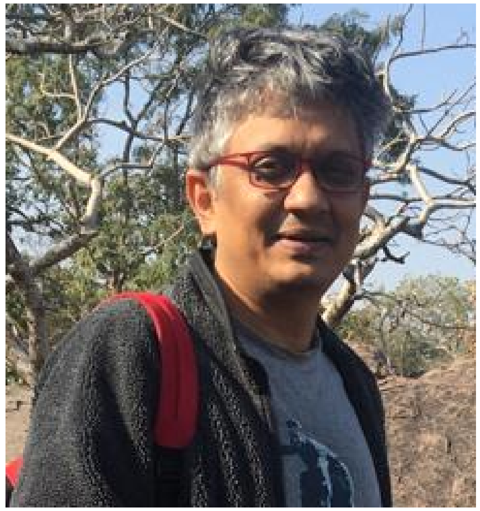
DENSE RANDOM GRAPHS: DYNAMICS OF GRAPHONS

Frank den Hollander
Leiden University, The Netherlands

Spring School in Probability: Complex Networks,
Technische Universität Darmstadt, Germany, 2-6 March 2020.



Siva Athreya

Adrian Roellin

Annals of Applied Probability 2020

§ DYNAMIC GRAPHS AND GRAPHONS

Sequences of dense dynamic random graphs

$$\{G_n(t)\}_{t\geq 0}, \qquad n \in \mathbb{N},$$

and their $n \to \infty$ limits called dynamic random graphons

$$\{h_t\}_{t\geq 0},$$

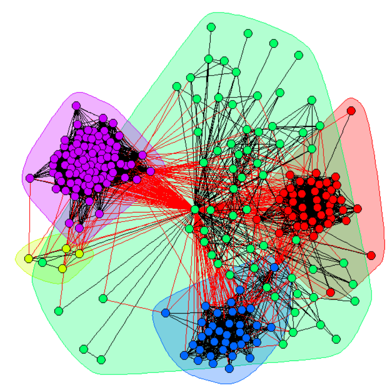
both live in $\widetilde{\mathcal{W}}$ via discrete embedding.

KEY CHALLENGE:

> Construct examples where the limit dynamics is a diffusion.

MOTIVATION: evolution of complex networks.

- Social networks:
  friendships come and go.
- Mobile networks:
  calls switch on and off.
- Partnership networks:
  business relations evolve over time.



PAST WORK:

It is easy to construct random dynamics of random graphs in $\mathcal{G}_n$ for fixed n. However, it is a challenge to construct non-trivial limit dynamics for dense graphs as $n \to \infty$.

- Crane 2016 constructed a limit dynamics based on Aldous-Hoover theory. The exchangeability of vertices leads to either jump processes or deterministic flows.
- Černý & Klimovsky 2018 elaborated on Crane's work and provided a conceptual framework.

Diffusions are not captured through the lens of the Aldous-Hoover theory, which typically leads to random processes in graphon space that are not Markovian.

Are there no diffusions in graphon space?

OBSTACLE:

As $n \to \infty$, subgraph counts tend to becomes sharply concentrated around their average. Consequently, typical random dynamics lead to deterministic flows on graphon space.

CONVERGENCE:

When does $G_n(t) \to h_t$ as $n \to \infty$ in graphon space?

The notion of graphon convergence is built on convergence of subgraph densities:

$$t(F, G_n(t)) = \frac{|\text{copies of } F \text{ in } G_n(t)|}{|\text{copies of } F \text{ in complete graph } K_n|}$$

$$\to \int_{[0,1]^k} \mathrm{d}x_1 \cdots \mathrm{d}x_k \prod_{i \sim j \text{ in } F} h_t(x_i, x_j)$$

$$= t(F, h_t)$$

for all F finite simple graphs on k vertices.

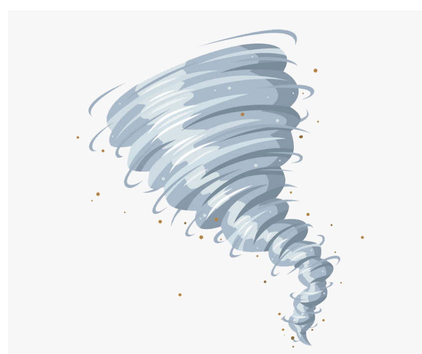
§ A KEY EXAMPLE

A natural choice for the random dynamics is:

- Edges evolves independently.
- Edges remains present during a random time.
- Edges remains absent during a random time.

However, this leads to a deterministic limit dynamics.

Something more dramatic is needed!



STANDARD MORAN MODEL:

- Consider n individuals, each carrying type 0 or type 1.
- At rate 1 an individual randomly draws an individual from the population (possibly itself) and adopts its type.
- Let $X^n(s)$ be the number of individuals of type 0 at time s, and $Y^n(s) = \frac{1}{n}X^n(ns)$ their fraction at time ns.

It is known that if

$$Y^n(0) \Rightarrow Y(0), \qquad n \to \infty,$$

then

$$(Y^n(s))_{s\geq 0} \Rightarrow (Y(s))_{s\geq 0}, \qquad n \to \infty,$$

where the limit is the Fisher-Wright diffusion

$$\mathrm{d}Y(s) = \sqrt{Y(s)(1-Y(s))}\,\mathrm{d}W(s)$$

with $(W(s))_{s\geq 0}$ standard Brownian motion.

- $G_n(ns)$: discrete dynamics with finite n.

  - At any time $s \geq 0$, vertices i and j are connected by an edge if and only if their types are the same.

  - For any connected finite simple graph F on k vertices, the subgraph density of F in $G_n(ns)$ is

$$t(F, G_n(ns)) = \frac{|\text{copies of } F \text{ in } G_n(ns)|}{|\text{copies of } F \text{ in complete graph } K_n|}$$

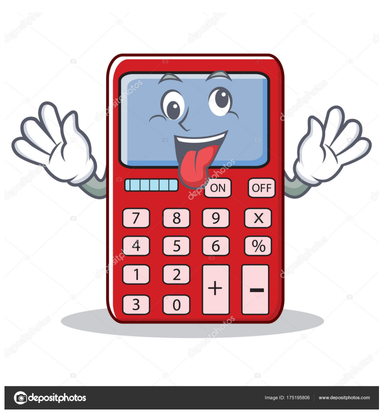
$$= \frac{\binom{X^n(ns)}{k} + \binom{n - X^n(ns)}{k}}{\binom{n}{k}}.$$

- $G_n(ns)$: discrete dynamics with $n \to \infty$.
  - For any connected finite simple graph F on k vertices

$$t(F, G_n(ns)) = \frac{\binom{X^n(ns)}{k} + \binom{n - X^n(ns)}{k}}{\binom{n}{k}}$$

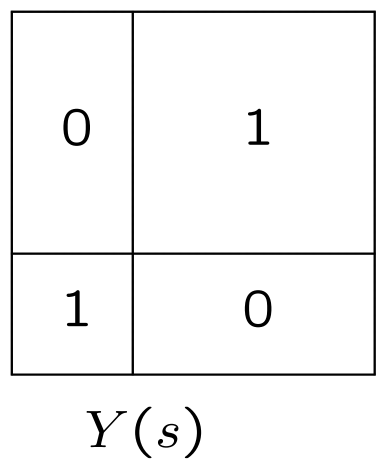
$$= \frac{(X_n(ns))_k}{(n)_k} + \frac{(n - X_n(ns))_k}{(n)_k}$$

$$\longrightarrow Y(s)^k + (1 - Y(s))^k.$$



- h_s: $[0,1]^2 \to [0,1]$: graphon dynamics.
  - For any connected graph F on k vertices, the subgraph density of F in h_s is

$$t(F, h_s) = Y(s)^k + (1 - Y(s))^k.$$



The graphon dynamics is a diffusive 2-block model.

TECHNICALITIES: $G_n(ns) \to h_s$ as $n \to \infty$

- Show that $t_F(G_n(ns))$ converges weakly to $t_F(h_s)$ for fixed s.
- Establish tightness of the process $(t_F(G_n(ns)))_{s \geq 0}$ in graphon space.
- Deduce process convergence on graphon space.
- Check that $t_F(h_s)$ is adapted to the filtration generated by $Y(s)$, and is a Markov process.
- Compute the modulus of continuity of $(t_F(h_s))_{s \geq 0}$ to conclude that $(h_s)_{s \geq 0}$ is diffusive.

Note that the graphon dynamics is a measurable function of the Fisher-Wright diffusion.

§ EXTENSION TO MULTI-TYPE

The multi-type Moran model has $m+1$ types, labelled $0,\ldots,m$.

- Consider n individuals, each carrying one of the types.
- At rate 1 an individual randomly draws an individual from the population (possibly itself) and adopts its type.
- Let $X_\ell^{m,n}(s)$, $0 \le \ell \le m-1$, be the number of individuals of type ℓ at time s, and let

$$X^{m,n}(s) = \left\{X_0^{m,n}(s), \ldots, X_{m-1}^{m,n}(s)\right\}$$

be the vector of type counts. Then the number of individuals of type m at time s equals $X_m^{m,n}(s) := n - \sum_{\ell=0}^{m-1} X_\ell^{m,n}(s)$.

Consider the space-time rescaling

$$Y^{m,n}(s) = \frac{1}{n} X^{m,n}(ns), \qquad s \geq 0,$$

which consists of m components

$$Y^{m,n}(s) = \left\{ Y_0^{m,n}(s), \ldots, Y_{m-1}^{m,n}(s) \right\},$$

representing the fractions of individuals of types $0, \ldots, m-1$ at time ns. The fraction of individuals of type m at time ns equals $Y_m^{m,n}(s) = 1 - \sum_{\ell=0}^{m-1} Y_\ell^{m,n}(s)$.

It is known that if

$$Y^{m,n}(0) \Rightarrow Y^m(0), \qquad n \to \infty,$$

then

$$(Y^{m,n}(s))_{s \geq 0} \Rightarrow (Y^m(s))_{s \geq 0}, \qquad n \to \infty.$$

The limit process consists of m components

$$Y^m(s) = \Big\{Y_0^m(s), \ldots, Y_{m-1}^m(s)\Big\},$$

takes values in the m-dimensional simplex

$$\mathcal{S}^m = \Big\{x = (x_0, \ldots, x_{m-1}) \in \mathbb{R}^m : \\ x_\ell \geq 0 \text{ for all } 0 \leq \ell \leq m-1,\ \textstyle\sum_{\ell=0}^{m-1} x_\ell \leq 1\Big\},$$

and is referred to as the multi-type Fisher-Wright diffusion.

Simplex

- $G_n(ns)$: discrete dynamics with finite n.
  - At any time $s \geq 0$, vertices i and j are connected by an edge if and only if their types are the same.
- h_s: $[0,1]^2 \to [0,1]$: graphon dynamics.

0	0	1
0	1	0
1	0	0

$Y^1(s)$ $Y^1(s) + Y^2(s)$

The graphon dynamics is a diffusive $(m+1)$-block model.

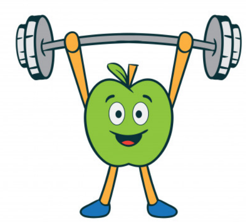
§ FITNESS LANDSCAPE

Connection probabilities of vertices can be modulated by a random fitness landscape $H^{m,n}\colon [0,1] \to [0,1]$ for the types.

ASSUMPTIONS:

(I) The connection probabilities of vertices depend on their types and their fitness, and are given by a continuous function $r\colon [0,1] \times [0,1] \to [0,1]$.

(II) The fitness landscape changes dynamically over time: $H_s^{m,n}(x)$ is the fitness of type $x = \ell/(m+1)$ at time ns.



Construct $G_n(s)$ at time $s \geq 0$ by connecting i and j when

$$U_{ij}^n < r\Big(H_s^{m,n}(T_i^{m,n}(s)), H_s^{m,n}(T_j^{m,n}(s))\Big),$$

where

- $\{U_{ij}^n \colon n \in \mathbb{N}, 1 \leq i < j \leq n\}$ are independent uniform random variables on $[0,1]$.
- $T_i^{m,n}(s)$ is the scaled type of vertex i at time ns, i.e., if at time ns the type of vertex i equals ℓ, $0 \leq \ell \leq m$, then $T_i^{m,n}(s) = \ell/(m+1)$.

THEOREM: *Fix* $m \in \mathbb{N}$. *Suppose that*

$$(Y^{m,n}(s), H_s^{m,n})_{s\geq 0} \Rightarrow (Y^m(s), H_s^m)_{s\geq 0}, \qquad n \to \infty.$$

Then

$$(G_n(ns))_{s\geq 0} \Rightarrow (\tilde{h}_s^m)_{s\geq 0}, \qquad n \to \infty,$$

where the equivalence class $\tilde{h}_s^m$ *has a representative* h_s^m *of the form*

$$h_s^m(x,y) = r\Big(H_s^m\big(\bar{F}_s^m(x)\big), H_s^m\big(\bar{F}_s^m(y)\big)\Big)$$

with $\bar{F}_s^m$ *denoting the generalised inverse of the cumulative type distribution function* F_s^m *given by*

$$F_s^m(x) = \sum_{\ell=0}^{\lfloor (m+1)x \rfloor} Y_\ell^m(s), \quad x \in [0,1].$$

The importance of adding fitness is that now also the heights of the graphons inside the blocks become random instead of 0 or 1.

e_s	f_s	c_s
d_s	b_s	f_s
a_s	d_s	e_s

$Y^1(s) \quad Y^1(s) + Y^2(s)$

The graphon dynamics is a diffusive $(m+1)$-block model with diffusive heights.

§ EXTENSION TO INFINITELY MANY TYPES

Define the empirical type distribution

$$Z^m(s) = \sum_{\ell=0}^{m} Y_\ell^m(s)\, \delta_{\ell/(m+1)}$$

and note that

$$Z^m(s;[0,x]) = F_s^m(x), \qquad x \in [0,1].$$

It is known that if

$$Z^m(0) \Rightarrow Z(0), \qquad m \to \infty,$$

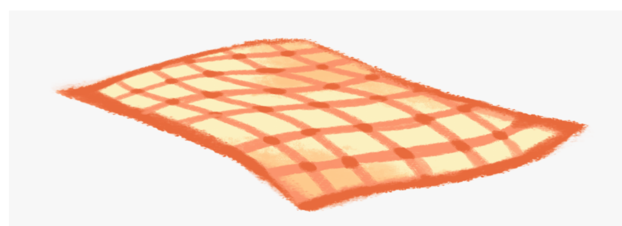
then

$$(Z^m(s))_{s\geq 0} \Rightarrow (Z(s))_{s\geq 0}, \quad m \to \infty.$$

The limit process takes values in $\mathcal{P}([0,1])$, i.e., the set of probability measures on $[0,1]$ endowed with the topology of weak convergence, and is referred to as the Fleming-Viot diffusion. Its cumulative type distribution function is

$$Z(s;[0,x]) = F_s(x), \qquad x \in [0,1].$$

With the help of a similar fitness landscape as before, we obtain a graphon dynamics that behaves like some sort of measure-valued diffusion.



§ CONCLUSION

A rich class of non-trivial diffusions in graphon space arises, well beyond the narrow setting of block models.

A formal generator for the Markov process in graphon space can be written down.

Lots remains to be investigated!


AAAHHHH!