Darmstadt Short Course: Connectivity and components of random Euclidean graphs

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Abstract

The random Euclidean graph $G(n, r)$ has n vertices uniformly distributed in the unit square, with edges between any two vertices distant less than r from each other. This course is concerned with the asymptotic behaviour of the graph $G(n, r(n))$ (or the Poissonized version thereof) in the large-n limit with a given sequence $r(n)$. We shall prove two types of phase transition:

If $nr(n)^2/\log n$ tends to a constant c, then the graph is disconnected with high probability for $c < 1/\pi$, but connected w.h.p. for $c > 1/\pi$.

If instead $nr(n)^2$ tends to a constant b, then the graph enjoys a 'giant component' containing a positive proportion of the vertices, asymptotically in probability, if and only if b exceeds a certain critical value.

On the way we shall explain a number of key results and ideas in the theory of point processes and continuum percolation, which are needed to derive the above two results.

1 Introduction and Preliminaries

1.1 Overview of the course

Given $d \in \mathbb{N}$ and finite $\mathcal{X} \subset \mathbb{R}^d$, and $r > 0$, the *geometric graph* $G(\mathcal{X}, r)$ has vertex set $\mathcal{X}$ and edge set $\{\{x, y\} : \|x - y\| \leq r\}$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$.

Motivation for this: radio stations/communications. Trees/disease. Stars/constellations. Topological data analysis.

A *random* geometric graph (RGG) is obtained by taking $\mathcal{X}$ to be a random set of points. Other terminology for the same thing: random Euclidean graph. The RGG terminology used here dates back at least to 2003 [5].

Let $\xi_1, \xi_2, \ldots$ be independent random d-vectors, uniformly distributed over the set $B(1) := [-1/2, 1/2]^d$ (a box of side 1). Set

$$\mathcal{X}_n := \{\xi_1, \ldots, \xi_n\}.$$

One reason to study RGGs is to explore 'typical' properties of geometric graphs. Another reason is to assess statistical tests based on the graph $G(\mathcal{X}_n, r_n)$, for example tests for uniformity.

In this course we consider the RGG $G(\mathcal{X}_n, r_n)$ with $(r_n)_{n\geq 1}$ a specified sequence of distance parameters. For simplicity we assume from now on that $d = 2$, although many of the ideas here can be extended to higher dimensions .

Notation. Many of the results described in this course are asymptotic results as $n \to \infty$. Unless stated otherwise, any limiting statement in the sequel is as $n \to \infty$. Also, for positive real-valued sequences a_n and b_n we use the following asymptotic notational conventions:

- $a_n = O(b_n)$ means $\limsup(a_n/b_n) < \infty$.
- $a_n = \Theta(b_n)$ means that both $a_n = O(b_n)$ and $b_n = O(a_n)$.
- $a_n \sim b_n$ means $a_n/b_n \to 1$.
- 'With high probability' or 'w.h.p.' means 'with probability tending to 1 as $n \to \infty$'.

We investigate the following questions for $G(\mathcal{X}_n, r_n)$, asymptotically as $n \to \infty$:

- How large does r_n have to be for $G(\mathcal{X}_n, r_n)$ to be connected w.h.p.?
- How large does r_n have to be for $G(\mathcal{X}_n, r_n)$ to have a component containing a non-vanishing proportion of the vertices, w.h.p.?

One interpretation of the RGG is as a (crude) model of a spatial epidemic, starting from a single infected individual. The first question above relates to whether the entire population becomes infected; the second question relates to whether a significant proportion of the population become infected.

To get a feel for the answers we might expect to these questions, let us first do some preliminary computations on vertex degrees. For $k \in \mathbb{N}$ let $N_k(n)$ denote the number of vertices of degree k in $G(\mathcal{X}_n, r_n)$.

Exercise 1.1. Prove that if $r_n \to 0$, then $\mathbb{E}[\text{Degree}(\xi_1)] \sim n\pi r_n^2$.

That is, if $r_n \to 0$, the expected number of edges incident to a 'typical vertex' of $\mathcal{X}_n$ goes like $n\pi r_n^2$.

Now consider $N_0(n)$, the number of isolated vertices. We have the following 'back of the envelope' computation: if $n\pi r_n^2 \sim \alpha \log n$ as $n \to \infty$, then

$$\mathbb{E}[N_0(n)] = n\mathbb{P}[\text{Degree}(\xi_1) = 0] \approx n(1 - \pi r_n^2)^{n-1} \approx n\exp(-(n-1)\pi r_n^2) \approx n^{1-\alpha},$$

which suggests the following:

Exercise 1.2. Prove that if $n\pi r_n^2/\log n \to \alpha \in (0,\infty)$, then $\mathbb{E}[N_0(n)] \to \infty$ if $\alpha < 1$, and $\mathbb{E}[N_0(n)] \to 0$ if $\alpha > 1$.

This suggests that if $\alpha < 1$ then $G(\mathcal{X}_n, r_n)$ is unlikely to be connected (because there are lots of isolated vertices), but if $\alpha > 1$ it has a chance to be connected (because there are probably no isolated vertices). It turns out that once we are past the obstacle of there being isolated vertices, we do indeed have a connected graph with high probability (i.e. with probability tending to 1 as $n \to \infty$). Thus, an answer to the first question above is as follows.

Theorem 1.3. *If $n\pi r_n^2/\log n \to \alpha \in (0,\infty)$ then if $\alpha > 1$ then $G(\mathcal{X}_n, r_n)$ is connected with high probability but if $\alpha < 1$ then $G(\mathcal{X}_n, r_n)$ is not connected with high probability.*

For the second question, it turns out we should consider the case where with $nr_n^2 \to \lambda$ for some $\lambda \in (0,\infty)$. We call this the *thermodynamic limit* (bulk limit). In this case, by Exercise 1.1. the 'average degree' approximates to $\pi\lambda$. More information about the degree profile is provided by the next exercise.

Exercise 1.4. Prove that in the thermodynamic limit $n^{-1}\mathbb{E}[N_k(n)] \to \mathbb{P}[Z = k]$, where Z is Poisson with parameter $\lambda\pi$.

The last result shows that in the thermodynamic limit, these graphs do not exhibit any kind of heavy tailed or 'scale free' behaviour of the degree profile.

An answer to the second question is as follows. The $p_\infty(\lambda)$ appearing in this result is a continuum percolation function, that we shall discuss in more detail later on.

Theorem 1.5. *If $nr_n^2 \to \lambda \in (0,\infty)$ as $n \to \infty$, then the order of the largest component of $G(\mathcal{X}_n, r_n)$, divided by n, converges in probability to a limit $p_\infty(\lambda)$. There is a critical value $\lambda_c \in (0,\infty)$ such that $p_\infty(\lambda) = 0$ for $\lambda \le \lambda_c$ and $p_\infty(\lambda) > 0$ for $\lambda > \lambda_c$.*

It is of interest to compare this random graph model with others, such as the **Erdös-Rényi random graph** $G(n,p)$. This is defined as follows. There are n vertices, and for each pair of vertices, an edge between them is included with probability p, independently of the other pairs. Given a sequence $(p_n)_{n\ge1}$, it is well known that:

- If $p_n \sim \alpha(\log n)/n$ as $n \to \infty$, for some constant α, then if $\alpha > 1$ the graph $G(n,p_n)$ is connected w.h.p. If $\alpha < 1$ the graph $G(n,p_n)$ is disconnected w.h.p.
- If $p_n \sim \beta/n$ as $n \to \infty$, for some contant β, then the order of the largest component of $G(n,p_n)$, divided by n, converges in probabity to a limit which is strictly positive if and only if $\beta > 1$.

Theorems 1.3 and 1.5 are provide analogues, for the RGG, to the above results for the Erdös-Rényi graph $G(n,p)$. However the proofs are completely different.

Proofs of Theorems 1.3 and 1.5 can be found in [5]. In the present course we shall provide proofs of 'Poissonized' versions of these theorems (as described in the next subsection). The proofs we provide here are adapted from the methods in [5], but with various changes which make the proofs here more self-contained than for the corresponding results in [5].

1.2 Poissonization

A set $\mathcal{X} \subset \mathbb{R}^2$ is said to be *locally finite* if $\mathcal{X}(B) < \infty$ for all bounded $B \subset \mathbb{R}^2$, where $\mathcal{X}(B)$ means the number of points of $\mathcal{X}$ in B. Given a bounded Borel-measurable function $g : \mathbb{R}^2 \to [0,\infty)$, a *Poisson process* in $\mathbb{R}^2$ with *intensity function* g is a random, locally finite subset $\mathcal{P}$ of S such that for all Borel $A, A_1, \ldots, A_k \subset S$;

$$\mathcal{P}(A) \sim \mathrm{Po}\left(\int_A g(x)dx\right);$$
$$\mathcal{P}(A_1), \ldots, \mathcal{P}(A_k) \text{ are independent for } A_1, \ldots, A_k \text{ disjoint}$$

where $\mathrm{Po}(t)$ is the Poisson distribution with parameter t, $t \in (0, \infty)$, and a $\mathrm{Po}(\infty)$ random variable takes the value $+\infty$ almost surely.

In the special case where $g = \lambda \mathbf{1}_S$, for some constant $\lambda > 0$ and some Borel set $S \subseteq \mathbb{R}^2$, we refer to $\mathcal{P}$ as a *homogeneous Poisson process* in S with *intensity* λ.

For $s > 0$, set $B(s) := [-s/2, s/2]^2$. Recall that $\xi_1, \xi_2, \cdots$ are independent and uniform over $B(1)$. Let $N_{\lambda s^2}$ be Poisson distributed with parameter λs^d, independent of $(\xi_1, \xi_2, \ldots)$, and set

$$\mathcal{H}_{\lambda,s} := \{s\xi_1, \ldots, s\xi_{N_{\lambda s^2}}\}. \tag{1.1}$$

If $\xi_i = \xi_j$ for some $i \neq j$, then $\mathcal{H}_{\lambda,s}$ should be seen as a *multiset* rather than a set, but this happens with probability zero.

Then $\mathcal{H}_{\lambda,s}$ is a homogeneous Poisson point process in $B(s)$ with intensity λ.

Exercise 1.6. Prove this.

We also write $\mathcal{P}_n$ for $\mathcal{H}_{n,1}$. The 'Poissonized' version of the RGG $G(\mathcal{X}_n, r_n)$ is the graph $G(\mathcal{P}_n, r_n)$, and we shall later prove results along the lines of Theorems 1.3 and 1.5 for these graphs.

We now state some basic facts about Poisson processes.

Theorem 1.7. (Superposition theorem) *Suppose $\mathcal{P}$ is a Poisson process in $\mathbb{R}^2$ with bounded intensity function $g(\cdot)$ and $\mathcal{P}'$ is a Poisson process in $\mathbb{R}^2$ with bounded intensity function $g'(\cdot)$, independent of $\mathcal{P}$. Then $\mathcal{P} \cup \mathcal{P}'$ is a Poisson process in $\mathbb{R}^2$ with intensity function $g(\cdot) + g'(\cdot)$.*

Exercise 1.8. Prove this.

Theorem 1.9. (Thinning theorem) *Suppose $\mathcal{P}$ is a Poisson process in $\mathbb{R}^2$ with intensity function $g(\cdot)$ and $0 < p < 1$. For each point X of $\mathcal{P}$, let X be accepted with probability p and rejected if not accepted, independently of all other points; let $\mathcal{P}'$ be the point process of accepted points. Then $\mathcal{P}'$ is a Poisson process in $\mathbb{R}^2$ with intensity function $pg(\cdot)$.*

Exercise 1.10. Prove this.

In the proof of the next result we use notation

$$(n)_{(k)} := n(n-1)\cdots(n-k+1) \text{ for } n, k \in \mathbb{N}$$

(the so-called 'descending factorial').

Theorem 1.11. (Mecke formula.) *Let $k \in \mathbb{N}$. Let $\lambda, s > 0$. For any measurable real-valued function f, defined on the product of $(\mathbb{R}^2)^k$ and the space of finite subsets of $B(s)$, for which the following expecation exists,*

$$\mathbb{E} \sum_{X_1,\ldots,X_k \in \mathcal{H}_{\lambda,s}}^{\neq} f(X_1, X_2, \ldots, X_k, \mathcal{H}_{\lambda,s} \backslash \{X_1, \ldots, X_k\}) = \lambda^k \int_{B(s)} dx_1 \cdots \int_{B(s)} dx_k \mathbb{E} f(x_1, \ldots, x_k, \mathcal{H}_{\lambda,s})$$

where $\sum^{\neq}$ means the sum is over ordered k-tuples of distinct points of $\mathcal{H}_{\lambda,s}$.

Proof. We use the representation (1.1) and condition on $N_{\lambda s^2}$, the total number of points. Set $\mu := \lambda s^2$. With all integrals below being over $B(1)$,

$$\begin{aligned}\mathbb{E} \sum_{X_1,\dots,X_k \in \mathcal{H}_{\lambda,s}}^{\neq} f(X_1, X_2, \dots, X_k, \mathcal{H}_{\lambda,s} \setminus \{X_1, \dots, X_k\}) \\ = \sum_{m=k}^{\infty} \left(e^{-\mu} \frac{\mu^m}{m!} \right) (m)_k \int_{B(1)} dx_1 \cdots \int_{B(1)} dx_m f(sx_1, \dots, sx_k, \{sx_{k+1}, \dots, sx_m\}) \\ = \mu^k \int dx_1 \cdots \int dx_k \sum_{m=k}^{\infty} \left(\frac{e^{-\mu}\mu^{m-k}}{(m-k)!} \right) \int dy_1 \cdots \int dy_{m-k} f(sx_1, \dots, sx_k, \{sy_1, \dots, sy_{m-k}\}) \\ = \mu^k \int dx_1 \cdots \int dx_k \sum_{r=0}^{\infty} \left(\frac{e^{-\mu}\mu^{r}}{r!} \right) \int dy_1 \cdots \int dy_r f(sx_1, \dots, sx_k, \{sy_1, \dots, sy_r\}) \\ = \mu^k \int dx_1 \cdots \int dx_k \mathbb{E} f(sx_1, \dots, sx_k, \mathcal{H}_{\lambda,s})\end{aligned}$$

where in the third line we made the substitution $y_j = x_{k+j}$ for $k < j \leq m$, and in the fourth line we set $r = m - k$. Changing variable in the last line to $x_i' := sx_i$ for $1 \leq i \leq k$ then yields the result. □

For $\lambda > 0$, let $\mathcal{H}_\lambda$ denote a homogeneous Poisson process of intensity λ in the whole of $\mathbb{R}^2$.

Exercise 1.12. Prove such an object exists, i.e. that there exists a system of random variables $(\zeta_1, \zeta_2, \dots)$ on a suitable probability space such that $\mathcal{H}_\lambda := \{\zeta_1, \zeta_2, \dots\}$ has the defining properties of the Poisson process. *Hint: first extend the superposition theorem to a union of countably many Poisson processes.*

Theorem 1.13 (Translation and rotation invariance)**.** *Let $\lambda > 0$. Then for any translation τ of $\mathbb{R}^2$ (i.e. a mapping $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $x \mapsto x + y$ for some fixed $y \in \mathbb{R}^2$) and any rotation ρ of $\mathbb{R}^2$, the point process $\tau(\mathcal{H}_\lambda)$ is also a homogeneous Poisson process in $\mathbb{R}^2$ of intensity λ, as is $\rho(\mathcal{H}_\lambda)$.*

Exercise 1.14. Prove this.

The next result extends the Mecke formula to the the infinite Poisson process $\mathcal{H}_\lambda$ (here we consider only the case with $k = 1$ and f translation invariant). Given $\mathcal{X} \subset \mathbb{R}^2$ and $y \in \mathbb{R}^2$, write $\mathcal{X} + x$ for $\{x + y : x \in \mathcal{X}\}$. Also $\mathbf{o} := (0, 0)$, the origin in $\mathbb{R}^2$.

Theorem 1.15. (Mecke formula for infinite Poisson process) *Suppose $h(x; \mathcal{X})$ is a bounded measurable real-valued function defined on all pairs of the form $(x, \mathcal{X})$ with $\mathcal{X}$ a locally finite subset of $\mathbb{R}^2$. Assume that h is translation-invariant, meaning that $h(x; \mathcal{X}) = h(\mathbf{o}; \mathcal{X} + (-x))$ for any $(x, \mathcal{X})$. Then*

$$\mathbb{E} \sum_{x \in \mathcal{H}_\lambda \cap B(s)} h(x; \mathcal{H}_\lambda \setminus \{x\}) = \lambda s^2 \mathbb{E}\left[h(\mathbf{o}; \mathcal{H}_\lambda)\right]. \tag{1.2}$$

Proof. Consider $\mathcal{H}_\lambda$ as the union of two independent Poisson processes, namely, $\mathcal{H}_{\lambda,s}$ (a homogeneous Poisson process of intensity λ on $B(s)$) and $\tilde{\mathcal{H}}_{\lambda,s}$ (a homogeneous Poisson process of intensity λ on $\mathbb{R}^d \setminus B(s)$). Then, by Theorem 1.11,

$$\mathbb{E}\left[\sum_{x \in \mathcal{H}_\lambda \cap B(s)} h(x; \mathcal{H}_\lambda \setminus \{x\}) | \tilde{\mathcal{H}}_{\lambda,s} \right] = \lambda \int_{B(s)} \mathbb{E}[h(x; \mathcal{H}_{\lambda,s} \cup \tilde{\mathcal{H}}_{\lambda,s}) | \tilde{\mathcal{H}}_{\lambda,s}] \mathrm{d}x,$$

and taking the expectation of both sides and using Theorem 1.13, we obtain (1.2). $\square$

2 Connectivity

Let $\mathcal{K}$ be the class of connected graphs, and let

$$\rho'_n = \min\{r : G(\mathcal{P}_n, r) \in \mathcal{K}\}$$

which is a random variable determined by the configuration of $\mathcal{P}_n$. It is called the *connectivity threshold*. Similarly define

$$\rho_n = \min\{r : G(\mathcal{X}_n, r) \in \mathcal{K}\}.$$

In this section we prove the following result.

Theorem 2.1. *It is the case that*

$$n\pi(\rho'_n)^2/\log n \xrightarrow{P} 1. \tag{2.1}$$

Remarks.

(i) The corresponding result also holds with ρ'_n replaced by ρ_n (this is equivalent to Theorem 1.3, and is covered by in the exercises in these notes), and also with almost sure convergence (proving this is beyond the scope of these notes).

(ii) A further extension of (2.1) is the following convergence in distribution result: for any $t \in \mathbb{R}$,
$$\lim_{n\to\infty} \mathbb{P}[n\pi(\rho'_n)^2 - \log n \leq t] \to \exp(-e^{-t}).$$
The proof of this is beyond our scope here, but can be found in [5].

(iii) Higher-dimensional analogues to these results can also be found in [5]. There are some discrepancies because boundary effects become more important in higher dimensions.

Given r_n, let δ'_n denote the minimum degree of $G(\mathcal{P}_n, r_n)$.
For $x \in \mathbb{R}^2$ and $r > 0$ define the disk

$$D(x,r) := \{y \in \mathbb{R}^2 : \|y - x\| \leq r\}.$$

Also let Leb(.) denote area (2-dimensional Lebesgue measure).

Theorem 2.2. *If $n\pi r_n^2/\log n = \alpha < 1$ for all $n \geq 2$ then $\mathbb{P}[\delta'_n = 0] \to 1$.*

Proof. Let N_n here denote the number of vertices of $\mathcal{P}_n \cap B(1/2)$ of degree zero in $G(\mathcal{P}_n, r_n)$ (we restrict to the smaller square $B(1/2)$ to avoid boundary effects). Then by the Mecke formula,

$$\mathbb{E}[N_n] = n \int_{S(1/2)} \mathbb{P}[\mathcal{P}_n(D(x,1/2)) = 0]dx = (n/4)\exp(-n\pi r_n^2) = (1/4)n^{1-\alpha},$$

which tends to infinity as $n \to \infty$.

Also $N_n(N_n-1)$ is the number of ordered pairs (x,y) of distinct points in $\mathcal{P}_n \cap S(1/2)$ such that $(\mathcal{P}_n \setminus \{x,y\})(D(x,1/2) \cup D(y,1/2)) = 0$. By the Mecke formula

$$\mathbb{E}[N_n(N_n-1)] = n^2 \int_{S(1/2)} \int_{S(1/2)} \exp(-n\text{Leb}(D(x,r) \cup D(y,r)))dydx.$$

Splitting the inner integral according to whether or not $y \in D(x,2r)$ yields

$$\begin{aligned}\mathbb{E}[N_n(N_n-1)] &\leq (n/4)^2 \exp(-2nr_n^2) + (n^2/4)\pi(4r^2)\exp(-nr_n^2) \\ &= (\mathbb{E}[N_n])^2 + O(n^{1-\alpha}\log n),\end{aligned}$$

and therefore $\mathbb{E}[N_n^2]/(\mathbb{E}[N_n])^2 \to 1$. Hence, $\text{Var}(N_n/\mathbb{E}[N_n]) \to 0$ so $N_n/\mathbb{E}[N_n] \to 1$ in probability. Thus $\mathbb{P}[N_n = 0] \to 0$, but if $N_n > 0$ then $\delta'_n = 0$ so the result follows. □.

Corollary 2.3. *Given $\varepsilon > 0$ we have $\mathbb{P}[n\pi(\rho'_n)^2/\log n > 1-\varepsilon] \to 1$.*

Proof. Assume $\varepsilon < 1$. Set $r_n = ((1-\varepsilon)\log n/(n\pi))^{1/2}$, so $n\pi r_n^2/\log n = 1-\varepsilon$. Let δ'_n be the minimum degree of $G(\mathcal{P}_n, r_n)$. If the minimum degree of a graph of order greater than 1 is zero, then it is not connected; hence

$$\begin{aligned}\mathbb{P}[n\pi(\rho'_n)^d/\log n < 1-\varepsilon] &= \mathbb{P}[G(\mathcal{P}_n, r_n) \in \mathcal{K}] \\ &\leq \mathbb{P}[\delta'_n > 0] + \mathbb{P}[\mathcal{P}_n(B(1)) \leq 1],\end{aligned}$$

which tends to zero by Theorem 2.2. □

Exercise 2.4. Let δ_n denote the minimum degree of $G(\mathcal{X}_n, r_n)$. Show that Theorem 2.2 holds with δ'_n replaced by δ_n, and deduce that $\mathbb{P}[N\pi\rho_n^2/\log n > 1-\varepsilon] \to 1$.

To complete the proof of Theorem 2.1, it suffices to prove the following:

Theorem 2.5. *Suppose $(r_n)_{n\in\mathbb{N}}$ is such that*

$$n\pi r_n^2/\log n = \alpha > 1, \quad \forall n \geq 2. \tag{2.2}$$

Then $\mathbb{P}[G(\mathcal{P}_n, r_n) \in \mathcal{K}] \to 1$.

The proof of this requires a series of lemmas. It proceeds by discretization of space.

Assume $d = 2$ and r_n is given, satisfying (2.2). Let $\varepsilon \in (0, 1/9)$ be chosen in such a way that

$$(1-\varepsilon)\alpha((1-3\varepsilon)^2 - 2\varepsilon) > 1+\varepsilon. \tag{2.3}$$

Divide $B(1)$ into squares of side εr_n; actually we should use squares of side $1/\lfloor(1/\varepsilon r_n)\rfloor$ so they fit exactly, but to ease notation we shall ignore this minor technicality and assume/pretend that $1/(\varepsilon r_n)$ is an integer for all n.

Let $\mathcal{L}_n$ be the set of centres of these squares (a finite lattice). Then $|\mathcal{L}_n| = \Theta(n/\log n)$.

List the squares as $Q_i, 1 \leq i \leq |\mathcal{L}_n|$, and the corresponding centres of squares (i.e., the elements of $\mathcal{L}_n$) as $q_i, 1 \leq i \leq |\mathcal{L}_n|$.

Given $U \subset \mathcal{P}_n$, let us say $q_i \in \mathcal{L}_n$ is *U-occupied* if $U \cap (Q_i) \neq \emptyset$. Let $\mathcal{O}_n(U)$ be the set of sites $q_i \in \mathcal{L}_n$ that are U-occupied.

Lemma 2.6. *Let $U \subset \mathcal{P}_n$ be such that $G(U, r_n) \in \mathcal{K}$. Then also $G(\mathcal{O}_n(U), r_n(1+2\varepsilon)) \in \mathcal{K}$.*

Proof. Suppose $x, y \in U$ with $\{x, y\}$ an edge of $G(U, r_n)$. Choose $q_i, q_j \in \mathcal{L}_n$ with $x \in Q_i$ and $y \in Q_j$. Then by the triangle inequality we have

$$\|q_i - q_j\| \leq \|q_i - x\| + \|x - y\| + \|y - q_j\| \leq r_n\varepsilon + r_n + r_n\varepsilon = r_n(1 + 2\varepsilon),$$

so either $i = j$ or $\{q_i, q_j\}$ is an edge of $G(\mathcal{O}_n(U), r_n(1 + 2\varepsilon))$.

Given $q_k, q_\ell \in \mathcal{O}_n(U)$, pick $u \in U \cap Q_k$ and $v \in U \cap Q_\ell$. Then there is a path in $G(U, r_n)$ from u to v and by the above, taking the box centres of the successive points in this path provides a path in $G(\mathcal{O}_n(U), r_n(1 + 2\varepsilon))$ from q_k to q_ℓ. Hence $G(\mathcal{O}_n(U), r_n(1 + 2\varepsilon))$. is connected, □

Let $\mathcal{A}_{n,m}$ denote the set of $\sigma \subset \mathcal{L}_n$ with m elements such that $G(\sigma, r_n(1 + 2\varepsilon)) \in \mathcal{K}$ (sometimes called 'lattice animals').

Let $\mathcal{A}^2_{n,m}$ be the set of $\sigma \in \mathcal{A}_{n,m}$ such that $\text{dist}(\sigma, \partial B(1)) > 2r_n$, i.e. all elements of σ are distant at least $2r_n$ from the boundary of $B(1)$.

Let $\mathcal{A}^1_{n,m}$ be the set of $\sigma \in \mathcal{A}_{n,m}$ such that σ is distant less than $2r_n$ from *just one edge* of $B(1)$.

Let $\mathcal{A}^0_{n,m} := \mathcal{A}^0_{n,m} \setminus (\mathcal{A}^2_{n,m} \cup \mathcal{A}^1_{n,m})$, the set of $\sigma \in \mathcal{A}_{n,m}$ such that σ is distant less than $2r_n$ from *two edges* of $B(1)$ (i.e. near a corner of $B(1)$).

The counting argument in the next lemma is sometimes called a *Peierls argument.*

Lemma 2.7. *Given $m \in \mathbb{N}$, there is constant $C = C(m)$ such that*

$$|\mathcal{A}_{n,m}| \leq C(n/\log n), \qquad |\mathcal{A}^1_{n,m}| \leq C(n/\log n)^{1/2}, \qquad |\mathcal{A}^0_{n,m}| \leq C$$

for all n.

Proof. Fix m. Consider how many ways there are to choose $\sigma \in \mathcal{A}_{n,m}$.

There are at most r_n^{-2} choices, and hence $O(n/\log n)$ choices, for the first element of σ in the lexicographic ordering. Having chosen the first element of σ, there are a bounded number of ways to choose the rest of σ.

Consider how many ways there are to choose $\sigma \in \mathcal{A}^1_{n,m}$. In this case there are $O(r_n^{-1}) = O((n/\log n)^{1/2})$ ways to choose the first element of σ (distant at most $2r_n$ from the boundary of $[0, 1]^2$), and then a bounded number of ways to choose the rest of σ.

Finally consider how many ways there are to choose $\sigma \in \mathcal{A}^0_{n,m}$. In this case there are $O(1)$ ways to choose the first element of σ, and then a bounded number of ways to choose the rest of σ. □

For $n \in \mathbb{N}$, let $\mathcal{K}_n(\mathcal{P}_n)$ be the collection of vertex sets of the components of $G(\mathcal{P}_n, r_n)$ (a partition of $\mathcal{P}_n$). Given $\sigma \subset \mathcal{L}_n$, let E_σ be the event that there exists $U \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\mathcal{O}_n(U) = \sigma$.

Lemma 2.8. *Assume r_n satisfy* (2.2) *and ε has been chosen to satisfy* (2.3)*. Let $m \in \mathbb{N}$. Then*

$$\sup_{\sigma \in \mathcal{A}^2_{n,m}} \mathbb{P}[E_\sigma] = O(n^{-(1+\varepsilon)}). \tag{2.4}$$

Also

$$\sup_{\sigma \in \mathcal{A}^1_{n,m}} \mathbb{P}[E_\sigma] \leq n^{-(1+\varepsilon)/2} \tag{2.5}$$

and

$$\sup_{\sigma\in\mathcal{A}^o_{n,m}} (\mathbb{P}[E_\sigma]) \le n^{-(1+\varepsilon)/4}. \tag{2.6}$$

Proof. Given $\sigma \in \mathcal{A}^2_{n,m}$, let q_i (respectively q_j) be the lexicographically first (resp. last) element of σ. Let D^-_σ be the part of $D(q_i, r_n(1-3\varepsilon))$ lying to the left of Q_i. Let D^+_σ be the part of $D(q_j, r_n(1-3\varepsilon))$ lying to the right of Q_j.

We claim that if E_σ occurs, then $\mathcal{P}_n(D^-_\sigma) = 0$ and $\mathcal{P}_n(D^+_\sigma) = 0$. Indeed, if E_σ occurs and $\mathcal{P}_n(D^-_\sigma) \neq 0$, we can choose $z \in \mathcal{P}_n \cap D^-_\sigma$, and also $U \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\mathcal{O}_n(U) = \sigma$, and also $y \in U \cap Q_i$. Then

$$\|z-y\| \le \|z-q_i\| + \|q_i - y\| \le r_n(1-3\varepsilon) + \varepsilon r_n < r_n,$$

so also $y \in U$, but then taking k such that $z \in q_k$, we have $k \in \mathcal{O}_n(U)$ but also q_k to the left of q_i, a contradiction. This shows the first part of the claim, and we can argue similarly for D^+_σ. By the claim,

$$\begin{aligned}\mathbb{P}[E_\sigma] &\le \mathbb{P}[\mathcal{P}_n(D^-_\sigma \cup D^+_\sigma) = 0] \\ &\le \exp(-n[\pi(r_n(1-3\varepsilon))^2 - 2\varepsilon r_n^2]) \\ &\le \exp\left[-n\left(\frac{\alpha \log n}{n}\right)((1-3\varepsilon)^2 - 2\varepsilon)\right]\end{aligned}$$

By (2.3), this is less than $n^{-1-\varepsilon}$, completing the proof of (2.4).

To prove (2.5). Take $\sigma \in \mathcal{A}^1_{n,m}$. Consider just the case where σ is near to the left edge of $B(1)$. Define D^+_σ as above. Then

$$\begin{aligned}\mathbb{P}[E_\sigma] &\le \mathbb{P}[\mathcal{P}_n(D^+_\sigma) = 0] \\ &\le \exp(-(n/2)\pi(r_n(1-3\varepsilon))^2 - 2\varepsilon r_n^2) \\ &\le \exp\left[-n\left(\frac{\alpha \log n}{2n}\right)((1-3\varepsilon)^2 - 2\varepsilon)(1-\varepsilon)\right]\end{aligned}$$

and by (2.3) this is less than $n^{-(1+\varepsilon)/2}$ completing the proof of (2.5).

The proof of (2.6) is similar. $\square$

Lemma 2.9. *Let $m \in \mathbb{N}$. Then $\mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n) : |\mathcal{O}_n(U)| = m] \to 0$ as $n \to \infty$.*

Proof. By Lemma 2.6, if $U \in \mathcal{K}_n(\mathcal{P}_n)$ with $|\mathcal{O}_n(U)| = m$, then $\mathcal{O}_n(U) \in \mathcal{A}_{n,m}$. Hence by Lemma 2.8,

$$\begin{aligned}\mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n) : |\mathcal{O}_n(U)| = m] &\le \sum_{\sigma\in\mathcal{A}_{n,m}} \mathbb{P}(E_\sigma) \\ &\le |\mathcal{A}^2_{n,m}| n^{-(1+\varepsilon)} + |\mathcal{A}^1_{n,m}| n^{-(1+\varepsilon)/2} + |\mathcal{A}^0_{n,m}| n^{-(1+\varepsilon)/4}\end{aligned}$$

and using Lemma 2.7 we find that this tends to zero. $\square$

For $U \subset \mathbb{R}^2$ define

$$U^r := \cup_{x\in U} D(x,r).$$

Write diam_∞ for diameter in the ℓ_∞ norm. For any bounded connected $A \subset \mathbb{R}^2$, $\mathrm{diam}_\infty(A)$ is the smallest possible side length of a rectilinear square containing A.

Lemma 2.10. *Let $Q = [-1/2, 1/2]^2$, $Q^o = (-1/2, 1/2)^2$ and $\partial Q = Q \setminus Q^o$. Let $r > 0$ and suppose U, V are disjoint finite nonempty subsets of Q^o such that no two distinct points x, y of $U \cup V$ satisfy $\partial D(x, r) \cap \partial D(y, r) \cap \partial Q \neq \emptyset$. Suppose the sets U^r and V^r are connected and $U^r \cap V^r = \emptyset$.*

Then there exists a connected set $\Gamma \subset Q^o \cap \partial(U^r)$ with

$$\mathrm{diam}_\infty(\Gamma) \geq \min(\mathrm{diam}_\infty(U^r), \mathrm{diam}_\infty(V^r)). \tag{2.7}$$

Proof. Given $x \in U$, let us define an *exposed arc* of the circle $\partial D(x, r)$ to be a portion of this circle that is not covered by any of the other disks, i.e. a connected component of $(\partial D(x, r)) \setminus \cup_{y \in U \setminus \{x\}} D(y, r)$.

Then $\partial_Q(U^r)$ (the boundary of U^r relative to Q) consists of all the exposed arcs of the circles $\partial D(x, r), x \in U$, together with some vertices of degree 2 (wherever two exposed arcs meet) or degree 1 (wherever an exposed arc meets ∂Q). The exposed arcs and vertices can be seen as a finite plane graph with all vertices of degree 1 or 2.

Such a graph must split into a finite collection of cycles, each of which is a Jordan curve, along with some curves (paths) which start and end at points in ∂Q. These cycles and curves are all disjoint from each other. Denote these cycles and curves by $\Gamma_1, \ldots, \Gamma_m$.

The set V^r lies in a single component of the complement of $\partial_Q(U^r)$ and the boundary of this component (relative to Q) is one of the curves $\Gamma_1, \ldots, \Gamma_m$, without loss of generality Γ_1. Then taking $\Gamma = \Gamma_1$, we have that any continuous path in Q^o from V^r to U^r must pass through Γ.

If $\mathrm{diam}_\infty(\Gamma) < \min(\mathrm{diam}_\infty(U^r), \mathrm{diam}_\infty(V^r))$, then we can find a closed rectilinear square S containing Γ of side $\mathrm{diam}_\infty(\Gamma)$, but also can find $x \in U^r \cap Q^o \setminus S$ and $y \in V^r \cap Q^o \setminus S$. But then we could find a continuous path in Q^o from x to y avoiding S, contradicting our earlier conclusion that any path from U^r to V^r must pass through Γ. Therefore (2.7) holds. □

Given $K \in \mathbb{N}$, let $F_K(n)$ be the event that there exist distinct $U, V \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\min(|\mathcal{O}_n(U)|, |\mathcal{O}_n(V)|) \geq K$.

Lemma 2.11. *There exists $K \in \mathbb{N}$ such that $\mathbb{P}[F_K(n)] \to 0$ as $n \to \infty$.*

Proof. Suppose $F_K(n)$ occurs. Then there exist distinct $U, V \in \mathcal{K}_n(\mathcal{P}_n)$, such that $\min(|\mathcal{O}_n(U)|, |\mathcal{O}_n(V)|) \geq K$. Let $U' := U^{r_n/2}$, and $V' := V^{r_n/2}$. By Lemma 2.10, there is a connected set $\Gamma \subset \partial U' \cap (0, 1)^2$ with $\mathrm{diam}_\infty(\Gamma) \geq \min(\mathrm{diam}_\infty(U'), \mathrm{diam}_\infty(V'))$.

Let τ be the set of $q_i \in \mathcal{L}_n$ such that $Q_i \cap \Gamma \neq \emptyset$.

Then τ is $*$-connected in $\mathcal{L}_n$, i.e. for any two sites x, y in τ, there is a path $(x_0, x_1, \ldots, x_k)$ with $x_0 = x$, $x_k = y$ and $\|x_i - x_{i-1}\|_\infty = \varepsilon r_n$ for $1 \leq i \leq k$.

Also, for each $q_i \in \tau$ we claim $\mathcal{P}_n(Q_i) = 0$. Indeed, any such Q_i contains part of the boundary of U', so if there were a point of $\mathcal{P}_n$ in Q_i it would be distant at most $r_n((1/2) + 2\varepsilon)$ from U and therefore would actually be in U so Q_i would *not* include any of the boundary of U', a contradiction.

Next we claim the isoperimetric inequality $|\tau| \geq K^{1/2}$. To see this note that $\cup_{i:q_i \in \mathcal{O}_n(U)} Q_i$ is contained in U', and therefore with Leb(.) denoting area,

$$\mathrm{Leb}(U') \geq \mathrm{Leb}(\cup_{i:q_i \in \mathcal{O}_n(U)} Q_i) \geq K\varepsilon^2 r_n^2.$$

Also $\mathrm{diam}_\infty(U') \geq (\mathrm{Leb}(U'))^{1/2}$, and hence $\mathrm{diam}_\infty(U') \geq K^{1/2}\varepsilon r_n$. Likewise $\mathrm{diam}_\infty(V') \geq K^{1/2}\varepsilon r_n$ and hence $\mathrm{diam}_\infty(\Gamma) \geq K^{1/2}\varepsilon r_n$. Since $|\tau| \geq \mathrm{diam}_\infty(\Gamma)/(\varepsilon r_n)$, we have the claim.

Let $\mathcal{A}'_{n,m}$ be the set of $*$-connected subsets of $\mathcal{L}_n$ with m elements. By a similar argument to the proof of Lemma 2.7 (see also [5, Lemma 9.3]), there are finite constants γ and C (we can take $\gamma = 2^8$ for example) such that

$$|\mathcal{A}'_{n,m}| \leq C(n/\log n)\gamma^m.$$

Set

$$\phi_n := \mathbb{P}[\mathcal{P}_n(Q_i) = 0] = \exp(-n(\varepsilon r_n)^2) = \exp[-\varepsilon^2(\alpha/\pi)(\log n)1],$$

where the last line comes from (2.2). Then

$$\begin{aligned}\mathbb{P}[F_K(n)] &\leq \sum_{m \geq K^{1/2}} C(n/\log n)\gamma^m \phi_n^m \\ &\leq 2C(n/\log n)(\gamma n^{-\varepsilon^2/\pi})^{K^{1/2}} \\ &= 2C\gamma^{K^{1/2}} n^{1-\varepsilon^2\pi^{-1}K^{1/2}}/\log n\end{aligned}$$

which tends to zero provided K is chosen large enough so that $(\varepsilon^2/\pi)K^{1/2} > 1$. □

Proof of Theorem 2.5. Choose $K \in \mathbb{N}$ as in Lemma 2.11. Then by Lemma 2.6 we have that

$$\begin{aligned}&\mathbb{P}[G(\mathcal{P}_n, r_n) \notin \mathcal{K}] \leq \mathbb{P}[\exists U, V \in \mathcal{K}_n(\mathcal{P}_n), U \neq V] \\ &\leq \left(\sum_{m=1}^{K} \mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n), |\mathcal{O}_n(U)| = m]\right) + \mathbb{P}[F_K(n)].\end{aligned}$$

By Lemmas 2.9. and 2.11, this tends to zero. □

Exercise 2.12. Show that if $(r_n)_{n\in\mathbb{N}}$ satisfy (2.2), then $\mathbb{P}[G(\mathcal{X}_n, r_n) \in \mathcal{K}] \to 1$.

Together with Exercise 2.4, this shows that Theorem 2.1 holds with ρ'_n replaced by ρ_n.

Exercise 2.13. Let $k \in \mathbb{N}$. A graph G with more than $k+1$ vertices is said to be k-vertex-connected if for any two vertices there are at least k vertex-disjoint paths between them. Equivalently, it is said to be k-vertex-connected if there is no way to remove $k-1$ vertices that disconnects the graph.

Let $\rho_n^{(k)}$ be the minimum r such that $G(\mathcal{P}_n, r)$ is k-vertex-connected. Show that for $d = 2$, (2.1) holds with ρ'_n replaced by $\rho_n^{(k)}$.

3 Percolative ingredients

This chapter contains further preliminaries which will be useful in proving results about large components of random geometric graphs.

3.1 Bernoulli and k-dependent percolation

Motivated mainly by the study of random physical media, percolation theory is the study of connectivity properties of random sets in space. Lattice percolation in particular has been much studied. See e.g. [1, 2]. In the present context, its importance arises from various dicretizations of continuum processes. The most useful lattice percolation model for us is *site percolation on the triangular lattice*.

The triangular lattice is the the graph $G(\mathbb{T},1)$, where $\mathbb{T}$ is the set in $\mathbb{R}^2$ defined by $\mathbb{T} := \{m(1,0)+n(1/2,\sqrt{3}/2) : m,n \in \mathbb{Z}\}$. *Site percolation* on $\mathbb{T}$ is defined as follows. Given $p \in [0,1]$, let $Z^p = (Z^p_x, x \in \mathbb{T})$ be a family of mutually independent Bernoulli(p) random variables. The sites $x \in \mathbb{T}$ for which $Z^p_x = 1$ are denoted *open* and the sites $x \in \mathbb{T}$ for which $Z^p_x = 0$ are denoted *closed*. Let $\mathcal{B}_p$ denote the (random) set of open sites in $\mathbb{T}$; here $\mathcal{B}$ stands for 'Bernoulli' and we shall sometimes refer either to Z^p or to $\mathcal{B}_p$ as a *Bernoulli process*.

The components of the graph $G(\mathcal{B},1)$ (i.e. the maximal connected subsets of $\mathcal{B}$) are denoted the *open clusters* (or just *clusters*) in $\mathcal{B}$. We shall avoid using the term 'cluster' for random geometric graphs in the continuum.

The *cluster at the origin* for $\mathcal{B}_p$ is the open cluster in $\mathcal{B}_p$ containing the origin $\mathbf{o}$ (or the empty set if $\mathbf{o}$ is closed). Let $\theta(p)$ denote the probability that this cluster is infinite. Then $\theta(p)$ is nondecreasing in p, so there is a critical value p_{c} of p such that if $p < p_{\mathrm{c}}$ then $\theta(p) = 0$ and if $p > p_{\mathrm{c}}$ then $\theta(p) > 0$. If $p < p_{\mathrm{c}}$ then the Bernoulli process $\mathcal{B}_p$ is *subcritical*, while if $p > p_{\mathrm{c}}$ the Bernoulli process $\mathcal{B}_p$ is *supercritical*. It is well known that

$$0 < p_{\mathrm{c}} < 1, \tag{3.1}$$

and also

$$\theta(p) \to 1 \quad \text{as } p \to 1. \tag{3.2}$$

These results carry over to many other lattice percolation models in 2 or more dimensions; see e.g. [2] or [1]. More specific to the triangular lattice $\mathbb{T}$ is the deeper fact that

$$p_c = 1/2. \tag{3.3}$$

Also of use to us later is k-dependent percolation, where $k \in \mathbb{N}$ is fixed. We say that $Z = (Z_x, x \in \mathbb{T})$ is a (weakly) *k-dependent Bernoulli random field* on $\mathbb{T}$, if it is a collection of $\{0,1\}$-valued random variables in the same probability space, such that for any finite $A \subset \mathbb{T}$ with $\|x-y\| > k$ for any distinct $x, y \in A$, the random variables $(Z_x, x \in A)$ are mutually independent. Let $C_0(Z)$, the cluster at the origin for Z, be defined the same way as before.

Theorem 3.1. *Let $k \in \mathbb{N}$. There exists $p^*_c(k) \in (0,1)$ such that for any k-dependent Bernoulli random field Z with $\mathbb{P}[Z_x = 1] \geq p_c$ for all $x \in \mathbb{T}$, we have $\mathbb{P}[|C_0(Z)| = \infty] > 0$.*

Remark. Clearly this implies the second inequality of (3.1). We do not prove the rest of (3.1), (3.2) and (3.3) here.

Proof of Theorem 3.1. Let $p \in (0,1)$ and let Z be a k-dependent Bernoulli random field on $\mathbb{T}$ with $\mathbb{P}[Z_x = 1] \geq p$ for all $x \in \mathbb{T}$.

For each $x \in \mathbb{T}$ let H_x be the hexagon given by the Voronoi cell of x with respect to $\mathbb{T}$, i.e. the closure of the set of points $y \in \mathbb{R}^2$ lying closer to x than to any other point of $\mathbb{T}$. Let us say H_x is *occupied* if $Z_x = 1$ and *vacant* if $Z_x = 0$.

If C_0 is finite, define the *exterior boundary* ΔC_0 as follows. Let $C^*_0 := \cup_{x \in C_0} H_x$. Then ∂C^*_0 is a union of edges, each of which is the boundary between two neighbouring hexagons, one of which is occupied and the other is vacant.

The edges of ∂C^*_0 can be viewed as the edges of a finite graph (in fact, a subgraph of the dual lattice to $G(\mathbb{T},1)$), where each vertex has degree 2. Therefore this graph consists of a finite collection of cycles. Thus ∂C_0^* consists of a finite collection of disjoint Jordan curves. Since C_0^* is connected, one of these Jordan curves contains all the others: we call this the *exterior boundary* of C_0^* and denote it by $\partial_{\mathrm{ext}} C_0^*$.

Each edge of the polygon $\partial_{\mathrm{ext}} C_0^*$ has an occupied hexagon on one side of it, and a vacant hexagon on the other side. Let ΔC_0 be the union of $x \in \mathbb{T}$ such that H_x is a vacant hexagon adjacent to an edge of $\partial_{\mathrm{ext}} C_0$. Since $\partial_{\mathrm{ext}} C_0^*$ is connected, so ΔC_0 is also connected (i.e. $G(\Delta C_0, 1)$ is connected).

Thus if $|C_0(Z)| < \infty$, then for some m there exists a connected set of m closed sites in $\mathbb{T}$ including at least one vertex distant at most m from $\mathbf{o}$.

Let $\mathcal{A}_m$ be the class of connected subsets of $\mathbb{T}$ with m elements, at least one of them distant at most m from $\mathbf{o}$ (here we say $\sigma \subset \mathbb{T}$ is connected if $G(\sigma, 1)$ is connected). By a similar argument to Lemma 2.7, there are finite constants γ and c such that $|\mathcal{A}_m| \leq cm^2\gamma^m$ for all m. Moreover, setting $\beta := \beta(k) := |\{z \in \mathbb{T} : \|z\| \leq k\}|$, for any $\sigma \in \mathcal{A}_m$ we can find a collection of at least $\lceil m/\beta \rceil$ elements of σ, all of them distant more than k from each other (**Exercise**). Note also that $\beta \leq (k+1)^2$ by a packing argument.

Therefore using the k-dependence, and the union bound we obtain that

$$\begin{aligned}\mathbb{P}[|C_0| < \infty] &\leq \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{A}_m} \mathbb{P}[\cap_{z \in \sigma} \{Z_z = 0\}] \\ &\leq \sum_m (cm^2\gamma^m)(1-p)^{m/\beta} \\ &\leq c \sum_m m^2 (\gamma q^{1/\beta})^m,\end{aligned}$$

where we set $q = 1 - p$. By taking q small enough (i.e. p close enough to 1) we can arrange that $\gamma q^{1/\beta} < 1$, so that the above sum converges. Then by taking q even smaller we can arrange that the sum is less than 1, and this gives the result. □

3.2 Continuum percolation

In its simplest form, continuum percolation can loosely be characterized as the study of large components of the infinite graph $G(\mathcal{H}_\lambda; 1)$. Equivalently, one may study the connected components of the set $\mathcal{H}_\lambda^+$, where for $\mathcal{X} \subset \mathbb{R}^2$ we set

$$\mathcal{X}^+ := \cup_{x \in \mathcal{X}} D(x, \frac{1}{2}).$$

Continuum percolation is of interest in its own right; for example, the balls centred at the points of $\mathcal{H}_\lambda$ could represent pores in a piece of rock, or regions accessible to radio transmitters. The basic continuum percolation model readily lends itself to generalizations such as balls of random radius, but we shall concentrate here on the basic model.

For $\lambda, s > 0$, as before let $B(s) := [-s/2, s/2]^2$ and let $\mathcal{H}_{\lambda,s}$ be a homogeneous Poisson process of intensity λ on $B(s)$.

Observe that $G(\mathcal{H}_{\lambda,s}, 1)$ is isomorphic to $G(s^{-1}\mathcal{H}_{\lambda,s}, s^{-1})$ (where for $a \in \mathbb{R}$ and $\mathcal{X} \subset \mathbb{R}^2$ we set $a\mathcal{X} = \{ax : x \in \mathcal{X}\}$), and by the representation (1.1) we have

$$s^{-1}\mathcal{H}_{\lambda,s} = \{\xi_1, \ldots, \xi_{N_{\lambda s^2}}\} = \mathcal{H}_{1,\lambda s^2} = \mathcal{P}_{\lambda s^2}.$$

Thus, taking $n = \lambda s^2$ and $r_n = s^{-1}$, we have that $G(\mathcal{H}_{\lambda,s}, 1)$ is isomorphic to $G(\mathcal{P}_{\lambda s^2}, s^{-1}) = G(\mathcal{P}_n, r_n)$ with r_n taken so $nr_n^2 = \lambda$.

Let $\mathcal{H}_{\lambda,0}$ denote the point process $\mathcal{H}_\lambda \cup \{\mathbf{o}\}$, where $\mathbf{o}$ is the origin in $\mathbb{R}^2$. For $k \in \mathbb{N}$, let $p_k(\lambda)$ denote the probability that the component of $G(\mathcal{H}_{\lambda,0}; 1)$ containing the origin is of order k; see (3.7) below for a formula for $p_k(\lambda)$. The *continuum percolation probability* $p_\infty(\lambda)$ is the probability that $\mathbf{o}$ lies in an infinite component of the graph $G(\mathcal{H}_{\lambda,0}; 1)$, and is defined by

$$p_\infty(\lambda) = 1 - \sum_{k=1}^{\infty} p_k(\lambda).$$

Exercise 3.2. Suppose $0 < \lambda < \lambda'$. Show that $p_\infty(\lambda) \leq p_\infty(\lambda')$.

The *critical value* (continuum percolation threshold) λ_{c} is defined by

$$\lambda_{\mathrm{c}} = \inf\{\lambda > 0 : p_\infty(\lambda) > 0\}. \tag{3.4}$$

The value of λ_{c} depends on the dimension d. The following result is fundamental.

Theorem 3.3. *We have* $0 < \lambda_{\mathrm{c}} < \infty$*, and* $p_\infty(\lambda) \to 1$ *as* $\lambda \to \infty$.

Proof. Let C_0 denote the vertex set of the component containing $\mathbf{o}$ in $G(\mathcal{H}_{\lambda,0}, 1)$. Then $p_\infty(\lambda) = \mathbb{P}[|C_0| = \infty]$.

Divide $\mathbb{R}^2$ into regular hexagons $H_1, H_2, \ldots$ of side 1, one of them centred at $\mathbf{o}$. Then $\mathcal{H}_\lambda(H_i)$ is Poisson with parameter λA for each i, where A is the area of each hexagon. Also for $i \neq j$, if $x \in H_i, y \in H_j$ with $\|x - y\| \leq 1$, then H_i and H_j must be adjacent. Let us say H_i is *open* iff $\mathcal{H}(H_i) > 0$.

If $|C_0| = \infty$ there must be an infinite path of open regular hexagons from $\mathbf{o}$. Therefore if $\lambda > \lambda_c$, we have $1 - \exp(-\lambda A) \geq p_c$. Hence $\lambda \geq A^{-1} \log(1/(1 - p_c))$ for all $\lambda > \lambda_c$, so $\lambda_c \geq A^{-1} \log(1/(1 - p_c))$. Since $p_c > 0$ by (3.1) we have $\lambda_c > 0$.

Now instead divide $\mathbb{R}^2$ into hexagons, denoted H'_i, of side a, where a is chosen so that $\mathrm{diam}(H'_i \cup H'_j) = 1$ for neighbouring hexagons. Denote by A' the area of the new hexagons. With the new size of hexagons, if there is an infinite path of open hexagons starting at $\mathbf{o}$, then $|C_0| = \infty$. Thus

$$p_\infty(\lambda) \geq \theta(1 - \exp(-\lambda A')). \tag{3.5}$$

Therefore if $1 > \exp(-\lambda A') > p_c$, then $\lambda \geq \lambda_c$, and $p_c < 1$ by (3.1). Hence $\lambda_c \leq (A')^{-1} \log(1/(1 - p_c)) < \infty$.

Finally, as $\lambda \to \infty$ we have $(1 - \exp(-\lambda A')) \to 1$, and therefore $p_\infty(\lambda) \to 1$ by (3.5) and (3.2). ■

Exercise 3.4. Use the above proof, and (3.3), to find upper and lower bounds for λ_c.

Exercise 3.5. Show that as a function of λ, the percolation probability $p_\infty(\lambda)$ is right continuous.

In fact it is known that $p_\infty(\lambda)$ is *continuous* in λ. Moreover, it is known [3] that $p_\infty(\lambda)$ is infinitely differentiable in λ, except at $\lambda = \lambda_c$.

Exact values for λ_{c} or for $p_\infty(\lambda)$ are not known. Simulation studies, such as Quintanilla *et al.* (2000), indicate that $1 - \mathrm{e}^{-\lambda_{\mathrm{c}}\pi/4} \approx 0.676$ so that $\lambda_{\mathrm{c}} \approx 1.44$, while rigorous bounds $0.696 < \lambda_{\mathrm{c}} < 3.372$ are given in Meester and Roy (1996, Chapter 3.9).

It is known that $p_\infty(\lambda_{\mathrm{c}}) = 0$ (we shall prove this later). It is not known if this holds in all dimensions.

Next, we give a formula for $p_k(\lambda)$.

Theorem 3.6. (Formula for $p_k(\lambda)$) *Given $x_0, x_1, \ldots, x_k \in \mathbb{R}^2$, let the function $h(x_0, x_1, \ldots, x_k)$ take the value 1 if $G(\{x_0, x_1, \ldots, x_k\}; 1)$ is connected. Otherwise, set $h(x_0, x_1, \ldots, x_k) = 0$. Also, set*

$$A(x_0, x_1, \ldots, x_k) := \mathrm{Leb}\left(\bigcup_{i=0}^{k} B(x_i; 1)\right), \tag{3.6}$$

the area of the union of balls of radius 1 centred at $x_0, x_1, \ldots, x_k$. Then, for $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} p_{k+1}(\lambda) &= (1/k!)\lambda^k \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} h(\mathbf{o}, x_1, \ldots, x_k) \\ &\quad \times \exp(-\lambda A(\mathbf{o}, x_1, \ldots, x_k)) \mathrm{d}x_1 \cdots \mathrm{d}x_k. \end{aligned} \tag{3.7}$$

Proof. Set $x_0 = \mathbf{o}$. Define $f(x_1, \ldots, x_k, \mathcal{X})$ to be $h(x_0, x_1, \ldots, x_k)\mathbf{1}\{\mathcal{X}(\cup_{i=0}^k B(x_i, 1)) = 0\}$. Set $s = 2k + 4$. By the Mecke formula (Theorem 1.11),

$$\begin{aligned} p_{k+1}(\lambda) &= k!^{-1}\mathbb{E} \sum_{x_1,\ldots,x_k \in \mathcal{H}_\lambda}^{\neq} f(x_1, \ldots, x_k, \mathcal{H}_\lambda \setminus \{x_1, \ldots, x_k\}) \\ &= k!^{-1}\mathbb{E} \sum_{x_1,\ldots,x_k \in \mathcal{H}_{\lambda,s}}^{\neq} f(x_1, \ldots, x_k, \mathcal{H}_{\lambda,s} \setminus \{x_1, \ldots, x_k\}) \\ &= k!^{-1}\lambda^k \int_{B(s)} dx_1 \cdots \int_{B(s)} dx_k h(x_0, x_1, \ldots, x_k) \mathbb{P}[\mathcal{H}_{\lambda,s}(\cup_{i=0}^k B(x_i, 1)) = \emptyset] \end{aligned}$$

which yields the formula. □

We say a real-valued function f, defined on locally finite point configurations $\mathcal{X} \subset \mathbb{R}^d$, is *increasing* if $f(\mathcal{X}) \leq f(\mathcal{Y})$ whenever $\mathcal{X} \subset \mathcal{Y}$. We say f is *decreasing* if $-f$ is increasing. Given $\lambda > 0$, we say E is an increasing (resp. decreasing) event on $\mathcal{H}_\lambda$ if $\mathbf{1}_E$ is an increasing (resp. decreasing) function of $\mathcal{H}_\lambda$.

Theorem 3.7 (Harris-FKG inequality)**.** *Suppose f, g are measurable bounded increasing real-valued functions defined on locally finite point configurations in $\mathbb{R}^2$. Then $\mathbb{E}[f(\mathcal{H}_\lambda)g(\mathcal{H}_\lambda)] \geq \mathbb{E}[f(\mathcal{H}_\lambda)]\mathbb{E}[g(\mathcal{H}_\lambda)]$. The same inequality holds if f and g are both decreasing.*

Proof. We prove this only for functions of the following form (which is the only case we shall use). See any of [1], [7] or [4] for proof in more generality. Let $A_1, A_2, A_3, B_1, B_2, B_3$ be sets sets in $\mathbb{R}^2$ that are either squares, or disks, or line segments. Let f be the indicator of the statement that there is a path in $\mathcal{H}_\lambda^+ \cap A_3$ from A_1 to A_2, and define g while g is the indicator of the statement that there is a path in $\mathcal{H}_{\lambda,s}^+ \cap B_3$ from B_1 to B_2.

Divide $\mathbb{R}^2$ into squares of side 2^{-n}. Let $\mathcal{L}_n$ be the set of centres of those squares distant at most 1 from $A_3 \cup B_3$ (a finite lattice).

List these squares as $Q_i, 1 \leq i \leq |\mathcal{L}_n|$, and the corresponding centres of squares (i.e., the elements of $\mathcal{L}_n$) as $q_i, 1 \leq i \leq |\mathcal{L}_n|$.

Let us say $q_i \in \mathcal{L}_n$ is *occupied* if $\mathcal{H}_{\lambda,s}(Q_i) > 0$. Let $\mathcal{O}_n$ be the (random) set of sites $q_i \in \mathcal{L}_n$ that are occupied and let $\mathcal{O}_n^+ := \cup_{q \in \mathcal{O}_n} D(q, 1/2)$.

Let $f_n(\mathcal{H}_\lambda)$ be the indicator that there is a path in $\mathcal{O}_n^+ \cap A_3$ from A_1 to A_2, and define g_n similarly. Then $f_n(\mathcal{H}_\lambda) \to f(\mathcal{H}_\lambda)$ and $g_n(\mathcal{H}_\lambda) \to g(\mathcal{H}_\lambda)$ almost surely, as $n \to \infty$. Also $f_n(\mathcal{H}_\lambda)$ and $f_n(\mathcal{H}_\lambda)$ are increasing functions defined on a finite family of Bernoulli random variables, and therefore we have $\mathbb{E}[f_n(\mathcal{H}_\lambda)g_n(\mathcal{H}_\lambda)] \geq \mathbb{E}[f_n(\mathcal{H}_\lambda)]\mathbb{E}[g_n(\mathcal{H}_\lambda)]$ by the 'usual' Harris-FKG inequality. Taking the limit and using dominated convergence gives the result for f, g increasing, and it is then easy to deduce the result for f, g decreasing. □

Corollary 3.8 (Square Root trick)**.** *Let $\lambda > 0$, $k \in \mathbb{N}$, $\varepsilon \in (0, 1)$. Suppose for $i = 1, \ldots, k$ we have increasing events A_i defined on $\mathcal{H}_\lambda$. such that $\mathbb{P}[\cup_{i=1}^k A_i] > 1 - \varepsilon$.*

Then $\max_{1 \leq i \leq k} \mathbb{P}[A_i] > 1 - \varepsilon^{1/k}$.

Proof. Set $M = \max_{1 \leq i \leq k} \mathbb{P}[A_i]$. The events A_i^c are all decreasing, so by the FKG inequality

$$\varepsilon > \mathbb{P}[\cap_{i=1}^k A_i^c] \geq \prod_{i=1}^k \mathbb{P}[A_i^c] \geq (1 - M)^k,$$

so that $1 - M < \varepsilon^{1/k}$ and $M > 1 - \varepsilon^{1/k}$. □

4 Uniqueness of the infinite component

Fix $\lambda > 0$ and let N be the number of infinite components of the graph $G(\mathcal{H}_\lambda; 1)$.

Exercise 4.1. Show that if $p_\infty(\lambda) = 0$, then $\mathbb{P}[N = 0] = 1$.

Our main result in this section deals with the other case, where $p_\infty(\lambda) > 0$.

Theorem 4.2. *Suppose $p_\infty(\lambda) > 0$. Then $\mathbb{P}[N = 1] = 1$.*

This is important for what comes later, so we shall give a proof (a continuum version of the classic Burton-Keane proof, which generalizes easily to higher dimensions). We prove it from first principles without appealing to any general ergodic theorem. We prepare with some lemmas; assume from now on that λ is such that $p_\infty(\lambda) > 0$.

For $n \in \mathbb{N}$, let $D_n := D(\mathbf{o}, n)$. Let N_n be the number of infinite components of $G(\mathcal{H}_\lambda, 1)$ having at least one vertex in D_n. For $m \in \mathbb{N}$ with $m > n$, let $U_{n,m}$ be the event that there is a unique component of $G(\mathcal{H}_\lambda \cap D_{m+1}, 1)$ meeting both D_n and D_m^c.

Exercise 4.3. Prove that $\mathbb{P}[U_{n,m} \triangle \{N_n = 1\}] \to 0$ as $m \to \infty$.

Hint: if $\{N_n = 1\} \setminus U_{n,m}$ occurs then there is either a finite component of $G(\mathcal{H}_\lambda, 1)$ that intersects both D_n and D_m^c, or a pair $x, y \in D_n \cap \mathcal{H}_\lambda$ such that there is a path from x to y but no such path within D_{m+1}. If $U_{n,m} \setminus \{N_n = 1\}$ occurs then there is a finite component that intersects both D_n and D_m^c.

Lemma 4.4. *Either $\mathbb{P}[N = 1] = 0$, or $\mathbb{P}[N = 1] = 1$.*

Proof. (Sketch) Clearly $N_n \to N$ as $n \to \infty$. Thus $|\mathbf{1}_{\{N_n=1\}} - \mathbf{1}_{\{N=1\}}| \to 0$ as $n \to \infty$, almost surely. Taking expectations and using the Dominated Convergence theorem yields

$$\mathbb{P}[\{N_n = 1\} \triangle \{N = 1\}] \to 0 \qquad \text{as } n \to \infty.$$

Combined with Exercise 4.3 this shows that given $\varepsilon > 0$ we can find $n, m \in \mathbb{N}$ with $n < m$ such that $\mathbb{P}[U_{n,m} \triangle \{N = 1\}] < \varepsilon$.

Now suppose $\mathbb{P}[N = 1] \in (0, 1)$. Fix n, m with $n < m$ such that $\mathbb{P}[U_{n,m}] \approx \mathbb{P}[N = 1]$.

Next define event $U'_{n,m}$ in the same way but in terms of the shifted Poisson process $\mathcal{H}_\lambda + (3m, 0)$. Then $\mathbb{P}[U'_{n,m}] \approx \mathbb{P}[N = 1]$ since the event $\{N = \infty\}$ is shift-invariant.

Moreover $\mathbb{P}[U_{n,m} \cap U'_{n,m}] \approx \mathbb{P}[N = \infty]$. But also $U_{n,m}$ and $U'_{n,m}$ are independent, so

$$\mathbb{P}[N = 1] \approx \mathbb{P}[U_{n,m} \cap U'_{n,m}] = \mathbb{P}[U_{n,m}]\mathbb{P}[U'_{n,m}] \approx \mathbb{P}[N = 1]^2$$

and this yields a contradiction. □

Exercise 4.5. Fill in the details of the above sketch proof.

Lemma 4.6. $\mathbb{P}[N = 2] = 0$.

Proof. By contradiction. Suppose $\mathbb{P}[N = 2] > 0$. If $N = 2$ let C_∞ and C'_∞ denote the vertex sets of the two infinite components of $G(\mathcal{H}_\lambda, 1)$. For $n \in \mathbb{N}$ define the event

$$E_n := \{N = 2\} \cap \{C_\infty \cap D_n \neq \emptyset\} \cap \{C'_\infty \cap D_n \neq \emptyset\}.$$

Then $E_n \subset E_{n+1}$ and $\cup_n E_n = \{N = 2\}$. Therefore we can and do choose m such that $\mathbb{P}[E_m] > 0$.

Now let $\mathcal{H}'_\lambda$ be a new point process defined as follows.

Step 1: Generate $\mathcal{H}_\lambda$.

Step 2: Retain each point of $\mathcal{H}_\lambda \cap D_m$ with probability $1/2$, and discard it otherwise, independently of evertything else. (Retain all points of $\mathcal{H}_\lambda \setminus D_m$).

Step 3: Superimpose a further (independent) homogeneous Poisson process Z intensity $\lambda/2$ in D_n on top of the retained points from the previous step.

Then we claim $\mathcal{H}'_\lambda$ is another homogeneous Poisson process of intensity λ in $\mathbb{R}^d$. Indeed, by the Thinning theorem, the retained points in D_m form a homogeneous Poisson process of intensity $\lambda/2$ there, and by the superposition theorem these, together with the added points from Step 3, make a new homogeneous Poisson process of intensity λ in D_n.

Let F be the event that in Step 2 above, all points of $\mathcal{H}_\lambda \cap D_m$ are retained.

Let G be the event that in Step 3 above, the union of balls of radius $1/2$ on points of Z contains the whole of D_n.

Then the event $E_m \cap F \cap G$ has strictly positive probability. But if this event occurs, then $[N' = 1]$, where N' is the number of infinite components in $G(\mathcal{H}'_\lambda, 1)$. Therefore $\mathbb{P}[N = 1] = \mathbb{P}[N' = 1] > 0$, and hence $\mathbb{P}[N = 1] = 1$ by the previous lemma. Hence $\mathbb{P}[N = 2] = 0$, which is a contradiction. □

Given $k \in \mathbb{N}$, define a *k-graph* to be a finite graph with all of its vertices in D_k. We refer to vertices of such a graph that lie in $D_k \setminus D_{k-1}$ as *exit points*. Given a connected k-graph G, let us say a vertex x of G is a *branch-point* if removing x splits G into exactly three components, each of which contains at least one exit point.

Lemma 4.7. *Let $k, j \in \mathbb{N}$. If G is a connected k-graph with j branch-points, then G has at least $j + 2$ exit points.*

Proof. By induction on j. By definition, if G has a single branch-point then it has at least 3 exit points, so the result holds for $j = 1$. Let $r \in \mathbb{N}$ with $r \geq 2$, and suppose the result holds for all $j < r$.

Suppose G is a connected k-graph with r branch points. Then we can (and do) pick a pair of neighbouring branch-points x_1, x_2 in G, i.e. branch points such that there is a path from x_1 to x_2 not passing through any other branch point.

Denote by $\langle x_1, x_2 \rangle$ the intersection of the component of $G \setminus \{x_1\}$ containing x_2, with the component of $G \setminus \{x_2\}$ containing x_1. This could consist of a single edge (if x_1 and x_2 are adjacent vertices in G) or be a larger collection of vertices and edges.

Removing $\langle x_1, x_2 \rangle$ from G leaves a graph with two components, denoted G_1 and G_2 (both of which are k-graphs), with $x_i \in G_i$ for $i = 1, 2$. For $i = 1, 2$ let j_i be the number of branch-points of G_i.

Then $j_1 + j_2 = r - 2$, since x and y are now no longer branch-points but every other branch-point of G is a branch-point either of G_1 or of G_2.

For $i = 1, 2$, G_i has at least two exit points since it contains two of the components of $G \setminus \{x_i\}$. Moreover, if $j_i \geq 1$ then by the inductive hypothesis G_i has at least $j_i + 2$ exit points.

Hence the total number of exit points of G is at least $(j_1 + 2) + (j_2 + 2) = r + 2$, completing the induction. □

Proof of Theorem 4.2. Assume $p_\infty(\lambda) > 0$; then $\mathbb{P}[N \geq 1] > 0$. By Lemma 4.6 we have $\mathbb{P}[N = 2] = 0$, so it suffices to prove $\mathbb{P}[N \geq 3] = 0$. We shall prove this by contradiction, so assume $\mathbb{P}[N \geq 3] > 0$.

Given $n \in \mathbb{N}$, and $x \in \mathbb{R}^2$, let $E_n(x)$ be the event that there are at least 3 infinite components of $G(\mathcal{H}_\lambda, 1)$ having vertex sets intersecting $D_n(x)$ (where $D_n(x) := D(x, n)$). Let $F_n(x)$ be the event that there are exactly 3 points of $\mathcal{H}_\lambda \cap D_n(x)$, and each of these points lies in a different infinite component of $G(\mathcal{H}_\lambda, 1)$. Set $E_n := E_n(\mathbf{o})$ and $F_n := F_n(\mathbf{o})$.

We claim that If $\mathbb{P}[E_n] > 0$ then $\mathbb{P}[F_n] > 0$.

Exercise 4.8. Prove the preceding claim. *Hint: this can be done by a similar resampling argument to the one used in proving Lemma 4.6.*

Also $E_n \subset E_{n+1}$ and $\cup_n E_n = \{N \geq 3\}$. Therefore we can and do fix $m \in \mathbb{N}$ with $\mathbb{P}[E_m] > 0$, and then also $\mathbb{P}[F_m] > 0$ by the preceding claim.

Given $k \in \mathbb{N}$ with $k \geq m + 2$, we modify the graph $G(\mathcal{H}_\lambda, 1)$ as follows. Let S_k be the (random) set of $x \in (3m\mathbb{Z}^d) \cap D_{k-2-m}$ such that $F_m(x)$ occurs.

Suppose $x \in S_k$. Then there are 3 distinct infinite components of $G(\mathcal{H}_\lambda, 1)$ meeting $D_m(x)$, each of which has precisely one vertex lying in $D_m(x)$. Denote these three vertices by u_x, v_x, w_x. Then add a single vertex at x, and add three edges, namely $\{x, u_x\}$, $\{x, v_x\}$ and $\{x, w_x\}$.

We do this for each $x \in S_k$. Also we remove all vertices of $\mathcal{H}_\lambda \cap D_{k+1}^c$ (and edges incident to them). Denote the resulting graph by $\mathcal{G}_k$. Then $\mathcal{G}_k$ is a k-graph, and each $x \in S_k$ is a branch point of the component of $\mathcal{G}_k$ containing x.

Hence by Lemma 4.7 (applied to each component of $\mathcal{G}_k$), the number of exit points of $\mathcal{G}_k$ is at least $|S_k| + 2$. Hence

$$\mathbb{E}[\mathcal{H}_\lambda(D_k \setminus D_{k-1})] \geq \mathbb{E}[|S_k|].$$

However, $\mathbb{E}[\mathcal{H}_\lambda(D_k \setminus D_{k-1})] = O(k)$ as $k \to \infty$, whereas

$$\mathbb{E}[|S_k|] = |3m\mathbb{Z}^2 \cap D_{k-2-m}|\mathbb{P}[E_m] = \Theta(k^2) \quad \text{as } k \to \infty,$$

yielding a contradiction. □

5 The largest component

In this section we aim to prove a Poissonized version of Theorem 1.5, concerning the theromodynamic limit $nr_n^2 \to \lambda$. In fact we just take $nr_n^2 = \lambda$.

Recall that $\mathcal{H}_{\lambda,s}$ is a homogeneous Poisson process of intensity λ in $B(s) = [-s/2, s/2]^2$. As observed in Section 3.2, $G(\mathcal{H}_{\lambda,s}, 1)$ is isomorphic to $G(\mathcal{P}_n, r_n)$ with r_n taken so $nr_n^2 = \lambda$, and $n = \lambda s^2$. Therefore it suffices to consider the components of $G(\mathcal{H}_{\lambda,s}, 1)$.

For any finite graph G, let $L_j(G)$ denote the order of its jth-largest component, that is, the jth-largest of the orders of its components, or zero if it has fewer than j components.

5.1 The subcritical case

Theorem 5.1. *Suppose $\lambda > 0$ with $p_\infty(\lambda) = 0$. Then*

$$s^{-2}L_1(G(\mathcal{H}_{\lambda,s}; 1)) \xrightarrow{P} 0 \qquad \text{as} \quad s \to \infty, \tag{5.1}$$

Proof. Suppose $p_\infty(\lambda) = 0$. For any locally finite $\mathcal{X} \subset \mathbb{R}^2$ and $x \in \mathcal{X}$, let $C_x(\mathcal{X})$ denote the vertex set of the component of $G(\mathcal{X}, 1)$ containing x.

Let $\varepsilon > 0$. Let N_s be the number of $x \in \mathcal{H}_{\lambda,s}$ such that $|C_x(\mathcal{H}_{\lambda,s})| \geq \varepsilon s^2$. If $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq \varepsilon s^2$, then $N_s \geq \varepsilon s^2$. Hence by Markov's inequality and the Mecke formula,

$$\begin{aligned}\mathbb{P}[L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq \varepsilon s^2] \leq (\varepsilon s^2)^{-1}\mathbb{E}[N_x] = (\varepsilon s^2)^{-1} \int_{B(s)} \mathbb{P}[|C_x(\mathcal{H}_{\lambda,s} \cup \{x\})| \geq \varepsilon s^2] \\ \leq (\varepsilon s^2)^{-1} \int_{B(s)} \mathbb{P}[|C_x(\mathcal{H}_{\lambda} \cup \{x\})| \geq \varepsilon s^2] \\ = \varepsilon^{-1} \sum_{k \geq \varepsilon s^2} p_k(\lambda)\end{aligned}$$

which tends to zero as $s \to \infty$. Therefore $s^{-2}L_1(G(\mathcal{H}_{\lambda,s}, 1))$ converges in probaility to zero, and hence so does $s^{-2}L_2(G(\mathcal{H}_{\lambda,s}, 1))$. ☐

Exercise 5.2. Adapt the preceding argument to show that if $\limsup_{n\to\infty}(nr_n^2) < \lambda_c$, then
(i) $n^{-1}L_1(G(\mathcal{P}_n, r_n)) \to 0$;
(ii) $n^{-1}L_1(G(\mathcal{X}_n, r_n)) \to 0$.

Exercise 5.3. Suppose $\lambda > \lambda_c$, and let $k \in \mathbb{N}$. Let $N_k(s)$ (respectively $N_{\leq K}(s)$) be the number of $x \in \mathcal{H}_{\lambda,s}$ such that $|C_x(\mathcal{H}_{\lambda,s})| = k$ (resp. such that $|C_x(\mathcal{H}_{\lambda,s})| \leq K$).

(i) Use the Mecke formula to show that $\lim_{s\to\infty} s^{-2}\mathbb{E}[N_k(s)] = \lambda p_k(\lambda)$, *and also, that* $\lim_{s\to\infty} \mathrm{Var}[s^{-2}N_k(s)] = 0$.

(ii) Deduce from the above that $(\lambda s^2)^{-1}N_{\leq K}(s) \xrightarrow{P} \sum_{k=1}^{K} p_k(\lambda)$.

(iii) Using (1.1), show that $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \leq \max(K, N_{\lambda s^2} - N_{\leq K}(s))$.

(iv) Let $\varepsilon > 0$. *Deduce that* $\mathbb{P}[(\lambda s^2)^{-1}L_1(G(\mathcal{H}_{\lambda,s}), 1) > p_\infty(\lambda) + \varepsilon] \to 0$ *as* $s \to \infty$.

(v) Show that $\mathbb{P}[(\lambda s^2)^{-1}L_1(G(\mathcal{H}_{\lambda,s}, 1)) + L_2(G(\mathcal{H}_{\lambda,s}, 1)) > p_\infty(\lambda) + \varepsilon] \to 0$ *as* $s \to \infty$.

For $\lambda < \lambda_c$, it is proved in [5] that the limit $\zeta(\lambda) := \lim_{n\to\infty}(-\log(p_n(\lambda)^{1/n}))$ exists and lies in $(0, \infty)$, and moreover $L_1(G(\mathcal{H}_{\lambda,s}, 1))/\log s$ converges in probability to $d/\zeta(\lambda)$, as $s \to \infty$. These results are beyond the scope of these notes.

5.2 Renormalization

In this section we use notation $D_r(x)$ for the disk $D(x,r)$ and D_r for $D(\mathbf{o},r)$. Also let $S_r = [-r,r]^2$, and let $\mathbf{e} := (1,0)$. Given $\lambda, K, L, M \in (0,\infty)$ with $L > K, M > 2K$, define the following events:

- $U_{K,L,\lambda}$ is the event that there is a unique component of $\mathcal{H}_\lambda^+ \cap D_L$ that meets both D_K and ∂D_L.
- $F_{K,M,\lambda}$ is the event that there is a path in $\mathcal{H}_\lambda^+ \cap D_{3M}$ from D_K to $D_K(M\mathbf{e})$.

Proposition 5.4. *Suppose $p_\infty(\lambda) > 0$ and let $\varepsilon \in (0,1)$. There exist constants $K > 0$ and $M > 3K$ such that $\mathbb{P}[U_{K,M/3,\lambda}] > 1-\varepsilon$ and $\mathbb{P}[F_{K,M,\lambda}] > 1-\varepsilon$.*

We shall use this to show that $p_\infty(\lambda_c) = 0$, and (later) to establish the limiting behaviour of $s^{-2}L_1(G(\mathcal{H}_{\lambda,s},1))$ for $\lambda > \lambda_c$. The point is that we can use it to compare $G(\mathcal{H}_\lambda, 1)$ with a dependent percolation process on the lattice $M\mathbb{T}$.

Proof of Proposition 5.4. We adapt an argument in [6]. Let $\varepsilon_1 = (1/3)\varepsilon^{32}$. Choose K such that $\mathbb{P}[D_K \leftrightarrow \infty] > 1-\varepsilon_1$. Since $\varepsilon > \varepsilon_1$, we can and do choose $n_1 \in \mathbb{N}$ with $n_1 > K$ such that

$$\mathbb{P}[U_{K,n,\lambda}] \geq 1-\varepsilon, \quad \forall n \geq n_1 \tag{5.2}$$

Exercise 5.5. *Prove this.* Hint: Using Theorem 4.2, the argument is similar to Exercise 4.3.

Now for $n \geq n_1$, and for $0 \leq \alpha \leq \beta \leq n$, let $E_n(\alpha,\beta)$ be the event that there is a path in $\mathcal{H}_\lambda^+ \cap S_n$ from D_K to $\{n\} \times [\alpha,\beta]$. Note that $\mathbb{P}[D_K \leftrightarrow \partial S_n] \geq \mathbb{P}[D_K \leftrightarrow \infty] > 1-\varepsilon_1$. Using the square root trick we can deduce that

$$\mathbb{P}[E_n(0,n)] > 1-\varepsilon_1^{1/8}. \tag{5.3}$$

Next, note that for fixed n, we have that as a function of α, $\mathbb{P}[E_n(0,\alpha)]$ increases continuously from a value less than 1 at $\alpha = 0$ to a value of $\mathbb{P}[E_n(0,n)]$ at $\alpha = n$, while $\mathbb{P}[E_n(\alpha,n)]$ decreases continuously from a value of $\mathbb{P}[E_n(0,n)]$ at $\alpha = 0$ to a value less than 1 at $\alpha = n$. Therefore we can and do choose $\alpha_n \in (0,n)$ such that $\mathbb{P}[E_n(0,\alpha_n)] = \mathbb{P}[E_n(\alpha_n,n)]$.

Exercise 5.6. *Prove the various assertions in the preceding lines.*

Since $E_n(0,n) = E_n(0,\alpha_n) \cup E_n(\alpha_n,n)$, by (5.3) and a further application of the square root trick we obtain that

$$\mathbb{P}[E_n(\alpha_n,n)] = \mathbb{P}[E_n(0,\alpha_n)] > 1-\varepsilon_1^{1/16}. \tag{5.4}$$

By yet another application of the square root trick we obtain that

$$\max(\mathbb{P}[E_n(0,\alpha_n/2)], \mathbb{P}[E_n(\alpha_n/2,\alpha_n)]) > 1-\varepsilon_1^{1/32}$$

so we can and do choose y_n, with either $y_n = \alpha_n/4$ or $y_n = 3\alpha_n/4$, such that

$$\mathbb{P}[E_n(y_n - \alpha_n/4, y_n + \alpha_n/4)] > 1-\varepsilon_1^{1/32}. \tag{5.5}$$

Set $n_2 = 3n_1$. We claim that there exists $N \geq n_2$ such that $\alpha_{3N} \leq 4\alpha_N$. Indeed, if this were not true then we would have for all $k \geq 1$ that $\alpha_{3^k n_2} \geq 4^k \alpha_{n_2}$, but since $\alpha_n \leq n$ for all n, this would imply $3^k n_2 \geq 4^k \alpha_{n_2}$ so that $(4/3)^k \leq n_2/\alpha_{n_2}$ for all k, which is not true (here we use the fact that $\alpha_{n_2} > 0$), justifying the claim.

Choose (deterministic) $N \geq n_2$ such that $\alpha_{3N} \leq 4\alpha_N$. Then by (5.2)

$$\min(\mathbb{P}[U_{K,N,\lambda}], \mathbb{P}[U_{K,3N,\lambda}]) > 1 - \varepsilon_1, \tag{5.6}$$

and by (5.4) and (5.5), setting $\varepsilon_2 := \varepsilon_1^{1/32}$ we have

$$\min(\mathbb{P}[E_N(\alpha_N, N)], \mathbb{P}[E_{3N}(y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4)]) > 1 - \varepsilon_2. \tag{5.7}$$

Now set $x = (2N, y_{3N})$. Let $S_N(x) = S_N + x$. Define the vertical intervals

$$I = \{3N\} \times [y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4],$$
$$J^+ = \{3N\} \times [y_{3N} + \alpha_N, y_{3N} + N],$$
$$J^- = \{3N\} \times [y_{3N} - N, y_{3N} - \alpha_N].$$

Let A^+ be the event that there is a path from $D_K(x)$ to J^+ in $\mathcal{H}_\lambda^+ \cap S'_N$, and let A^- be the event that there is a path from $D_K(x)$ to J^- in $\mathcal{H}_\lambda^+ \cap S_N(x)$. Then $\mathbb{P}[A^+] = \mathbb{P}[A^-] = \mathbb{P}[E_N(\alpha_N, N)]$.

By (5.7) and the union bound,

$$\mathbb{P}[A^+ \cap A^- \cap E_{3N}(y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4)] > 1 - 4\varepsilon_2 = 1 - \varepsilon.$$

If the above event holds, then since $\alpha_{3N}/4 \leq \alpha_N$, there is a path in $\mathcal{H}_\lambda^+ \cap S_{3n}$ from D_K to D'_K. Since $S_{3N} \subset D_{5N} \subset D_{3\|x\|}$, there is hence a path in $\mathcal{H}_\lambda^+ \cap D_{3\|x\|}$ from D_K to $D_K(x)$.

Set $M := \|x\|$. By rotation invariance we therefore have

$$\mathbb{P}[F_{K,M,\lambda}] > 1 - \varepsilon.$$

Also $M \geq 2N \geq 2n_2 = 6n_1$, so $M/3 \geq n_1$ so that $\mathbb{P}[U_{K,M/3,\lambda}] > 1 - \varepsilon$ by (5.2), so the proof is complete. □

Theorem 5.7. *It is the case that* $p_\infty(\lambda_c) = 0$.

Proof. It suffices to show that if $p_\infty(\lambda) > 0$, then $\lambda > \lambda_c$, so choose λ with $p_\infty(\lambda) > 0$. Let $\varepsilon > 0$ with $\varepsilon < (1 - p_c^*(7))/7$ and $p_c^*(7)$ given in Theorem 3.1. Using Proposition 5.4, choose K, M such that $0 < K < M/3$ and $\min(\mathbb{P}[U_{K,M/3,\lambda}], \mathbb{P}[F_{K,M,\lambda}]) > 1 - \varepsilon$.

Since events $U_{K,M/3,\lambda}$ and $F_{K,M,\lambda}$ are determined by $\mathcal{H}_\lambda \cap D_{3M+1}$, both $\min(\mathbb{P}[U_{K,M/3,\lambda}]$ and $\mathbb{P}[F_{K,M,\lambda}]$ vary continuously with λ (**Exercise** - prove this). Therefore we can find $\mu < \lambda$ such that $\min(\mathbb{P}[U_{K,M/3,\mu}], \mathbb{P}[F_{K,M,\mu}]) > 1 - \varepsilon$.

For each $x, y \in \mathbb{T}$ with $\|x - y\| = 1$, let U_x denote the event that there is a unique component of $\mathcal{H}_\mu^+ \cap D_{M/3}(Mx)$ that meets both $D_K(Mx)$ and $\partial D_{M/3}(Mx)$. Let F_{xy} denote the event that there is a path in $\mathcal{H}_\mu^+ \cap D_{3M}(Mx)$ that meets both $D_K(Mx)$ and $D_K(My)$.

By translation and rotation invariance of $\mathcal{H}_\mu$, $\mathbb{P}[U_x] > 1 - \varepsilon$ for each x, and $\mathbb{P}[F_{xy}] > 1 - \varepsilon$ for each (x, y).

For $x \in \mathbb{T}$, let us say $X_x = 1$ if event U_x occurs, and also F_{xy} occurs for each of the six $y \in \mathbb{T}$ with $\|y - x\| = 1$; otherwise set $X_x = 0$. Then by the union bound $\mathbb{P}[X_x = 1] \geq 1 - 7\varepsilon$. Then $(X_x, x \in \mathbb{T})$ is a 7-dependent Bernoulli random field with $\mathbb{P}[X_x = 1] > 1 - 7\varepsilon$ for each $x \in \mathbb{T}$. By Theorem 3.1, this percolates, i.e. there is an infinite path of sites $x \in \mathbb{T}$ with $X_x = 1$, with positive probability.

However, the existence of such an infinite path implies the existence of an infinite component of $G(\mathcal{H}_\mu, 1)$, and therefore $p_\infty(\mu) > 0$. Hence $\mu \geq \lambda_c$ and thus $\lambda > \lambda_c$ as required. □

5.3 Rectangle crossings

Suppose $R = [a, b] \times [c, d]$ with $a < b$ and $c < d$. We seay a set $S \subset \mathbb{R}^2$ is *1-crossing* (respectively *2-crossing*) for R if there is a continuous path in $S \cap R$ from the left edge of R to the right edge (resp. from the top edge to the bottom edge).

In this section we establish upper bounds for the probability of non-existence of a component of $(\mathcal{H}^+_{\lambda;s} \cap R)^+$ that is 1-crossing for certain rectangles R. Given $a, b, \lambda > 0$, define the rectangle $R(a, b) := [-a/2, a/2] \times [-b/2, b/2]$, and the event

$$\mathsf{Cr}(\lambda, a, b) = \{\mathcal{H}^+_\lambda \text{ is } 1-\text{crossing for } R(a, b)\}.$$

Lemma 5.8. *Let $\mu > \lambda_c$, $\nu > 0$. Then there exists $c > 0$ such that for all large enough t,*

$$1 - \mathbb{P}[\mathsf{Cr}(\mu, t^2, t)] \leq \exp(-ct)$$

Proof. Let events $U_{K,m} := U_{K,M,\mu}$ and $F_{K,M} := F_{K,M,\mu}$ be defined as in the previous subsection:

- $U_{K,M}$ is the event that there is a unique component of $\mathcal{H}^+_\mu \cap D_{M+1}$ that meets both D_K and ∂D_M.
- $F_{K,M}$ is the event that there is a path in $\mathcal{H}^+_\mu \cap D_{5M}$ from D_K to $D_K(M\mathbf{e})$.

Let $\varepsilon = (1/12)^{81}$, so that such that $6\varepsilon^{1/81} = 1/2$. Using Proposition 5.4, choose $K > 0, M > 3K$ such that $\mathbb{P}[U_{K,M/3}] > 1 - \varepsilon$ and $\mathbb{P}[F_{K,M}] > 1 - \varepsilon$.

Define the random field $(X_x)_{x \in \mathbb{T}}$ as in the proof of Theorem 5.7 (we refer to sites $x \in \mathbb{T}$ with $X_x = 1$ as being open). If there is an open path in $\mathbb{T}$ that is 1-crossing for the lattice rectangle $R((t^2/M)+4, (t/M)-2) \cap \mathbb{T}$ then $\mathsf{Cr}(\mu, t^2, t)$ occurs. But if there is no such open path, there must be a closed path in $\mathbb{T}$ that is 2-crossing for $R((t^2/M)+4, (t/M)-2) \cap \mathbb{T}$. Since $(X_x)_{x \in \mathbb{T}}$ is a 7-dependent Bernoulli random field, by a Peierls argument similar to that in the proof of Theorem 3.1, the probably of this occurring is bounded by

$$(2t^2/M) \sum_{m \geq t/(2M)} 6^m \varepsilon^{m/81} \leq (4t^2/M)(1/2)^{t/(2M)},$$

and the result follows. $\square$

Now for $\lambda, a, b > 0$ define the event

$$\mathsf{Cr}^*(\lambda, a, b) = \{(\mathcal{H}_\lambda \cap R(a, b))^+ \text{ is } 1-\text{crossing for } R(a, b)\},$$

which differs from $\mathsf{Cr}(\lambda, a, b)$ because only disks with centres inside $R(a, b)$ are allowed to be used.

Lemma 5.9. *Suppose $\lambda > \lambda_c$. Then there is a constant $c > 0$ such that for all large enough a,*

$$1 - \mathbb{P}[\mathsf{Cr}^*(\lambda, a, a/3)] \leq \exp(-ca^{1/2}).$$

Proof. Take $\mu \in (\lambda_c, \lambda)$. By the superposition theorem (Theorem 1.7), we may assume that $\mathcal{H}_\lambda$ is obtained as the union of two independent homogeneous Poisson processes $\mathcal{H}_\mu$ and $\mathcal{H}_{\lambda-\mu}$.

Given a, divide $R(a, a/4)$ lengthwise into strips (rectangles) of dimensions $a \times a^{1/2}$. Take alternate strips in the subdivision (to avoid depencences) and denote these by $T_{a,1}, \ldots, T_{a,\nu_a}$ Then ν_a, the number of strips conidered, is $\Theta(a^{1/2})$.

For $1 \leq i \leq \nu_a$, if $\mathcal{H}_\mu^+$ is 1-crossing for $T_{a,i}$, then $(\mathcal{H}_\mu \cap R(a, a/3))^+$ is 1-crossing for the slightly shorter rectangle (denoted $T'_{a,i}$) obtained by moving the left edge of $T_{a,i}$ by 1/2 to the right, and moving the right edge of $T_{a,i}$ by 1/2 to the left

Let $G_{i,a}$ be the event that $\mathcal{H}_\mu^+$ is 1-crossing for $T'_{i,a}$. By Lemma 5.8, for a large we have

$$\mathbb{P}[G_{1,a}^c] \leq \exp(-ca^{1/2}). \tag{5.8}$$

Let $H_{i,a}$ be the event that in addition to event $G_{i,a}$ occurring, there is a continuum path in $(\mathcal{H}_\lambda \cap T_{i,a})^+$ from the left edge to the right edge of $T_{i,a}$. We assert that there is a constant $\delta > 0$, independent of a, such that for all $i \leq \nu_a$ we have

$$P[H_{i,a} | G_{i,a}] \geq \delta. \tag{5.9}$$

Indeed, given a point x on the left edge of $T'_{i,a}$, the probability that there exist two points X, Y of $\mathcal{H}_{\lambda-\mu}$ such that there is a path to the left edge of $T_{i,a}$ through $\{X\mathcal{Y}\}^+$ is bounded away from zero. Likewise for the right edge.

If event $G_{i,a} \cap H_{i,a}$ occurs for any $i \leq \nu_a$, then $\mathsf{Cr}^*(B(a, a/3))$ occurs. Hence by (5.8) and (5.9), we have for all large enough a that

$$\begin{aligned}\mathbb{P}[\mathsf{Cr}^*(R(a, a/3))^c] &\leq \mathbb{P}[\cup_{i=1}^{\nu_a} G_{i,a}^c] + \mathbb{P}[\cap_{i=1}^{\nu_a} H_{i,a}^c | \cap_{i=1}^{\nu_a} G_{i,a}] \\ &\leq \nu_a \exp(-ca^{1/2}) + (1-\delta)^{\nu_a},\end{aligned}$$

and since $\nu_a = \Theta(a^{1/2})$, this gives us the result. □

5.4 Proof of the giant component phenomenon

We are nearly ready to probe the Poisson version of Theorem 1.5 concerning the giant component, in the supercritical phase.

Proposition 5.10. *Suppose $\lambda > \lambda_c$. Let E_s denote the event that there is a unique component of $\mathcal{H}_{\lambda,s}^+$ having diameter greater than $6s^{1/2}$. Then $\mathbb{P}[E_s] \to 1$ as $s \to \infty$.*

Proof. Divide $B(s)$ into squares of side $s^{1/2}$ (in general these do not exactly fit, so we should really take squares of side $s/\lfloor s^{1/2} \rfloor$ to make them fit, but we ignore this minor issue). Let $R_{s,1}, R_{s,2}, \ldots R_{s,m_s}$ denot the collection of rectangles of aspect ratio 3, obtained by taking a horizontal or vertical line of three of the rectangles in the subdivision. Then $m_s = \Theta(s)$.

Then, by Lemma 5.9, there exists a constant $c > 0$ such that, for large enough s, and for $1 \leq i \leq m_s$,

$$P[G(\mathcal{H}_\lambda \cap H_{i,j}; 1) \text{ is crossing the long way for } R_{s,i}] > 1 - \mathrm{e}^{-cs^{1/4}}, \tag{5.10}$$

Therefore, if I_s denotes the intersection over all $i \leq s^{1/2}$ of the events described in (5.10), $P[I_s]$ exceeds $1 - s\exp(-cs^{1/4})$. But, on the event I_s, the long-way crossing components of $\mathcal{H}_\lambda^+ \cap R_{a,i}$ must all be part of the same big component of $\mathcal{H}_{\lambda,s}^+$ (since the long-way crossings for rectangles that intersect at right angles must overlap. Also on I_s, no other component can have diameter greater than $6s^{1/2}$ without intersecting this big component. □

We are now finally ready to complete the proof of the Poissonized version of Theorem 1.5, with extra information about the second largest component.

Theorem 5.11. *Suppose* $\lambda > \lambda_c$. *Then*

$$s^{-d} L_1(G(\mathcal{H}_{\lambda,s};1)) \xrightarrow{\mathbb{P}} \lambda p_\infty(\lambda) \quad \text{as } s \to \infty. \tag{5.11}$$

Also,

$$s^{-d} L_2(G(\mathcal{H}_{\lambda,s};1)) \xrightarrow{\mathbb{P}} 0 \quad \text{as } s \to \infty. \tag{5.12}$$

Proof. Let $N_{\geq 6s^{1/2}}$ be the number of points of $\mathcal{H}_{\lambda,s}$ lying in components $\mathcal{H}^+_{\lambda,s}$ of diameter more than $6s^{1/2}$. By the Mecke formula (**Exercise!**), as $s \to \infty$ we have

$$s^{-2}\mathbb{E}[N_{\geq 6s^{1/2}}] \to \lambda p_\infty(\lambda),$$

and moreover

$$s^{-4}\mathbb{E}[N_{\geq 6s^{1/2}}(N_{\geq 6s^{1/2}} - 1)] \to \lambda^2 p_\infty(\lambda)^2.$$

Therefore $\mathrm{Var}[s^{-2}N_{\geq 6s^{1/2}}] \to 0$, so by Chebyshev's inequality $s^{-2}N_{\geq \phi_s}$ converges in probability to $\lambda p_\infty(\lambda)$, as $s \to \infty$.

Hence, given $\varepsilon > 0$, we have

$$\mathbb{P}[N_{\geq 6s^{1/2}} \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)] \to 1 \quad \text{as } s \to \infty. \tag{5.13}$$

If the event E_λ, defined in Proposition 5.10, occurs then all those points of $\mathcal{H}_{\lambda,s}$, that lie in components of $\mathcal{H}^+_{\lambda,s}$ of diameter greater than $6s^{1/2}$, must lie in the same component of $G(\mathcal{H}_{\lambda,s}, 1)$. Therefore if also $N_{\geq 6s^{1/2}} \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)]$ we have $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)$. Therefore by (5.13) and Proposition 5.10 we obtain that

$$\mathbb{P}[L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)] \to 1.$$

Combined with Exercise 5.3(iv), this yields (5.11).

Using Exercise 5.3(v), we obtain that

$$\mathbb{P}[s^{-2}(L_1(G(\mathcal{H}_{\lambda,s})) + L_2(G(\mathcal{H}_{\lambda,s}))) > p_\infty(\lambda) + \varepsilon] \to 0,$$

and therefore $s^{-2}(L_1(G(\mathcal{H}_{\lambda,s})) + L_2(G(\mathcal{H}_{\lambda,s})))$ converges in probability to the same limit as $s^{-2}(L_1(G(\mathcal{H}_{\lambda,s}))$. This implies $s^{-2}L_2(G(\mathcal{H}_{\lambda,s}))$ must converge in probability to zero. □

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