Multiplicative chaos and cascades

Spring school, February 19–23, 2024, TU Darmstadt

Mini Courses:
Julien Barral
Eero Saksman

Invited Speakers:
Xiong Jin
Janne Junnila
Sebastian Mentemeier

Organization:
Frank Aurzada
Volker Betz
Matthias Meiners
Christian Mönch

Contents

1 General Information

1.1 Accommodation

The participants are recommended to stay in one of the following hotels, located in walking distance (15 minutes) to the lecture venue.

BEST WESTERN Darmstadt Mitte
Grafenstraße 31, 64283 Darmstadt
Tel: +49-6151-28100 info@hotel-darmstadt.bestwestern.de

HOTEL WELCOME
Karolinenplatz 4, 64289 Darmstadt
Tel: +49-6151-3914-0 info.dar@welcome-hotels.com

HOTEL FELIX
Kasinostraße 4, 64293 Darmstadt
Tel: +49-6151-3973720 darmstadt@felix-hotels.de

For directions please see the map on the back cover.

1.2 Registration

On Monday morning, starting from 8:00, registration is possible in the lobby of the lecture hall.

1.3 Lecture Hall

Location: Technische Universität Darmstadt. The registration and all lectures will take place in building S2|04, Hochschulstraße 8, 64289 Darmstadt in lecture hall S2|04 213. In the lecture hall, there are 2 large blackboards and 2 small blackboards and a projector.

1.4 Map & Points of Interest

The map can be found on the back cover.

1.5 Public Transportation

The closest bus and tram stops to the venue of the workshop are **Schloss** (trams: S2, S3, S9) and **Willy-Brandt-Platz** (trams: S4, S5, S6, S7, S8). Both stops are within 10 minutes walking distance to the lecture hall.

1.6 Food & Beverage

There are lots of good restaurants and bistros near TU Darmstadt.

Name	Address	Phone	Cuisine	Opening
Ratskeller	Marktplatz 8	26444	German	10:00 - 01:00
Pizzeria da Nino	Alexanderstr. 29	24220	Italian	18:00 - 23:00
Haroun's	Friedensplatz 6	23487	Oriental	11:00 - 22:30
Wellnitz	Lauteschlägerstr. 4	6699255	Bistro	12:00 - 24:00
Cafe Extrablatt	Marktplatz 11	5998820	Bistro	08:30 - 23:30
Ristorante Sardegna	Kahlertstraße 1	23029	Italian	11:30 - 14:45

1.7 Conference Dinner

On Tuesday, February 20, 2024, there will be a conference dinner at the Restaurant *Ratskeller*, Marktplatz 8, 64283 Darmstadt, starting at 18:30, Tel: +49-6151-26444

1.8 Free Afternoon

On Wednesday, February 21, 2024, there will be a free afternoon.

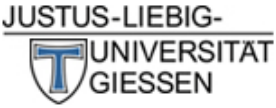
1.9 Contact Information

If you have any questions concerning the workshop, please feel free to contact one of the local organizers or the technical support:

- Prof. Dr. Frank Aurzada
  Office: S2-15, Room 341
  Phone: +49 6151 - 16 23375
- Prof. Dr. Volker Betz
  Office: S2-15, Room 340
  Phone: +49 6151 - 16 23370
- Office Department
  Office: S2-15, Room 339
  Phone:+49 6151 - 16 23380
- springschool@mathematik.tu-darmstadt.de

Acknowledgements

Financial support by the Department of Mathematics at Justus-Liebig-Universität Gießen, Department of Mathematics at Mathematics Johannes Gutenberg-Universität Mainz, and the Department of Mathematics at Technische Universität Darmstadt is acknowledged.



Time	Monday	Tuesday	Wednesday	Thursday	Friday
08:00	Registration				
09:00	Barral	Saksman	Barral	Saksman	Barral
10:30	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
11:00	Saksman	Barral	Saksman	Barral	Saksman
12:30	*Lunch*	*Lunch*	*Lunch*	*Lunch*	*Lunch*
14:00	Invited talk: Junnila	Invited talk: Mentemeier		Invited talk Jin	*End of* *Spring school*
14:45	Exercise session	Exercise session		Exercise session Saksman	
15:30	*Coffee break*	*Coffee break*		Coffee break	
16:00	Short talks: Bonnefont Chataignier Heizmann Petrova Schickentanz Stonner Verovkin		*Free* *Afternoon*	Exercise session Barral	
	17:45 Reception	18:30 Dinner			

Monday, 19 February 2024

Time	Speaker	*Title of Talk*
09:00-09:05	Welcome	
09:05-10:35	Julien Barral	*Mini course* *Geometric and dynamical aspects of Mandelbrot multiplicative cascades*
10:35-11:00		–Coffee break–
11:00-12:30	Eero Saksman	*Mini course* *Introduction to multiplicative chaos*
12:30-14:00		–Lunch break–
14:00-14:45	Janne Junnila	*Invited talk* *Noise-like properties and information content of imaginary chaos*
14:45-15:30		*Exercise session*
15:30-16:00		–Coffee break–
16:00-16:10	Short talk	Benjamin Bonnefont
16:10-16:20	Short talk	Louis Chataignier
16:20-16:30	Short talk	Nico Heizmann
16:30-16:40	Short talk	Iuliia Petrova
16:40-16:50	Short talk	Dominic T. Schickentanz
16:50-17:00	Short talk	Jakob Stonner
17:00-17:10	Short talk	Glib Verovkin
17:45-23:00	Reception	–Cheese & Wine–

Tuesday, 20 February 2024

Time	Speaker	*Title of Talk*
09:00-10:30	Eero Saksman	*Mini course* *Introduction to multiplicative chaos*
10:30-11:00		–Coffee break–
11:00-12:30	Julien Barral	*Mini course* *Geometric and dynamical aspects of Mandelbrot multiplicative cascades*
12:30-14:00		–Lunch break–
14:00-14:45	Sebastian Mentemeier	*Invited talk* *Multidimensional Mandelbrot's cascades*
14:45-15:30		*Exercise session*
15:30-16:00		–Coffee break–
18:30-22:00	Conference Dinner	– Ratskeller, Marktplatz 8 –

Wednesday, 21 February 2024

Time	Speaker	*Title of Talk*
09:00-10:30	Julien Barral	*Mini course* *Geometric and dynamical aspects of Mandelbrot multiplicative cascades*
10:30-11:00		–Coffee break–
11:00-12:30	Eero Saksman	*Mini course* *Introduction to multiplicative chaos*
12:30-14:00		–Lunch break–
		–Free afternoon – 14:30 Guided city tour Darmstadt –

Thursday, 22 February 2024

Time	Speaker	*Title of Talk*
09:00-10:30	Eero Saksman	*Mini course* *Introduction to multiplicative chaos*
10:30-11:00		–Coffee break–
11:00-12:30	Julien Barral	*Mini course* *Geometric and dynamical aspects of Mandelbrot multiplicative cascades*
12:30-14:00		–Lunch break–
14:00-14:45	Xiong Jin	*Invited talk* *On the fibres of planar self-similar sets with dense rotations*
14:45-15:30	Course Saksman	*Exercise session*
15:30-16:00		–Coffee break–
16:00-16:45	Course Barral	*Exercise session*

Friday, 23 February 2024

Time	Speaker	*Title of Talk*
09:00-10:30	Julien Barral	*Mini course* *Geometric and dynamical aspects of Mandelbrot multiplicative cascades*
10:30-11:00		–Coffee break–
11:00-12:30	Eero Saksman	*Mini course* *Introduction to multiplicative chaos*
12:30-14:30		–Lunch, end of the Spring School–

2 List of Talks

2.1 Mini Courses

Julien Barrel
Université Paris 13, France
Mini course: Geometric and dynamical aspects of Mandelbrot multiplicative cascades

When B. Mandelbrot introduced in the early seventies his simplified model for intermittent turbulence based on multiplicative cascades, he pointed out a series of questions, conjectures and properties about various aspects of these objects: the necessary and sufficient conditions for non degeneracy of the total mass of the associated limit measure, the possible renormalisation in case of degeneracy and the general form of the associated probability distributions invariant by random weighted means, finiteness of moments and tail of such distributions, dimension of the associated limiting non degenerate measure, statistical distribution of the mass along scales, and Hausdorff dimension of the topological support of the measure, obtained via what he called fractal curdling, and then fractal (self-similar) percolation. These problems, as well as similar or related ones associated with branching random walks, turned out, and continue to be, a source of remarkable developments and achievements; in particular, the original problems are now solved. Also, advances in multiplicative cascade theory are naturally and intimately related to some aspects of the development of multiplicative chaos theory, especially that of Gaussian multiplicative chaos.

I will discuss properties of Mandelbrot multiplicative cascades as acting on measures on a symbolic space, and their geometric realizations on the unit interval or the unit square as statistically self-affine measures, with a particular interest to the action on Bernoulli product measures.

In case of non degeneracy, I will consider the dimension, as well as the multifractal and thermodynamic properties of the limit measure, and the version of KPZ formula associated to such an object; I would like also to say a word about the various types of measures associated to the distributions invariant by random weighted mean (also called fixed point in the branching processes community).

Also, I will present an approach to the Hausdorff dimension of statistically self-affine Sierpinski carpets and their projections on the principal axes through a variational principle based on the Haudorff dimensions of planar Mandelbrot measures and their projections, and give some hint for the study of the higher dimensional case and more general geometric realizations.

If time permits, I will talk about the multiplicative cascades as defining a dynamics on some set of fixed points of the smoothing transform with a finite second moment and an associated functional central limit theorem.

It is recommended that the audience be a little familiar with the notions of Hausdorff and box dimensions, as well as that of various notions of dimensions associated with measures. Some references for this are (one includes information about Mandelbrot cascades that will be established during the lectures):

- P. Billingsley, Ergodic Theory and Information, John Wiley & Sons (1965).
- A. Fan, K.-S. Liu, H. Rao, Relationships between dimensions of a measure. https://www.math.cuhk.edu.hk/∼kslau/publication/067-Monatsh-v135-3.pdf
- K. Falconer, Fractal Geometry: Mathematical foundation and applications, Wiley (1990) (contains an short introduction to multifractals).
- Y. Heurteaux, Dimension of measures: the probabilistic approach. Publ. Mat., 51: 243–290, 2007 (contains an introduction to multifractals), https://lmbp.uca.fr/∼heurteau/measures.pdf
- Y. Heurteaux, An introduction to Mandelbrot cascades. New Trends in Applied Harmonic Analysis, A. Aldroubi, C. Cabrelli, S. Jaffard, U. Molter (Eds), Birkäuser; 67-105, 2016 (covers, in particular, essential parts of the original paper by Kahane and Peyrière "On some martingales of B. Mandelbrot"): https://lmbp.uca.fr/ heurteau/cours_cascades.pdf
- P. Mattila, Geometry of sets and measures in Euclidean spaces: fractals and rectifiability, Cambridge studies in advanced mathematics, 44, 1995.

Some familiarity with large deviations theory will be useful as well.
For those who would like to browse original B. Mandelbrot papers:

- B. Mandelbrot, Multifractal and 1/f Noise: wild self-affinity in Physics (1963–1976), Springer (1999), reproduces, among other works, B. Mandelbrot's contributions to turbulence dissipation modeling, and contains an English translation of Kahane-Peyrière (1976) paper.

Some of these papers are in free access:

- The following presents the model which motivated Kahane's development of Gaussian multiplicative chaos: https://users.math.yale.edu/mandelbrot/web_pdfs/064lognormalHypothesis.pdf

Next papers introduce and discuss the multiplicative cascades and related questions:

- https://users.math.yale.edu/mandelbrot/web_pdfs/comptes_rendus_73_I.pdf
- https://users.math.yale.edu/mandelbrot/web_pdfs/comptes_rendus_73_II.pdf
- English translation of the two previous notes:
  https://users.math.yale.edu/mandelbrot/web_pdfs/071iteratedRandomMultiplications.pdf
- https://users.math.yale.edu/mandelbrot/web_pdfs/070intermittentTurbulence.pdf
- Kahane and Peyrière's original preprint is avalaible here:
  http://sites.mathdoc.fr/PMO/PDF/K_KAHANE-68.pdf

Eero Saksman
University Helsinki, Finland
Mini course: Introduction to multiplicative chaos

Gaussian multiplicative chaos (GMC) are random measures whose basic theory was developed by Kahane in the 1980's. The original motivation was modelling turbulent flows. Starting from 20 years ago it has been gradually understood that these objects play an important role in various parts of random geometry, including Liouville quantum gravity and SLE, and they appear in other areas including some questions of probabilistic analytic number theory.

The aim of these lectures is to give a soft introduction to the basic properties of GMC and also touch on some more specialized topics like imaginary chaos and critical chaos. Also, if time permits, we will discuss some applications to random geometry or probabilistic number theory.

As prerequisites one requires basic knowledge of probability and standard analysis. Also, some familiarity of Gaussian fields (e.g. Brownian motion) would be useful, and a rudimentary knowledge of Hilbert spaces or elementary Fourier analysis could help.

The following article is somewhat outdated now but it gives an excellent review of many aspects of the theory: Gaussian multiplicative chaos and applications: A review. Rémi Rhodes and Vincent Vargas, Probab. Surveys 11: 315-392 (2014).

Xiong Jin

On the fibres of planar self-similar sets with dense rotations

University of Manchester, UK

In this talk I will present some known examples of planar self-similar sets whose dimension of fibres are known. Then I will talk about some new results I obtained using the percolation method. This relies on whether there exist interior points of the radial projection of percolated self-similar sets.

Janne Junnila

Noise-like properties and information content of imaginary chaos

University Helsinki, Finland

Imaginary multiplicative chaos is a version of Gaussian multiplicative chaos (GMC) where the intermittency parameter is purely imaginary. In the case of real GMC it is known that the resulting measures are multifractal, and that one can recover the underlying log-correlated field from the GMC measure. In this talk I will discuss these questions in the context of imaginary GMC, where the GMC distribution turns out to be monofractal and in certain ways noise-like. Nevertheless, in two and higher dimensions it is still possible to recover the field from the GMC (at least up to a constant). The talk is based on joint works with Aru, Baverez and Jego.

Sebastian Mentemeier

Multidimensional Mandelbrot's cascades

University Hildesheim

We are interested in properties of random d-vectors X that satisfy a distributional equation, namely, that X has the same law as $\sum_{i=1}^{N} T_i X_i$, where X_i are i.i.d. copies of X, and $T_1, T_2, \ldots$ is a given sequence of random nonnegative $d \times d$-matrices.

2.3 Further Speakers

Benjamin Bonnefont
The left tail of the subcritical derivative martingale in a branching random walk
University of Geneva, Switzerland

I'll present a work in which we obtain estimates on the left tail of the derivative martingale in a branching random walk. These results answer a question raised by Lacoin, Rhodes & Vargas for the derivative GMC in the particular context of a cascade.

Louis Chataignier
Asymptotics of the Overlap Distribution of Branching Brownian Motion
Institut de Mathématiques de Toulouse, France

We are interested in a question about branching Brownian motion, coming from spin glass theory: Given some point a between 0 and 1, if we pick two particles at time t according to the Gibbs measure at inverse temperature β, what is the probability that their last common ancestor died after time at? We will focus on the subcritical regime where $\beta^2 < 2$.

Nico Heizmann
Laplacian growth models on fractals
TU Chemnitz, Germany

In this talk we will introduce the three Laplacian growth models of Internal Diffusion Limited Aggregation, Rotor-Router Aggregation, and Divisible Sandpile. We will investigate their common scaling limit on the Sierpinski gasket.

Iuliia Petrova
Small ball probabilities for Gaussian processes
PUC-Rio, Brazil

In the talk we will consider a problem of small ball probabilities for Gaussian processes, which consists in finding the asymptotics of probability that a norm of a process is less than "epsilon" as "epsilon" tends to zero. This question arises in different areas: quantization of Gaussian vectors, metric entropy, etc. We will consider what is known in the general situation and talk about more advanced results in L_2-norm, for which the distribution is totally defined by eigenvalues of the covariance operator. For a wide class of Green Gaussian processes (with covariance function being a Green function for some ODE) we can use the powerful methods of spectral theory for ODEs to get the exact small ball probabilities.

Dominic T. Schickentanz
Brownian motion conditioned to spend limited time outside a bounded interval – an extreme example of entropic repulsion
TU Darmstadt, Germany

We show that a Brownian motion on $\mathbb{R}_{\geq 0}$ which is allowed to spend a total of $s > 0$ time units outside a bounded interval does not leave the interval at all. This can be seen as an extreme example of entropic repulsion. Moreover, we explicitly determine the exact asymptotic behavior of the probability that a Brownian motion on $[0,T]$ spends limited time outside a bounded interval, as $T \to \infty$. This is joint work with Frank Aurzada (Darmstadt) and Martin Kolb (Paderborn).

Jakob Stonner
Asymptotics of supercritical Crump-Mode-Jagers processes without Malthusian parameter
JGU Gießen, Germany

We study the asymptotics of a general (Crump-Mode-Jagers) branching process, which may be considered as a general population model where individuals independently give birth to offspring at times according to a point process on the positive real numbers. In 1981 Nerman proved convergence of supercritical general branching processes towards a martingale limit, which is non-degenerate only if there exists a so-called Malthusian parameter, a point at which the Laplace transform of the intensity measure of the reproduction point process takes on the value 1. In contrast, very little is known about the asymptotics if no Malthusian parameter exists. We consider the case where the Laplace transform drops from infinity immediately below 1 and work out the asymptotics of the mean, using defective renewal theory.

Glib Verovkin
Solutions to spatial kinetic-type equations
Universität Hildesheim, Germany

We consider a spatial kinetic-type evolution equation, which is used as a model for particle interactions in an ideal gas. Under some regularity assumptions, we investigate its time-dependent and stationary solutions. As key element in our research, we use a branching random walk, which is used to describe changes in particle's velocities after collisions. The talk is based on the work-in-progress with Sebastian Mentemeier.

3 List of Participants

Alban, Alexander JGU Mainz, Germany

Aurzada, Frank TU Darmstadt, Germany

Barrel, Julien Université Paris 13, France

Bazaes, Rodrigo Universität Münster, Germany

Betz, Volker TU Darmstadt, Germany

Birkner, Matthias JGU Mainz, Germany

Bonnefont, Benjamin University of Geneva, Switzerland

Chataignier, Louis Institut de Mathématiques de Toulouse, France

Freiberg, Uta TU Chemnitz, Germany

Giersbach, Sebastian Telmo JLU Gießen, Germany

Günther, Boris JLU Gießen, Germany

Hanigk, Pascal JGU Mainz, Germany

Heizmann, Nico TU Chemnitz, Germany

Helmer, Max TU Darmstadt, Germany

Ischebeck, Jasper Goethe–Universität Frankfurt, Germany

Jin, Xiong University of Manchester, United Kingdom

Junnila, Janne University of Helsinki, Finland

Klippel, Andreas TU Darmstadt, Germany

Kolesko, Konrad University of Wroclaw, Poland

Kraft, Mino Nicola TU Darmstadt, Germany

Meiners, Matthias JLU Gießen, Germany

Mentemeier, Sebastian Universität Hildesheim, Germany

Mittenbühler, Pascal Universität Paderborn, Germany

Mönch, Christian JGU Mainz, Germany

Neininger, Ralph Goethe-Universität Frankfurt, Germany

Petrova, Iuliia PUC-Rio, Brazil

Roth, Lukas TU Darmstadt, Germany

Saksman, Eero University of Helsinki, Finland

Schickentanz, Dominic T. TU Darmstadt, Germany

Schindler, Florian JLU Gießen, Germany

Schmidt, Tobias TU Darmstadt, Germany

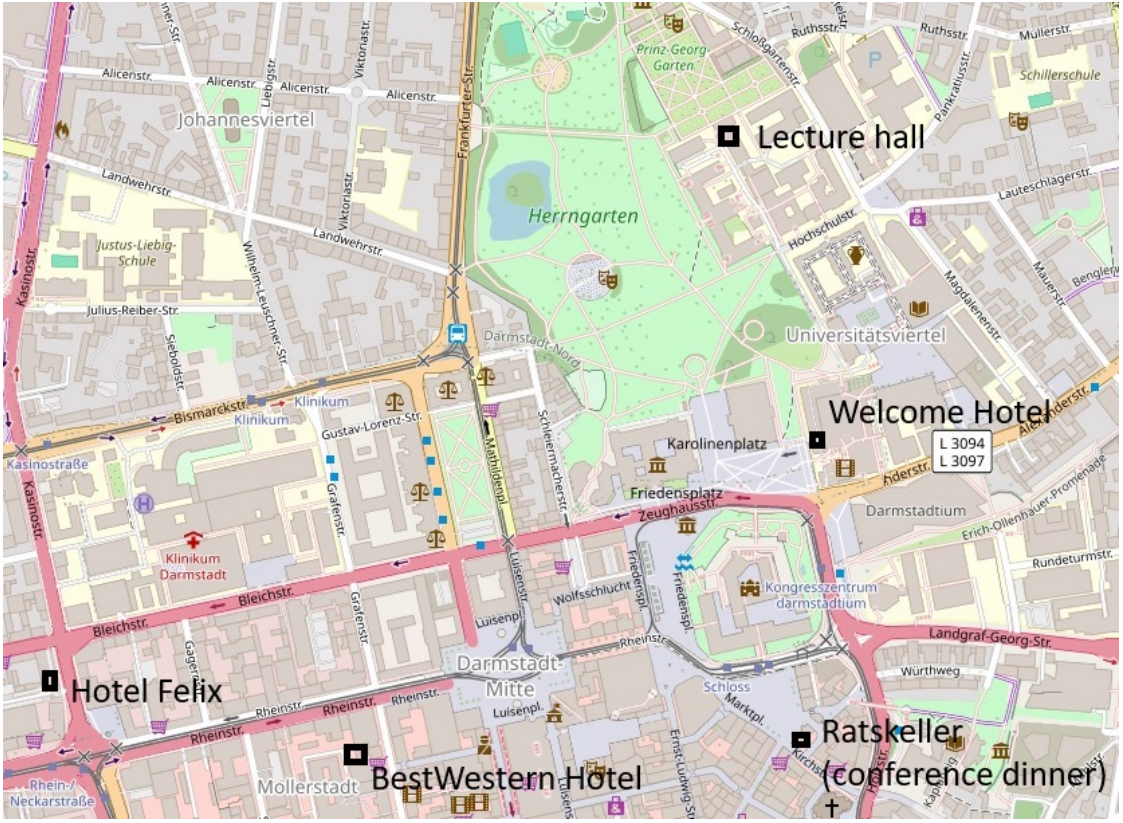
Stock, Laura JLU Gießen, Germany

Stonner, Jokob JLU Gießen, Germany

Tomic, Ivana JLU Gießen, Germany

Verovkin, Glib Universität Hildesheim, Germany

Wagner, Maren JLU Gießen, Germany


Lecture hall
Welcome Hotel
Hotel Felix
BestWestern Hotel
Ratskeller
(conference dinner)
+
Johannesviertel
Herrngarten
Prinz-Georg-Garten
Universitätsviertel
Darmstadt-Mitte
Mollerstadt
Darmstadtium
Kongresszentrum darmstadtium
Karolinenplatz
Friedensplatz
Zeughausstr.
Klinikum Darmstadt
Klinikum
Schloss
Marktpl.
Schillerschule
Justus-Liebig-Schule
Alicenstr.
Landwehrstr.
Frankfurter-Str.
Viktoriastr.
Liebigstr.
Wilhelm-Leuschner-Str.
Julius-Reiber-Str.
Sieboldstr.
Kasinostr.
Kasinostraße
Bismarckstr.
Gustav-Lorenz-Str.
Grafenstr.
Mathildenpl.
Schleiermacherstr.
Luisenstr.
Luisenpl.
Wolfsschlucht
Friedenspl.
Bleichstr.
Rheinstr.
Gagernstr.
Rhein-/Neckarstraße
Ruthsstr.
Müllerstr.
Pankratiusstr.
Schloßgartenstr.
Lauteschlagerstr.
Hochschulstr.
Magdalenenstr.
Mauerstr.
Alexanderstr.
Rundeturmstr.
Erich-Ollenhauer-Promenade
Landgraf-Georg-Str.
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L 3094
L 3097