Darmstadt Spring School Complex Networks

Detailed Course Outline

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The content of the summer school will mainly make use of selected material from the books (the latter in progress):

▷ Random Graphs and Complex Networks Volume 1 [1].
▷ Random Graphs and Complex Networks Volume 2 [2] (in progress).

We will refer to [1, Theorem 1.1] as [Vol 1, Theorem 1.1] and to [2, Theorem 1.1] as [Vol 2, Theorem 1.1]. We will further rely on several papers.

In the schedule below, we also add recommended exercises that we encourage the students to make. The exercises with stars are considered to be more difficult than the ones without, the exercises indicated with a minus are considered to be relatively simple. We encourage you to make these exercises, either during or after the school, as this will help you to absorb the material better.

The Spring School will treat the following material:

Lecture 1: Real-world networks and random graphs:

> In this first lecture, we discuss some of the empirical properties of real-world networks. Most real-world networks are sparse, in that they have a relatively small average degree, are small worlds, in that the average distances in them are not very large, and are scale-free, in that they have an enormous variability in the degrees of their elements. We then proceed by introducing the most commonly used random graph models for them.

This lecture treats the following topics:

Empirical networks: [Vol 1, Chapter 1].
Graphs and their degrees and connectivity structure in [Vol 1, Section 1.2].
Sparse and scale-free degree sequences in [Vol 1, Section 1.4.1].
Highly-connected graph sequences in [Vol 1, Section 1.4.2].
Small-world random graph sequences in [Vol 1, Section 1.4.3].
Further network statistics in [Vol 1, Section 1.5].

Random graphs:
▷ Generalized random graphs in [Vol 1, Chapter 6]:
Weight conditions in [Vol 1, Condition 6.4]
Degrees in [Vol 1, Theorem 6.10]
GRG conditioned on degrees is uniform in [Vol 1, Theorem 6.15]
▷ Configuration model in [Vol 1, Chapter 7]:
Construction and law graph in [Vol 1, Proposition 7.7]
Conditions on degrees in [Vol 1, Condition 7.8]

CM conditioned on simplicity in [Vol 1, Theorem 7.12]
Relation to uniform simple graphs and GRG in [Vol 1, Section 7.5]
▷ Preferential attachment models
Informal discussion degrees

Recommended exercises: [Vol 1, Ex. 1.1], [Vol 1, Ex. 4.1], [Vol 1, Ex. 4.4], [Vol 1, Ex. 4.5], [Vol 2, Ex. 1.1^{-}], [Vol 1, Ex. 6.1^{-}], [Vol 1, Ex. 6.2^{-}], [Vol 1, Ex. 6.8], [Vol 1, Ex. 7.3^{-}].
Optional exercises: [Vol 1, Ex. 6.3], [Vol 1, Ex. 6.4], [Vol 1, Ex. 7.1^{-}], [Vol 1, Ex. 7.4], [Vol 1, Ex. $7.23^{\star}$].

Lecture 2: Local weak convergence: theory:

In this lecture, we discuss the notion of *local weak convergence,* which is the main technique to investigate sparse random graphs. Local weak convergence is sometimes also called Benjamini-Schramm convergence, or the objective method of Aldous and Steele. Possibly, we will need to complete the description of random graph models from Lecture 1.

We further aim to cover the following topics:

Local weak convergence of deterministic graphs:
[Vol 2, Sections 2.1 and 2.2].

Theory of local weak convergence of random graphs:
[Vol 2, Sections 2.3 and 2.4].

Consequences of local weak convergence:
[Vol 2, Section 2.5].

Recommended exercises: [Vol 1, Ex. 8.3^{-}], [Vol 1, Ex. 8.4^{-}], [Vol 1, Ex. 8.9], [Vol 2, Ex. 2.1], [Vol 2, Ex. 2.1], [Vol 2, Ex. 2.8], [Vol 2, Ex. 2.10], [Vol 2, Ex. 2.16].
Optional exercises: [Vol 1, Ex. $8.8^{\star}$], [Vol 1, Ex. 8.14], [Vol 2, Ex. 2.2], [Vol 2, Ex. 2.4], [Vol 2, Ex. 2.18].

Lecture 3: Local weak convergence of random graphs:

Here, we apply the theory from Lecture 2 to (some of) the random graph models introduced in Lecture 1. Often, the resulting limits are *trees*, more precisely, branching processes. We will thus also discuss trees and branching processes in some detail.

The main topics are the following:

Branching processes and trees:
Branching processes in [Vol 1, Chapter 3], and descriptions of trees in [Vol 2, Section 1.5].

Branching process comparisons for random graphs:
Comparison of neighborhoods in [Vol 2, Section 4.1], in particular, [Vol 2, Theorem 4.1, Lemma 4.2 and Lemma 4.3].

Recommended exercises: [Vol 1, Ex. 3.1], [Vol 1, Ex. 3.8], [Vol 2, Ex. 4.1$^-$], [Vol 2, Ex. 4.2], [Vol 2, Ex. 6.7*], [Vol 2, Ex. 7.7].
Optional exercises: [Vol 2, Ex. 7.6], [Vol 2, Ex. 7.15], [Vol 2, Ex. 7.17].

Lecture 4: The giant in random graphs is almost local:

In this lecture, we describe the relation between the giant component in (random) graphs and the survival probability of the local weak limit. It turns out that one relatively simple and intuitive additional assumption suffices to show that the proportion of vertices in the giant is indeed the survival probability of the local weak limit.

More precisely, we aim to cover the following topics:

Giant is almost local:
[Vol 2, Section 2.5].

Giant in random graph models:
Giant component for ERRG: [Vol 2, Section 2.5].
Giant component for CM: [Vol 2, Section 4.2], in particular [Vol 2, Theorem 4.4], and glimpse of proof.
Connectivity of CM in [Vol 2, Section 4.3], in particular [Vol 2, Theorem 4.15].

Recommended exercises: [Vol 2, Ex. 4.7], [Vol 2, Ex. 4.8], [Vol 2, Ex. 4.12].
Optional exercises: [Vol 2, Ex. 4.3], [Vol 2, Ex. 4.4], [Vol 2, Ex. 4.5], [Vol 2, Ex. 4.11].

Lecture 5: Small-world properties of random graphs:

We close the course by discussing the small-world properties of random graphs. We describe the two main regimes, namely, the small-world regime where typical distances are logarithmic in the number of vertices, and the ultra-small-world regime, where typical distances grow doubly logarithmically in the network size. We focus on the latter regime, as this is a possible explanation for the 'six degrees of separation' paradigm.

In more detail, we discuss the following topics:

Material:
Small-world properties (mainly results) in [Vol 2, Section 7.1], and relations to branching processes.
Branching processes with infinite-mean degrees in [Vol 2, Section 7.3], in particular, [Vol 2, Theorem 7.14].
Diameter of the core in infinite-variance CMs in [Vol 2, Theorem 7.9].

Recommended exercises: [Vol 1, Ex. 1.4*], [Vol 2, Ex. 2.28], [Vol 2, Ex. 2.29], [Vol 2, Ex. 6.1$^-$], [Vol 2, Ex. 6.2], [Vol 2, Ex. 7.10], [Vol 2, Ex. 7.13].
Optional exercises: [Vol 2, Ex. 6.28], [Vol 2, Ex. 7.15*], [Vol 2, Ex. 7.17].

References

[1] R. v. d. Hofstad. *Random graphs and complex networks. Volume 1.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, (2017).
[2] R. v. d. Hofstad. *Random graphs and complex networks. Volume 2.* (2018+). In preparation, see
`http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf`.