintegrals, we will restrict ourselves to the expected value of the mean width only. This shall be enough to give an idea about key concepts that appear in the study of random polytopes.

To keep the reader motivated, let us state here the main result that we are going to present in a ‘qualitative’, not so precise version. The main result that we are going to present in this note is the following (reduced) version of [14, Theorem 1.1] obtained by Nikolaos Dafnis, Apostolos Giannopoulos, and Antonis Tsolomitis in 2009. We will write  $w(\cdot)$  for the mean width and  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  for the random polytope with points  $X_1, \dots, X_N$  chosen independently and uniformly at random in  $K$ . We refer to Section 2 below for any undefined notion or notation and to Section 4 (more precisely Theorem 3.1) for the exact statement of the result.

**Theorem 1.1.** *Let  $n \lesssim N \leq \exp(\sqrt{n})$  and  $K \subseteq \mathbb{R}^n$  be a convex body (in isotropic position). Then the expected average width of a random polytope that is given as the symmetric convex hull of  $N$  points that are chosen independently and uniformly at random from  $K$  is about  $\sqrt{\log N} L_K$ , where  $L_K \in (0, \infty)$  is a constant that depends on the body  $K$ .*

This shows that if we increase the number of points we take in the body  $K$ , then the random polytope becomes wider on a logarithmic scale (on average).

The following notions (as you can guess already in parts) will play a crucial rôle:

- Isotropic convex body,
- Support function,
- Mean width,
- $L_q$ -centroid body.

Having said that let us start with the background material from the theory of asymptotic geometric analysis needed to answer the question above.

Let me already apologize here for any omissions with regard to references and important mathematical contributions by various other people. Those are of course not on purpose.

## 2. THE FUNDAMENTAL CONCEPTS

We present here the background from asymptotic geometric analysis needed to define our model of random polytopes and study their typical shape. For more details and a beautiful exposition about the state-of-the-art as well as the history of this area, we refer to the recent monographs [7] and [12].

**2.1. Isotropic convex bodies.** We will work in  $\mathbb{R}^n$  equipped with the standard Euclidean structure  $\langle \cdot, \cdot \rangle$ . A convex body  $K$  in  $\mathbb{R}^n$  is a compact and convex subset of  $\mathbb{R}^n$  that has non-empty interior. We denote the  $n$ -dimensional Lebesgue measure of a convex body  $K \subseteq \mathbb{R}^n$  by  $\text{vol}_n(K)$ . We

will write  $\mathbb{B}_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$  for the Euclidean ball and denote the corresponding unit sphere by  $\mathbb{S}^{n-1}$ . Later we use the fact (which follows from Stirling's formula) that  $\text{vol}_n(\mathbb{B}_2^n)^{1/n} \approx \frac{1}{\sqrt{n}}$ . We will denote the Haar probability measure on the sphere by  $\sigma = \sigma_n$ . Note that in this case,  $\sigma$  is simply the cone probability measure, that is, for any Borel set  $B \subseteq \mathbb{S}^{n-1}$ ,

$$\sigma(B) = \frac{\text{vol}_n(\{\alpha x : x \in B, \alpha \in [0, 1]\})}{\text{vol}_n(\mathbb{B}_2^n)}.$$

This measure is also sometimes referred to as the spherical Lebesgue measure.

**Definition 2.1.** One says that a convex body  $K \subseteq \mathbb{R}^n$  is isotropic or in isotropic position if  $\text{vol}_n(K) = 1$ ,  $K$  has its barycenter at the origin, i.e., for all  $\theta \in \mathbb{S}^{n-1}$ ,

$$\int_K \langle x, \theta \rangle dx = 0,$$

and  $K$  satisfies the so-called isotropic condition, namely: for all  $\theta \in \mathbb{S}^{n-1}$ ,

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2.$$

The constant  $L_K \in (0, \infty)$  is independent of the direction  $\theta \in \mathbb{S}^{n-1}$  and is called the isotropic constant of  $K$ .

In probabilistic terms, the isotropic condition says that the variance of all 1-dimensional marginals is the same. So roughly speaking the isotropic constant  $L_K$  measures the spread of a convex body  $K$ . A typical example of an isotropic convex body you may want to think of is a volume normalized  $\ell_p$ -ball ( $1 \leq p \leq \infty$ ).

Let us now collect some important facts about the isotropic constant and the isotropic position. The proofs of these facts can be found, for instance, in the wonderful monograph [12].

**Lemma 2.2.** (a) *Every convex body  $K \subseteq \mathbb{R}^n$  can be brought into isotropic position, which is unique up to orthogonal transformations, via an affine transformation.*

(b) *The Euclidean ball in  $\mathbb{R}^n$  minimizes the isotropic constant, i.e.,  $L_K \geq L_{\mathbb{B}_2^n} \geq c$ , where  $c \in (0, \infty)$  is an absolute constant.*

(c) *For each isotropic convex body  $K \subseteq \mathbb{R}^n$ , we have the following bounds on its circumradius,*

$$\sqrt{n} L_K \leq R(K) := \max_{x \in K} \|x\|_2 \leq cn L_K,$$

where  $c \in (0, \infty)$  is an absolute constant.

*Proof.* (a) We can assume that  $K$  is centered. We need to show that there exists  $A \in GL(\mathbb{R}^n)$  such that  $A(K)$  is isotropic. We consider the linear operator

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad y \mapsto \int_K \langle x, y \rangle x dx.$$

It is not hard to see that this operator is symmetric and positive definite, which means that  $T$  has a symmetric and positive definite square root  $S$  (i.e.,  $S$  is a linear operator such that  $S^2 = T$ ). We now prove that the

body  $S^{-1}(K)$  satisfies the isotropic condition (which means, up to volume normalization, we found the isotropic position). For any  $\theta \in \mathbb{S}^{n-1}$ , we observe that

$$\begin{aligned} \int_{S^{-1}(K)} \langle x, \theta \rangle^2 dx &= |\det(S^{-1})| \int_K \langle S^{-1}x, \theta \rangle^2 dx \\ &= \frac{1}{|\det(S)|} \int_K \langle x, S^{-1}\theta \rangle^2 dx \\ &= \frac{1}{|\det(S)|} \left\langle \int_K \langle x, S^{-1}\theta \rangle x dx, S^{-1}\theta \right\rangle \\ &= \frac{1}{|\det(S)|} \langle TS^{-1}\theta, S^{-1}\theta \rangle \\ &= \frac{1}{|\det(S)|} \|\theta\|_2 = \frac{1}{|\det(S)|}, \end{aligned}$$

where we used that  $T = S^2$  with a symmetric operator  $S$ . Normalizing the volume of  $S^{-1}(K)$ , we obtain the result.

(b) Let  $n \in \mathbb{N}$  and  $r_n \in (0, \infty)$  such that  $\text{vol}_n(r_n \mathbb{B}_2^n) = 1$ . Note that  $r_n \mathbb{B}_2^n$  is then isotropic. Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then,

$$nL_K^2 = \int_K \|x\|_2^2 dx = \int_{K \cap r_n \mathbb{B}_2^n} \|x\|_2^2 dx + \int_{K \setminus r_n \mathbb{B}_2^n} \|x\|_2^2 dx,$$

and, similarly,

$$nL_{\mathbb{B}_2^n}^2 = \int_{\mathbb{B}_2^n} \|x\|_2^2 dx = \int_{r_n \mathbb{B}_2^n \cap K} \|x\|_2^2 dx + \int_{r_n \mathbb{B}_2^n \setminus K} \|x\|_2^2 dx.$$

Obviously,

$$\int_{K \cap r_n \mathbb{B}_2^n} \|x\|_2^2 dx = \int_{r_n \mathbb{B}_2^n \cap K} \|x\|_2^2 dx$$

and, since  $\text{vol}_n(K \setminus r_n \mathbb{B}_2^n) = \text{vol}_n(r_n \mathbb{B}_2^n \setminus K)^4$  and because  $\|\cdot\|_2 > r_n$  on  $K \setminus r_n \mathbb{B}_2^n$  and  $\|\cdot\|_2 \leq r_n$  on  $r_n \mathbb{B}_2^n \setminus K$ , we obtain

$$\int_{K \setminus r_n \mathbb{B}_2^n} \|x\|_2^2 dx \geq \int_{r_n \mathbb{B}_2^n \setminus K} \|x\|_2^2 dx.$$

Therefore,

$$nL_K^2 \geq nL_{\mathbb{B}_2^n}^2.$$

It is left to compute and estimate  $L_{\mathbb{B}_2^n}$ . Integrating in polar coordinates, we obtain

$$L_{\mathbb{B}_2^n}^2 = \frac{1}{n} \int_{r_n \mathbb{B}_2^n} \|x\|_2^2 dx = \frac{\text{vol}_n(\mathbb{B}_2^n)}{n+2} r_n^{n+2} = \frac{\text{vol}_n(\mathbb{B}_2^n)^{-2/n}}{n+2} \approx 1.$$

This completes the proof of part (b).

(c) In this part we present an argument due to Kannan, Lovász and Simonovits [23], which gives the more precise bound  $R(K) \leq (n+1)L_K$ . Consider  $x \in K$  and define a function  $g := g_x : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  by

$$g(\xi) = \max\{t \geq 0 : x + t\xi \in K\}.$$

<sup>4</sup>Of course,  $[K \cap r_n \mathbb{B}_2^n] \cup [K \setminus r_n \mathbb{B}_2^n] = 1 = [r_n \mathbb{B}_2^n \cap K] \cup [r_n \mathbb{B}_2^n \setminus K]$ , which, taking volumes and using that the sets are disjoint, immediately implies the equality.

This function measures the distance from  $x \in K$  to the boundary of  $K$  looking in direction  $\xi \in \mathbb{S}^{n-1}$ . Note that, since  $\text{vol}_n(K) = \text{vol}_n(K - x)$ , by integrating in polar coordinates, we obtain

$$\begin{aligned} 1 &= \text{vol}_n(K - x) = n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbb{1}_{K-x}(r\xi) r^{n-1} dr d\sigma(\xi) \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbb{1}_K(x + r\xi) r^{n-1} dr d\sigma(\xi) \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \int_0^{g(\xi)} r^{n-1} dr d\sigma(\xi) = \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} g(\xi)^n d\sigma(\xi), \end{aligned}$$

that is,

$$\int_{\mathbb{S}^{n-1}} g(\xi)^n d\sigma(\xi) = \frac{1}{\text{vol}_n(\mathbb{B}_2^n)}. \quad (2.1)$$

We will use this equality in a moment. Since  $K$  is isotropic, using a linear shift of  $K$  by  $x$ , we have that for any  $\theta \in \mathbb{S}^{n-1}$ , (we use again integration in polar coordinates)

$$\begin{aligned} L_K^2 &= \int_K \langle y, \theta \rangle^2 dy \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \int_0^{g(\xi)} \langle r\xi + x, \theta \rangle^2 r^{n-1} dr d\sigma(\xi) \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \int_0^{g(\xi)} r^{n+1} \langle \xi, \theta \rangle^2 + 2r^n \langle \xi, \theta \rangle \langle x, \theta \rangle + r^{n-1} \langle x, \theta \rangle^2 dr d\sigma(\xi) \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \frac{g(\xi)^{n+2}}{n+2} \langle \xi, \theta \rangle^2 + \frac{2g(\xi)^{n+1}}{n+1} \langle \xi, \theta \rangle \langle x, \theta \rangle + \frac{g(\xi)^n}{n} \langle x, \theta \rangle^2 d\sigma(\xi) \\ &= n \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} \left( \frac{g(\xi)^n}{n(n+1)^2} \langle x, \theta \rangle^2 + g(\xi)^n \left[ \frac{g(\xi) \langle \xi, \theta \rangle}{\sqrt{n+2}} + \frac{\sqrt{n+2} \langle x, \theta \rangle}{n+1} \right]^2 \right) d\sigma(\xi) \\ &\geq \frac{\langle x, \theta \rangle^2}{(n+1)^2} \text{vol}_n(\mathbb{B}_2^n) \int_{\mathbb{S}^{n-1}} g(\xi)^n d\sigma(\xi) = \frac{\langle x, \theta \rangle^2}{(n+1)^2}, \end{aligned}$$

where the last equality follows from (2.1). But this shows that, for any  $x \in K$  and every  $\theta \in \mathbb{S}^{n-1}$ ,

$$\langle x, \theta \rangle \leq (n+1)L_K,$$

which means that, for any  $x \in K$ ,

$$\|x\|_2 = \sup_{\theta \in \mathbb{S}^{n-1}} |\langle x, \theta \rangle| \leq (n+1)L_K.$$

Now the lower bound. Since  $K$  is isotropic, the isotropic condition holds for any  $\theta \in \mathbb{S}^{n-1}$ . In particular, we may pick the standard unit vectors  $e_1, \dots, e_n$ . This means that for each  $1 \leq i \leq n$ ,

$$L_K^2 = \int_K x_i^2 dx.$$

Taking the sum, we obtain

$$nL_K^2 = \int_K \sum_{i=1}^n x_i^2 dx = \int_K \|x\|_2^2 dx \leq \max_{x \in K} \|x\|_2^2 = R(K)^2,$$

which shows the lower bound.  $\square$

We have just seen that the isotropic constant is bounded from below by some absolute constant. As already mentioned in the introduction, there is a famous conjecture regarding a universal upper bound on the isotropic constant, which first appeared in the work [10] of Bourgain.

**Conjecture 2.3** (Isotropic constant conjecture). *There exists an absolute constant  $C \in (0, \infty)$  such that for every  $n \in \mathbb{N}$  and every convex body  $K \subseteq \mathbb{R}^n$ ,*

$$L_K \leq C.$$

Using a result of Hensley [19], which shows that for any  $n \in \mathbb{N}$ , every convex body  $K \subseteq \mathbb{R}^n$  and any direction  $\theta \in \mathbb{S}^{n-1}$ ,

$$c_1 \frac{1}{L_K} \leq \text{vol}_n(K \cap \theta^\perp) \leq c_2 \frac{1}{L_K},$$

with  $c_1, c_2 \in (0, \infty)$  being absolute constants, one can show that this conjecture is in fact equivalent to the famous hyperplane conjecture (see, e.g., [12, pp. 107-108])<sup>5</sup>.

**Conjecture 2.4** (Hyperplane conjecture). *There exists an absolute constant  $c \in (0, \infty)$  such that for any  $n \in \mathbb{N}$  and every centered convex body  $K \subseteq \mathbb{R}^n$  of volume 1, we can find a direction  $\theta \in \mathbb{S}^{n-1}$  such that*

$$\text{vol}_n(K \cap \theta^\perp) \geq c.$$

**Remark 2.5.** The hyperplane conjecture has an affirmative answer for several classes of convex bodies such as unconditional convex bodies [10, 32], zonoids and duals of zonoids [8], bodies with a bounded outer volume ratio [32], or unit balls of Schatten  $p$ -classes [27]. The best general upper bound for the isotropic constant known up to now is due to Klartag [24] and gives an upper bound of order  $\sqrt[4]{n}$  with  $n$  being the space dimension. This improves by a logarithmic factor the previous bound of Bourgain [11].

**2.2. Random polytopes & the model of interest.** After no further improvement or solution to the hyperplane conjecture appeared people started looking for counterexamples. As already mentioned in the introduction, random polytopes often exhibit a certain maximal behavior (see, for instance [17] and [15]). It is therefore natural to investigate the following:

“Do random polytopes provide us with a counterexample to the hyperplane conjecture?”

For the following models of randomness, it was proved that with high probability the isotropic constant is bounded by an absolute constant (note that all proofs follow the idea of [25]):

- Klartag, Kozma [25]: Random polytopes spanned by independent Gaussian random vectors, i.e.,  $K_N = \text{conv}\{\pm G_1, \dots, \pm G_N\}$  ( $N \geq n$ ), where  $G_i = (g_1^i, \dots, g_n^i)$  with independent standard Gaussians  $g_1^i, \dots, g_n^i$ .

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<sup>5</sup>By  $\theta^\perp$  we denote the hyperplane orthogonal to  $\theta$ .

- Alonso-Gutiérrez [2]: Random polytopes  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  ( $N \geq n$ ) where  $X_1, \dots, X_N$  are chosen independently at random from the sphere  $\mathbb{S}^{n-1}$  with respect to the normalized spherical Lebesgue measure (= cone probability measure).
- Dafnis, Giannopoulos, and Guédon [13]: Random polytopes  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  ( $N \geq n$ ) where  $X_1, \dots, X_N$  are chosen independently and uniformly at random in a 1-unconditional isotropic convex body.
- Hörrmann, Prochno, and Thäle [21]: Random polytopes  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  ( $N \geq n$ ) where  $X_1, \dots, X_N$  are chosen independently at random from an  $\ell_p$ -sphere,  $\mathbb{S}_p^{n-1} = \{x \in \mathbb{R}^n : \|x\|_p = 1\}$ , with respect to the cone probability measure on  $\mathbb{B}_p^n$  ( $1 \leq p < \infty$ ).
- Prochno, Thäle, and Turchi [37]: Random polytopes  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  ( $N \geq n$ ) where  $X_1, \dots, X_N$  are chosen independently at random from the boundary of an unconditional convex body  $K$  with respect to the cone probability measure.

A notable exception (and so far the only one), using a different approach to show that the isotropic constant is bounded with high probability is the following paper:

- Hörrmann, Hug, Reitzner and Thäle [20]: Random polyhedra associated with a parametric class of Poisson hyperplane tessellations.

Having thus ‘seen’ several results on random polytopes and their isotropic constant, we want to come back to the original question about the typical shape of a random polytope. Before specifying this question to the present text, let us describe the model of random polytopes we are going to investigate.

Given an isotropic convex body  $K \subseteq \mathbb{R}^n$  and  $N \geq n$ , we let  $X_1, \dots, X_N$  be independent random vectors that are uniformly distributed in  $K$ . This means that for any Borel set  $A \subset \mathbb{R}^n$ ,

$$\mathbb{P}(X_1 \in A) = \text{vol}_n(A \cap K).$$

The random polytopes in the focus of our attention are defined as

$$K_N := \text{conv}\{\pm X_1, \dots, \pm X_N\},$$

where  $\text{conv}$  denotes the convex hull of these points. We now ask the following specific question that we will answer in these notes:

**“What is the expected ‘mean width’ of a random polytope  $K_N$  inside an isotropic convex body  $K$ ?”**

The mean width will be defined in the next subsection.

**2.3. The mean width of a convex body.** In this subsection we will present the notions of support function and mean width of a convex body. We also collect, without proof, some elementary facts and an inequality due to Urysohn, that we will need later to prove the lower bound on the mean width of a random polytope.



**Definition 2.6** (Support function). Let  $K \subseteq \mathbb{R}^n$  be a convex body. We define its support function to be the function

$$h_K : \mathbb{S}^{n-1} \rightarrow \mathbb{R}, \quad \theta \mapsto \max_{x \in K} \langle x, \theta \rangle.$$

Of course, we might as well define the support function on  $\mathbb{R}^n$ , but for the sphere  $\mathbb{S}^{n-1}$  we have a nice geometric interpretation, namely,  $h_K(\theta)$  is the (signed) distance of the supporting hyperplane of  $K$  in direction  $\theta$  from the origin. It is indeed the case that each convex body is uniquely determined by its support function and can be written as the intersection of its supporting half spaces.<sup>6</sup>

Let us collect some facts on support functions in the following lemma. For proofs (at least in parts) and further information, we refer the reader to [7, Appendix A, pp. 380-381].

**Lemma 2.7.** *Let  $C, K \subseteq \mathbb{R}^n$  be convex bodies. Then,*

(a)  $h_K$  is positively homogeneous (of degree 1) and subadditive, i.e.,

$$h_K(\alpha x) = \alpha h_K(x) \quad \text{for all } \alpha \geq 0 \text{ and } x \in \mathbb{R}^n$$

and

$$h_K(x + y) \leq h_K(x) + h_K(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

(b)  $h_K$  is order preserving. This means that if  $C \subseteq K$ , then

$$h_C \leq h_K,$$

and that if  $h_C \leq h_K$ , then  $C \subseteq K$ .

(c) Any positively homogeneous and subadditive function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  is the support function of some unique convex body  $K$ .

(d) A convex body is symmetric with respect to the origin ( $K = -K$ ) if and only if  $h_K(\theta) = h_K(-\theta)$  for all  $\theta \in \mathbb{S}^{n-1}$ .

(e)  $0 \in K$  if and only if  $h_K \geq 0$ .

Having now the notion of a support function at hand and keeping its nice geometric interpretation in mind, we will now define the mean width of a convex body  $K \subseteq \mathbb{R}^n$ . Before we do so, let us define the width of  $K$  in direction  $\theta \in \mathbb{S}^{n-1}$  as

$$\text{Width}(K, \theta) := h_K(\theta) + h_K(-\theta).$$

This is simply the distance of the two supporting hyperplanes of  $K$ , the one in direction  $\theta$  and the other in direction of  $-\theta$ . Therefore, it is geometrically just the width of the body  $K$  in direction of  $\theta$  (or equivalently  $-\theta$ ). The mean width will now be defined as the average width over the sphere. Please note that we actually define  $\frac{1}{2}$  times the mean width.

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<sup>6</sup>For a unit vector  $\theta \in \mathbb{S}^{n-1}$  the hyperplane orthogonal to  $\theta$  and at distance  $\gamma$  from the origin is given by  $H_\theta^\gamma := \{x \in \mathbb{R}^n : \langle x, \theta \rangle = \gamma\}$ . It divides the space in two halves, the half spaces. In our case,  $H_\theta^{h_K(\theta)}$  is such that  $K$  is contained in the half space  $\{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq h_K(\theta)\}$  and there is at least one point on the boundary of  $K$  which lies in  $H_\theta^{h_K(\theta)}$ . This explains the name supporting hyperplane for  $H_\theta^{h_K(\theta)}$ .

**Definition 2.8** (Mean width). Let  $K \subseteq \mathbb{R}^n$  be a convex body. We define the mean width of  $K$ , denoted by  $w(K)$ , to be

$$w(K) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \text{Width}(K, \theta) \, d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} h_K(\theta) \, d\sigma(\theta).$$

The mean width thus tells us how wide the body is on average. An important inequality, that can be proved via the so-called Steiner symmetrization method (see, e.g., [7, Theorem 1.5.11]), is Urysohn's inequality. We will make use of it to prove the lower bound on the mean width of our random polytopes.

**Lemma 2.9** (Urysohn's inequality). *Let  $K \subseteq \mathbb{R}^n$  be a convex body. Then*

$$w(K) \geq \left( \frac{\text{vol}_n(K)}{\text{vol}_n(\mathbb{B}_2^n)} \right)^{1/n}.$$

We will now come to an important notion in convex geometry, the so-called  $L_q$ -centroid bodies.

**2.4.  $L_q$ -centroid bodies.** The notion of an  $L_q$ -centroid body (although under a different normalization) was introduced by Erwin Lutwak and Gaoyong Zhang in [30]. Their study from an asymptotic point of view was initiated by Grigoris Paouris in his Ph.D. thesis resulting in the works [34, 33]. The asymptotic point of view on this class of bodies turned out to be an integral part in several other deep and wonderful discoveries of the last decade (see, e.g., [35]). Let us also refer to the important contributions by Emanuel Milman [31] and Klartag and Milman [26].

We now start rather bluntly with their definition and then try to build some intuition.

**Definition 2.10** ( $L_q$ -centroid body). Let  $K \subseteq \mathbb{R}^n$  be a convex body with  $\text{vol}_n(K) = 1$  and  $q \geq 1$ . Then the  $L_q$ -centroid body of  $K$ ,  $Z_q(K)$ , is defined via its support function

$$h_{Z_q(K)}(\theta) := \left( \int_K |\langle x, \theta \rangle|^q \, dx \right)^{1/q}, \quad \theta \in \mathbb{S}^{n-1}.$$

In the case that  $q = +\infty$ , we define the  $L_\infty$ -centroid body to be  $Z_\infty(K) = \text{conv}\{K, -K\}$ .

To allow a better understanding we provide a remark that highlights certain aspects of  $L_q$ -centroid bodies.

**Remark 2.11.** (a) Note that the function  $h_{Z_q(K)}$  is positively 1-homogeneous and, by Minkowski's inequality, subadditive. Therefore, by Lemma 2.7, there exists indeed a unique convex body having  $h_{Z_q(K)}$  as support function.

(b) Note that, again by Lemma 2.7, an  $L_q$ -centroid body is always symmetric with respect to the origin.

(c) A convex body  $K \subseteq \mathbb{R}^n$  is isotropic if and only if it is centered and  $Z_2(K) = L_K \mathbb{B}_2^n$ .

(d) When  $q = 1$ , then there is a neat interpretation of the centroid body of  $K$  which 'explains' the name and goes as follows: assume that  $K$  is symmetric and consider a direction  $\theta \in \mathbb{S}^{n-1}$  and the hyperplane orthogonal to

it. This hyperplane divides  $K$  into two pieces. Take one of the two halves and compute the centroid of it. This is what gives us the support function in the direction of  $\theta$ . Now we do this for every direction  $\theta \in \mathbb{S}^{n-1}$  and we obtain the centroid body.

(e) The  $L_q$ -centroid bodies can be compared with the floating bodies of a convex body, which have a really nice interpretation (we omit the details here). More precisely, we have the following relation (as shown by Paouris and Werner in [36]): there exist absolute constants  $c_1, c_2 \in (0, \infty)$  such that for any  $\delta \in (0, 1)$ ,

$$c_1 Z_{\log \frac{1}{\delta}}(K) \subseteq K_\delta \subseteq c_2 Z_{\log \frac{1}{\delta}}(K).$$

Let us collect some (comparably) simple properties of  $L_q$ -centroid bodies.

**Lemma 2.12.** *Let  $K \subseteq \mathbb{R}^n$  be a convex body with  $\text{vol}_n(K) = 1$ . Then the following hold:*

(a) *If  $K$  is isotropic, then  $w(Z_2(K)) = L_K$ .*

(b) *For all  $1 \leq p < q \leq \infty$ ,*

$$Z_p(K) \subseteq Z_q(K) \subseteq C \frac{q}{p} Z_p(K),$$

where  $C \in (0, \infty)$  is an absolute constant.

*Proof.* (a) We have

$$\begin{aligned} w(Z_2(K)) &= \int_{\mathbb{S}^{n-1}} h_{Z_2(K)}(\theta) \, d\sigma(\theta) \\ &= \int_{\mathbb{S}^{n-1}} \left( \int_K |\langle x, \theta \rangle|^2 \, dx \right)^{1/2} \, d\sigma(\theta) = L_K, \end{aligned}$$

where the latter equality follows from the isotropic condition. In fact, we could have also just used Remark 2.11 (c).

(b) The left inclusion follows directly from Hölder's inequality, meaning that for  $p \leq q$ ,

$$\|\langle \cdot, \theta \rangle\|_{L_p(K)} \leq \|\langle \cdot, \theta \rangle\|_{L_q(K)}. \quad (2.2)$$

The other inclusion follows from Borell's lemma ([12, Lemma 2.4.5]) and holds actually more generally for seminorms  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and not only the marginals  $\langle \cdot, \theta \rangle$ . A proof can be found, for instance, in [12, Theorem 2.4.6]. One obtains the estimate

$$\|\langle \cdot, \theta \rangle\|_{L_q(K)} \leq C \frac{q}{p} \|\langle \cdot, \theta \rangle\|_{L_p(K)}, \quad (2.3)$$

which immediately yields the right inclusion.  $\square$

**Remark 2.13.** (a) Assume that  $K \subseteq \mathbb{R}^n$  is symmetric. Then part (d) of Remark 2.11 shows that when  $q = 1$ , the centroid body  $Z_1(K)$  sits well inside  $K$ . However, part (b) in Lemma 2.12 shows that as  $q \rightarrow +\infty$  the  $L_q$ -centroid body increases and ultimately, when  $q = +\infty$ , coincides with  $K$  itself (since in the symmetric case  $Z_\infty(K) = K$ ).

(b) If  $K$  is isotropic, then looking at the proof of Lemma 2.12 (b) (and more precisely at (2.2) and (2.3)), we see that by choosing  $p = 2$ ,

$$L_K \leq \left( \int_K |\langle x, \theta \rangle|^q \, dx \right)^{1/q} \leq Cq L_K.$$

There are two very important and frequently used facts about  $L_q$ -centroid bodies and their volume radius and mean width, respectively. They follow from the works of Paouris [35] and Klartag and Milman [26] and are actually quite deep results. We inappropriately just state them in one lemma, but a proof or detailed discussion is simply beyond the scope of this introductory note.

**Lemma 2.14.** *Let  $K \subseteq \mathbb{R}^n$  be an isotropic convex body. Then we have the following:*

(a) *If  $1 \leq q \leq \sqrt{n}$ , then*

$$\text{vol}_n(Z_q(K))^{1/n} \approx \sqrt{\frac{q}{n}} L_K.$$

(b) *If  $1 \leq q \leq \sqrt{n}$ , then*

$$w(Z_q(K)) \approx \sqrt{q} L_K.$$

**Remark 2.15.** The lower bound on the volume of an  $L_q$ -centroid body, without the isotropic constant, was obtained by Erwin Lutwak, Deane Yang and Gaoyong Zhang [29]. The work [26] allowed to include the isotropic constant  $L_K$  in that lower bound. Paouris obtained the upper bound in [35] (see also [12, Theorem 5.1.17]). The estimates for the mean width follow from Paouris' works [34, 33, 35].

### 3. THE SHAPE OF A RANDOM POLYTOPE - MAIN RESULT

We are now prepared to state properly the main result that we are going to present and 'prove' in this note. It is actually a (reduced) version of the result [14, Theorem 1.1] obtained by Nikolaos Dafnis, Apostolos Giannopoulos and Antonis Tsolomitis in 2009.

**Theorem 3.1** (Dafnis, Giannopoulos, Tsolomitis). *Let  $n, N \in \mathbb{N}$  and  $K \subseteq \mathbb{R}^n$  be an isotropic convex body. If  $cn \leq N \leq \exp(\sqrt{n})$ , then*

$$c_1 \sqrt{\log \frac{N}{n}} L_K \leq \mathbb{E} w(K_N) \leq c_2 \sqrt{\log N} L_K,$$

where  $c, c_1, c_2 \in (0, \infty)$  are absolute constants.

**Remark 3.2.** (a) The result is obviously sharp if  $N \geq n^2$ . Actually for  $N \geq n^{1+\delta}$  ( $\delta > 0$ ) if we allow constants to depend on  $\delta$ .

(b) In [14], Dafnis, Giannopoulos, and Tsolomitis actually considered the whole sequence of quermassintegrals and also obtained nice high probability results.

(c) In the linear regime  $N \approx n$ , the result

$$\mathbb{E} w(K_N) \approx \sqrt{\log N} L_K$$

was proved by Alonso-Gutiérrez and Prochno in [5] by totally different means involving extremal order statistics and Orlicz norms, but only for the mean width (not the sequence of quermassintegrals).

(d) Note that the assumption  $N \leq e^{\sqrt{n}}$  guarantees that for the choice  $q = \log N$ , we have  $q \leq \sqrt{n}$ , which is needed in Lemma 2.14.

**3.1. The key idea and observation.** The central idea that allows one to determine the asymptotic shape of a random polytope inside an isotropic convex body  $K$  is to compare the random convex set with the  $L_q$ -centroid body of  $K$ . It originates in the work of Giannopoulos and Hartzoulaki [15] who studied random subspaces generated by vertices of the cube, that is, in other words, they studied the class of random symmetric  $\pm 1$ -polytopes. More precisely, if we consider the discrete cube  $\{-1, 1\}^n$  and let  $N > n$ , then such a random polytope is of the form

$$K_{\pm 1, N}^n = \text{conv}\{\pm X_1, \dots, \pm X_N\},$$

where the independent points  $X_1, \dots, X_N$  are chosen uniformly at random from  $\{-1, 1\}^n$ . The authors proved the following fact: for all  $n \geq n_0$  if  $N \geq n(\log n)^2$ , then

$$K_{\pm 1, N}^n \supseteq c \left( \sqrt{\log \frac{N}{n}} \mathbb{B}_2^n \cap \mathbb{B}_\infty^n \right) \supseteq c \sqrt{\frac{\log \frac{N}{n}}{n}} \mathbb{B}_\infty^n \quad (3.1)$$

with probability  $\geq 1 - e^{-n}$ ,  $c \in (0, \infty)$  being an absolute constant. The latter simply says that, with high probability,  $K_{\pm 1, N}^n$  contains a centered cube whose edges have length  $\sqrt{\log(N/n)}/\sqrt{n}$ .

The estimates from [15] were later generalized, improved, and the assumptions relaxed. This was done by Alexander Litvak, Alain Pajor, Mark Rudelson and Nicole Tomczak-Jaegermann [28] who considered a more general class of random polytopes including the important Bernoulli model discussed above and the Gaussian one<sup>7</sup>.

Let us now explain the key observation that was made by Dafnis, Giannopoulos, and Tsolomitis in [14]. They showed that (3.1) can be rewritten in terms of the  $L_q$ -centroid bodies of the unit volume cube  $[-\frac{1}{2}, \frac{1}{2}]^n$ . More precisely, consider the interpolation norm

$$K_{1,2}(x, t) = \inf_{y \in \mathbb{R}^n} \|y\|_1 + t \|x - y\|_2.$$

The so-called Holmstedt approximation shows that

$$K_{1,2}(x, t) \approx \sum_{j=1}^{\lfloor t^2 \rfloor} x_j^* + t \left( \sum_{j=\lfloor t^2 \rfloor + 1}^n (x_j^*)^2 \right)^{1/2},$$

where  $(x_j^*)_{j=1}^n$  is the non-increasing rearrangement of  $\{|x_1|, \dots, |x_n|\}$ . One can now show that for each  $\alpha \geq 1$  and all  $\theta \in \mathbb{S}^{n-1}$ ,

$$h_{\alpha \mathbb{B}_2^n \cap \mathbb{B}_\infty^n}(\theta) = K_{1,2}(\theta, \alpha),$$

---

<sup>7</sup>The random polytopes in their model arise as the absolute convex hull of the rows of a random matrix  $\Gamma_{n,N} := (\xi_{ij})_{i,j=1}^{N,n}$  with independent, symmetric entries satisfying  $\|\xi_{ij}\|_{L_2} \geq 1$  and  $\|\xi_{ij}\|_{\psi_2} \leq \gamma$  for some  $\gamma \geq 1$ . The proof is based on a lower bound (of the order  $\sqrt{N}$ ) for the smallest singular value of  $\Gamma_{n,N}$

which makes also clear the geometric interpretation of the interpolation norm. On the other hand, we know that for  $\theta \in \mathbb{S}^{n-1}$  and all  $q \geq 1$ ,

$$h_{Z_q([- \frac{1}{2}, \frac{1}{2}]^n)}(\theta) = \|\langle \cdot, \theta \rangle\|_{L_q([- \frac{1}{2}, \frac{1}{2}]^n)} \approx \sum_{j=1}^{\lfloor q \rfloor} x_j^* + \sqrt{q} \left( \sum_{j=\lfloor q \rfloor+1}^n (x_j^*)^2 \right)^{1/2}.$$

The latter can be found in [9] and is based on moment inequalities on  $\ell_p^n$ -balls that follow from work of Gluskin and Kwapień [18]. We therefore obtain that

$$h_{\sqrt{q} \mathbb{B}_2^n \cap \mathbb{B}_\infty^n}(\theta) = K_{1,2}(\theta, \sqrt{q}) \approx h_{Z_q([- \frac{1}{2}, \frac{1}{2}]^n)}(\theta),$$

which means that

$$\sqrt{q} \mathbb{B}_2^n \cap \mathbb{B}_\infty^n \approx Z_q\left([- \frac{1}{2}, \frac{1}{2}]^n\right).$$

Hence, looking again at (3.1), we do now have

$$K_{\pm 1, N}^n \supseteq c Z_{\log(N/n)}\left([- \frac{1}{2}, \frac{1}{2}]^n\right),$$

with high probability. This is the very place where one can come up with the idea to compare a random polytope, now a general one inside an isotropic convex body  $K$ , to the  $L_{\log(N/n)}$ -centroid body of  $K$ .

The comparison result they obtained is the following theorem which, for simplicity and since it is enough for our purpose, we do not state in its most general form. The original proof from [14] is a modification of the arguments from [28]. Please note that the nice probabilistic estimate presented arises from the remarkable work of Radosław Adamczak, Alexander Litvak, Alain Pajor, and Nicole Tomczak-Jaegermann [1]. The results they obtained are beautiful and at the same time the proofs are quite technical and involved and cannot be presented here.

**Theorem 3.3** (Comparison theorem). *Let  $n \in \mathbb{N}$  and assume that  $N \geq c_0 n$ . Then, for all  $q \leq c_1 \log \frac{N}{n}$ , we have*

$$K_N \supseteq c_2 Z_q(K)$$

*with probability greater than  $1 - e^{-c_3 \sqrt{N}}$ . Here  $c_0, c_1, c_2, c_3 \in (0, \infty)$  are absolute constants.*

**Remark 3.4.** What had to be bounded in the result of Dafnis, Giannopoulos, and Tsolomitis to obtain the inclusion of the centroid body with high probability was a certain tail probability of the operator norm of a random operator,

$$\mathbb{P}[\|T : \ell_2^m \rightarrow \ell_2^N\| \geq t\sqrt{N}],$$

where  $T(y) = (\langle X_1, y \rangle, \dots, \langle X_N, y \rangle)$ . Here one needs a deep result of Adamczak, Litvak, Pajor, and Tomczak-Jaegermann.

**Remark 3.5.** It is easy to see that one cannot expect a reverse inclusion of the form

$$K_N \subseteq CZ_q(K)$$

with probability close to 1 unless we choose  $q \approx n$ . In fact, using independence and the fact that  $X_1$  is distributed uniformly in  $K$ ,

$$\begin{aligned} \mathbb{P}[K_N \subseteq CZ_q(K)] &= (\mathbb{P}[X_1 \in CZ_q(K)])^N \\ &\leq \text{vol}_n(CZ_q(K))^N \leq \left( \tilde{C} \sqrt{\frac{q}{n}} L_K \right)^{nN}, \end{aligned}$$

where we used Lemma 2.14 (a) in the last estimate. If  $L_K$  is bounded, then we would need  $q \approx n$  to have a chance to get a probability close to 1.

We now have all the tools we need to present a ‘proof’ of the main result

#### 4. THE PROOF

In this section we will present the proof of Theorem 3.1. The previous sections have provided us with the necessary tools. As we will see, the lower bound on the mean width is essentially Urysohn’s inequality together with the comparison result of Theorem 3.3 and the volume estimate for the centroid bodies given in Lemma 2.14 (a). The upper bound follows from a combination of standard arguments, combined with the monotonicity of the centroid bodies and the bound on the mean width of an  $L_q$ -centroid body given in Lemma 2.14 (b).

**Proof of Theorem 3.1.** The lower bound: it follows from Urysohn’s lemma that

$$\mathbb{E} w(K_N) \geq \mathbb{E} \left( \frac{\text{vol}_n(K_N)}{\text{vol}_n(\mathbb{B}_2^n)} \right)^{1/n}.$$

The comparison with centroid bodies (Theorem 3.3) applied with  $q \approx \log \frac{N}{n}$ , says that, with probability greater than  $1 - \exp(-c\sqrt{N})$ ,

$$K_N \supseteq CZ_{\log \frac{N}{n}}(K),$$

where  $C \in (0, \infty)$  is an absolute constant. Let us now denote by  $I := \{\omega \in \Omega : K_N(\omega) \supseteq CZ_{\log(N/n)}(K)\}$  the event of inclusion. Recall that  $\text{vol}_n(\mathbb{B}_2^n) \approx n^{-1/2}$ . Then, using Urysohn’s estimate above and Lemma 2.14 (a), we obtain

$$\begin{aligned} \mathbb{E} w(K_N) &\geq \mathbb{E} \left( \frac{\text{vol}_n(K_N)}{\text{vol}_n(\mathbb{B}_2^n)} \right)^{1/n} \\ &\geq c_0 \sqrt{n} \mathbb{E} \text{vol}_n(K_N)^{1/n} \\ &\geq c_0 \sqrt{n} \mathbb{P}(I) \text{vol}_n(CZ_{\log(N/n)}(K))^{1/n} \\ &\geq c_1 \sqrt{n} \sqrt{\frac{\log(N/n)}{n}} L_K = c_1 \sqrt{\log \frac{N}{n}} L_K. \end{aligned}$$

Alternatively, one can avoid Urysohn’s inequality (we chose to present it here because it is needed as a tool when other quermassintegrals are considered).

We quickly outline the argument:

$$\begin{aligned}
\mathbb{E} w(K_N) &\geq \int_I w(K_N(\omega)) d\mathbb{P}(\omega) \\
&\geq \int_I w(CZ_{\log(N/n)}(K)) d\mathbb{P}(\omega) \\
&\geq C_1 \mathbb{P}(I) w(Z_{\log(N/n)}(K)) \\
&\geq C_2 \sqrt{\log \frac{N}{n}} L_K,
\end{aligned}$$

where we used Lemma 2.14 (b) instead of (a). This proves the lower bound in the main theorem.

The upper bound: We define the following set

$$A_N := \left\{ \theta \in \mathbb{S}^{n-1} : h_{K_N}(\theta) \leq e h_{Z_q(K)}(\theta) \right\}.$$

Then, for any realization in  $\omega \in \Omega$  (we omit this  $\omega$  as usual),

$$\begin{aligned}
w(K_N) &= \int_{\mathbb{S}^{n-1}} h_{K_N}(\theta) d\sigma(\theta) \\
&\leq e \int_{A_N} h_{Z_q(K)}(\theta) d\sigma(\theta) + \int_{A_N^c} h_{K_N}(\theta) d\sigma(\theta) \\
&\leq e \int_{A_N} h_{Z_q(K)}(\theta) d\sigma(\theta) + cnL_K \sigma(A_N^c),
\end{aligned}$$

where the last estimate used that, since  $K_N \subseteq K$ , we must have  $h_{K_N} \leq R(K)$ , while the circumradius satisfies  $R(K) \leq cnL_K$  by Lemma 2.2 (c). Taking expectation (and integrating over the full sphere above instead of  $A_N$  only), we get

$$\mathbb{E} w(K_N) \leq e w(Z_q(K)) + cnL_K \mathbb{E} \sigma(A_N^c).$$

Observe that by Markov's inequality,

$$\begin{aligned}
\mathbb{E} \sigma(A_N^c) &= \int_{\mathbb{S}^{n-1}} \mathbb{P}[h_{K_N}(\theta) > e h_{Z_q(K)}(\theta)] d\sigma(\theta) \\
&= \int_{\mathbb{S}^{n-1}} \mathbb{P}\left[ \max_{1 \leq i \leq N} |\langle X_i, \theta \rangle| > e h_{Z_q(K)}(\theta) \right] d\sigma(\theta) \\
&\leq N \mathbb{P}[|\langle X_1, \theta \rangle| > e h_{Z_q(K)}(\theta)] \\
&\leq \frac{N}{e^q}.
\end{aligned}$$

Therefore, using the latter estimate, the fact that  $L_K = w(Z_2(K))$  for our isotropic  $K$  (see Lemma 2.12 (a)), and that  $L_q$ -centroid bodies are lexicographically nested (see Lemma 2.12 (b)),

$$\mathbb{E} w(K_N) \leq e w(Z_q(K)) + c \frac{nN}{e^q} w(Z_2(K)) \leq \left( e + c \frac{nN}{e^q} \right) w(Z_q(K)).$$

Choosing  $q \approx \log N$  with large enough constants and using the mean width bound from Lemma 2.14 (b), we get

$$\mathbb{E} w(K_N) \leq C \sqrt{\log N} L_K.$$



This completes the proof of the theorem.  $\square$

### 5. APPENDIX ON THE GLUSKIN SPACES

Let us provide some more information on Gluskin’s probabilistic construction of  $n$ -dimensional spaces with a pathologically ‘bad’ behavior, see [17]. One chooses random vectors  $X_1, \dots, X_N$  independently and uniformly from the unit sphere  $\mathbb{S}^{n-1}$  in  $\mathbb{R}^n$ . For  $m \in \mathbb{N}$ , one then defines random symmetric polytopes  $K_N^n$  as follows:

$$K_N^n := \text{conv}\{\pm e_1, \dots, \pm e_n, \pm X_1, \dots, \pm X_N\},$$

where  $e_1, \dots, e_n \in \mathbb{R}^n$  are the standard unit vectors. Let the random spaces corresponding to this class of random polytopes be denoted by  $F_{K_N^n}$ . Of course, the  $N$ -fold product measure  $\sigma^{\otimes N}$  is a probability measure on the set of all those spaces. What Gluskin proved in [17] is that when  $N = 2n$  and  $\tilde{K}_N^n$  is an independent copy of  $K_N^n$ , then the Banach-Mazur distance of the corresponding spaces  $F_{K_N^n}$  and  $F_{\tilde{K}_N^n}$  is greater than or equal to  $cn$  with probability at least  $1 - 2^{-n^2}$ , where  $c \in (0, \infty)$  is an absolute constant.

### REFERENCES

- [1] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles. *J. Amer. Math. Soc.*, 23(2):535–561, 2010.
- [2] D. Alonso-Gutiérrez. On the isotropy constant of random convex sets. *Proc. Amer. Math. Soc.*, 136(9):3293–3300, 2008.
- [3] D. Alonso-Gutiérrez. A remark on the isotropy constant of polytopes. *Proc. Amer. Math. Soc.*, 139(7):2565–2569, 2011.
- [4] D. Alonso-Gutiérrez, A. E. Litvak, and N. Tomczak-Jaegermann. On the isotropic constant of random polytopes. *J. Geom. Anal.*, 26(1):645–662, 2016.
- [5] D. Alonso-Gutiérrez and J. Prochno. On the Gaussian behavior of marginals and the mean width of random polytopes. *Proc. Amer. Math. Soc.*, 143(2):821–832, 2015.
- [6] D. Alonso-Gutiérrez, J. Bastero, J. Bernus, and P. Wolff. On the isotropy constant of projections of polytopes. *J. Funct. Anal.*, 258(5):1452–1465, 2010.
- [7] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic Geometric Analysis. Part I*, volume 202 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
- [8] K. Ball. Normed spaces with a weak-Gordon-Lewis property. In *Functional analysis (Austin, TX, 1987/1989)*, volume 1470 of *Lecture Notes in Math.*, pages 36–47. Springer, Berlin, 1991.
- [9] F. Barthe, O. Guédon, S. Mendelson, and A. Naor. A probabilistic approach to the geometry of the  $l_p^n$ -ball. *Ann. Probab.*, 33(2):480–513, 2005.
- [10] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. *Amer. J. Math.*, 108(6):1467–1476, 1986.
- [11] J. Bourgain. On the distribution of polynomials on high dimensional convex sets. In *Geometric aspects of functional analysis*, volume 1469 of *Lecture Notes in Math.*, pages 127–137. Springer, Berlin, 1991.
- [12] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou. *Geometry of Isotropic Convex Bodies*, volume 196 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.
- [13] N. Dafnis, A. Giannopoulos, and O. Guédon. On the isotropic constant of random polytopes. *Adv. Geom.*, 10(2):311–322, 2010.
- [14] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. *J. Funct. Anal.*, 257(9):2820–2839, 2009.

- [15] A. Giannopoulos and M. Hartzoulaki. Random spaces generated by vertices of the cube. *Discrete Comput. Geom.*, 28:255–273, 2002.
- [16] A. Giannopoulos, L. Hioni, and A. Tsolomitis. Asymptotic shape of the convex hull of isotropic log-concave random vectors. *Adv. Appl. Math.*, 75:116–143, 2016.
- [17] E. D. Gluskin. The diameter of the Minkowski compactum is roughly equal to  $n$ . *Funktsional. Anal. i Prilozhen.*, 15(1):72–73, 1981.
- [18] E. D. Gluskin and S. Kwapien. Tail and moment estimates for sums of independent random variables with logarithmically concave tails. *Studia Math.*, 114(3):303–309, 1995.
- [19] D. Hensley. Slicing convex bodies—bounds for slice area in terms of the body’s covariance. *Proc. Amer. Math. Soc.*, 79(4):619–625, 1980.
- [20] J. Hörrmann, D. Hug, M. Reitzner, and C. Thäle. Poisson polyhedra in high dimensions. *Adv. Math.*, 281:1–39, 2015.
- [21] J. Hörrmann, J. Prochno, and C. Thäle. On the isotropic constant of random polytopes with vertices on an  $\ell_p$ -sphere. *The Journal of Geometric Analysis*, 2017.
- [22] M. Junge. Hyperplane conjecture for quotient spaces of  $\ell_p$ . *Forum Math.*, 6(5):617–636, 1994.
- [23] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995.
- [24] B. Klartag. On convex perturbations with a bounded isotropic constant. *Geom. Funct. Anal.*, 16(6):1274–1290, 2006.
- [25] B. Klartag and G. Kozma. On the hyperplane conjecture for random convex sets. *Israel J. Math.*, 170:253–268, 2009.
- [26] B. Klartag and E. Milman. Centroid bodies and the logarithmic Laplace transform – A unified approach. *J. Funct. Anal.*, 262(1):10–34, 2012.
- [27] H. König, M. Meyer, and A. Pajor. The isotropy constants of the Schatten classes are bounded. *Math. Ann.*, 312(4):773–783, 1998.
- [28] A. E. Litvak, A. Pajor, M. Rudelson, and N. Tomczak-Jaegermann. Smallest singular value of random matrices and geometry of random polytopes. *Adv. Math.*, 195(2):491–523, 2005.
- [29] E. Lutwak, D. Yang, and G. Zhang.  $L_p$  affine isoperimetric inequalities. *J. Differential Geom.*, 56(1):111–132, 2000.
- [30] E. Lutwak and G. Zhang. Blaschke-Santaló inequalities. *J. Differential Geom.*, 47(1):1–16, 1997.
- [31] E. Milman. On the mean-width of isotropic convex bodies and their associated  $L_p$ -centroid bodies. *Int. Math. Res. Not. IMRN*, (11):3408–3423, 2015.
- [32] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed  $n$ -dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.
- [33] G. Paouris. Concentration of mass and central limit properties of isotropic convex bodies. *Proc. Amer. Math. Soc.*, 133(2):565–575, 2005.
- [34] G. Paouris. On the  $\psi_2$ -behaviour of linear functionals on isotropic convex bodies. *Studia Math.*, 168(3):285–299, 2005.
- [35] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.
- [36] G. Paouris and E. Werner. Relative entropy of cone measures and  $\ell_p$  centroid bodies. *Proc. London Math. Soc.*, 104(2):253–286, 2012.
- [37] J. Prochno, C. Thäle, and N. Turchi. The isotropic constant of random polytopes with vertices on convex surfaces. *J. Complexity*, 2019.

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