Exercises

Let $b \geq 2$ be an integer and Σ be the symbolic space $\{0, \dots, b-1\}^{\mathbb{N}}$ endowed with the topology associated with cylinders as basis of open sets. Set $\Sigma^* = \bigcup_{n\geq 0}\{0, \dots, b-1\}^n$.

Recall the following inequality due to von Bahr and Esseen: If $p \in [1, 2]$, $N \geq 1$, and $X_1, \dots, X_N$ are centered and independent real valued random variables, then

$$\mathbb{E}|X_1 + \cdots + X_N|^p \leq 2(\mathbb{E}(|X_1|^p) + \cdots + \mathbb{E}(|X_N|^p)).$$

(1) Show that if μ is a non degenerate canonical Mandelbrot measure on $(\Sigma, \mathcal{B}(\Sigma))$ associated with a non negative random variable W of expectation 1, one has $(\mathbb{E}(\|\mu\|^q) < +\infty$ for all $q > 0)$ if and only if $(W \leq b$ almost surely and $\mathbb{P}(W = b) < 1/b)$. Then, treat the general case where the cascade is not necessarily canonical.

(2) Consider μ and μ' two Mandelbrot measures constructed simultaneously on $(\Sigma, \mathcal{B}(\Sigma))$ by considering a couple $(\widetilde{W} = (\widetilde{W}_0, \dots, \widetilde{W}_{b-1}), \widetilde{W} = (\widetilde{W}'_0, \dots, \widetilde{W}'_{b-1}))$ of random vectors, and a sequence $(\widetilde{W}(u), \widetilde{W}'(u))$ of independent copies of $(\widetilde{W}, \widetilde{W}')$.

(*a*) Show that if $\mathbb{P}(\forall\, 0 \leq i \leq b-1,\ W_i = 0 \Leftrightarrow W'_i = 0) = 1$, then $\mathbb{P}(\mu = 0)$ and $\mathbb{P}(\mu' = 0)$ are fixed points of the same polynomial function, and that if both μ and μ' are non degenerate, then $\mathbb{P}(\{\mu \neq 0\}\Delta\{\mu' \neq 0\}) = 0$.

(*b*) Show that $\lim_{n\to\infty} \mathbb{E}\Big(\sum_{|u|=n} \sqrt{\mu([u])\mu'([u])}\Big) = 0$, except if $\widetilde{W}' = \widetilde{W}$ and μ is non degenerate.

(*c*) (Mutual singularity) Suppose that μ is non degenerate. Show that conditional on $\{\mu \neq 0\}$, one has $W \neq W' \Rightarrow \mu' \perp \mu$.

Hint: Interpret $\sum_{|u|=n} \sqrt{\mu([u])\mu'([u])}$ as $\displaystyle\int_\Sigma \sqrt{\frac{\mu'([t|n])}{\mu([t|n])}}\,\mu(\mathrm{d}t)$.

(*d*) (Local dimension of μ' with respect to μ). Suppose that μ and μ' are non degenerate and $\mathbb{P}(\forall\, 0 \leq i \leq b-1,\ W_i > 0 \Rightarrow W'_i > 0) = 1$. Suppose also that $\mathbb{E}(Y^q) < \infty$ for some $q > 1$, $\mathbb{E}(Y'^{-\epsilon}|Y' > 0) < \infty$ for some $\epsilon > 0$, and $\mathbb{E}\Big(\sum_{i=0}^{b-1} \widetilde{W}_i |\log(\widetilde{W}'_i)|\Big) < \infty$. By using either a large deviation approach or the Peyrière measure associated with μ, show that for μ-almost every t, one has

$$\lim_{n\to\infty} \frac{\log(\mu'([t|n]))}{-n} = -\mathbb{E}\Big(\sum_{i=0}^{b-1} \widetilde{W}_i \log(\widetilde{W}'_i)\Big).$$

Then, revisit the previous question.

(e) (Continuity) Set $Y = \|\mu\|$ and $Y' = \|\mu'\|$. Suppose that both μ and μ' are non degenerate and that $\mathbb{E}(Y^h + Y'^h) < \infty$ for some $h > 1$. By using von Bahr and Esseen inequality and iterating if necessary the relations $Y = \sum_{i=0} \widetilde{W}_i Y(i)$ and $Y' = \sum_{i=0} \widetilde{W}'_i Y'(i)$, show that there exists a continuous function of $C = C(h, (\|\widetilde{W}_i\|_h)_{0,\dots,b-1}), (\|\widetilde{W}'_i\|_h)_{0,\dots,b-1})$ such that

$$\|Y - Y'\|_h \le C \sum_{i=0}^{b-1} \|\widetilde{W}_i - \widetilde{W}'_i\|_h.$$

(f) Suppose that μ is non degenerate. Set $\tau_{\widetilde{W}} : q \in \mathbb{R} \mapsto -\log \mathbb{E}(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i>0\}} \widetilde{W}_i^q)$. Let J be the interior of $\{q \in \mathbb{R} : \tau'_{\widetilde{W}}(q)q - \tau_{\widetilde{W}}(q) > 0\}$. For $q \in J$, denote by μ_q the non degenerate Mandelbrot measure associated with $\widetilde{W}_i(q) = (\widetilde{W}_i^q e^{\tau_{\widetilde{W}}(q)})_{0\le i\le b-1}$. Use the previous question to show that the process $(Y_q)_{q\in J}$ possesses a continuous modification.

(3) Under the same assumptions as in exercise (2), assume that μ is positive almost surely and use it to define an ultrametric distance over Σ. Find an assumption under which you can compute the dimension of μ' with respect to this metric.

(4) Show that if Q is the operator on finite non negative Borel measures on $(\Sigma, \mathcal{B}(\Sigma))$ associated with a canonical Mandelbrot cascade and ρ is an invariant measure on $(\Sigma, \mathcal{B}(\Sigma), \sigma)$, where σ is the left shift operation on Σ, then $\mathbb{E}(Q \cdot \varrho)$ is invariant.

(5) Suppose that μ is a on degenerate Mandelbrot measures constructed on $(\Sigma, \mathcal{B}(\Sigma))$ associated with a random vector $\widetilde{W} = (\widetilde{W}_0, \dots, \widetilde{W}_{b-1})$. Suppose that the components of $\widetilde{W}$ are bounded from below by a positive constant. Show that there are constants $c > 0$ and $\gamma \in (0,1)$ such that $\mathbb{E}(e^{-t\|\mu\|}) \le e^{-ct^\gamma}$ for all $t \ge 0$.

(6) Suppose that the vector $\widetilde{W} = (\widetilde{W}_0, \dots, \widetilde{W}_{b-1})$ used to define a Mandelbrot measure satisfies, $\tau_{\widetilde{W}}(1-) > 0$ and $\tau_{\widetilde{W}}(h) = -\infty$ for all $h > 1$. Fix $\delta \in (0,1)$. For $\gamma > 1$ and $x_0 > 1$, set $m(\gamma, x_0) = \mathbb{E}(\sum_{i=0}^{b-1} \mathbf{1}_{\{\widetilde{W}_i \le x_0\}} \widetilde{W}_i^\gamma)$. Check that if x_0 is large enough, then $m(\gamma, x_0) > 1$. Then set $\eta(\gamma, x_0) = \eta = (\eta_0, \dots, \eta_{b-1})$, where

$$\eta_i = d \left(\mathbf{1}_{\{\widetilde{W}_i \le x_0\}} \frac{\widetilde{W}_i^\gamma}{m(x_0, \gamma)} + \mathbf{1}_{\{\widetilde{W}_i > x_0\}} \widetilde{W}_i^{1-\delta} \right)$$

and d is a normalizing constant such that $\mathbb{E} \sum_{i=0}^{b-1} \eta_i = 1$.

Show that for all $0 \le i \le b-1$, one has $\eta_i \le \widetilde{W}_i$, and $\eta_i = 0$ if and only if $W_i = 0$. Also, prove that if x_0 is chosen big enough, for γ close enough to 1, one has $-\mathbb{E} \sum_{i=0}^{b-1} \eta_i \log \eta_i > 0$ and $\mathbb{E} \sum_{i=0}^{b-1} \eta_i^{1/(1-\delta)} < 1$.

(7) Suppose that μ is a on degenerate Mandelbrot measures constructed on $(\Sigma, \mathcal{B}(\Sigma))$ associated with a positive random vector $\widetilde{W} = (\widetilde{W}_0, \dots, \widetilde{W}_{b-1})$. Let J be the interior of $\{q \in \mathbb{R} : \tau'(q)q - \tau(q) > 0\}$. Fix $q \in J$.

(i) Use von Bahr and Esseen inequality to show that the martingale $\left(\sum_{|u|=n} W_{u_1}^q e^{\tau_{\widetilde{W}}(q)} \dots W_{u_1\cdots u_n}^q e^{\tau_{\widetilde{W}}(q)}\right)_{n\geq 1}$ and the sequence $\left(e^{n\tau_{\widetilde{W}}(q)} \sum_{|u|=n} \mu([u])^q\right)_{n\geq 1}$ have almost surely the same positive limit. Deduce from this observation that a.s. $\tau_\mu(q) = \tau_{\widetilde{W}}(q)$ for all $q \in J$.
(ii) Suppose that $q_+ = \sup(J) < \infty$ and $\tau'_{\widetilde{W}}(q_+-) < \tau_{\widetilde{W}}(q_+)/q_+$. Show that $\tau_{\widetilde{W}}(q_+) < \infty$ and $\tau_{\widetilde{W}}(q) = -\infty$ for all $q > q_+$. Then, use exercice (6), and inspire yourself from (i) to prove that if q is chosen in a right neighborhood of q_+, one has

$$\liminf_{n\to\infty} e^{n\frac{q}{q_+}\tau_{\widetilde{W}}(q_+)} \sum_{|u|=n} \mu([u])^q > 0$$

a.s. (non trivial).

(8) Show that the branching measure is the unique Mandelbrot measure of maximal Hausdorff dimension supported on a non-empty fractal percolation subset of Σ.