EXERCISES (crash course 'Introduction to Gaussian multiplicative chaos (TU Darmstadt 2/2024)

1. Let W be the hyperbolic white noise in the upper half plane $\mathbf{R}^2_+ = \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$, so that W is a centered Gaussian process indexed with Borel subset of $\mathbf{R}^2_+$ with the covariance structure

$$\mathbf{E}\, W(U_1)W(U_2) = \int_{U_1 \cap U_2} \frac{dxdy}{y^2},$$

where U_1, U_2 are Borel subsets of $\mathbf{R}^2_+$ (one should of course assume $\int_{U_i} \frac{dxdy}{y^2} < \infty$, but proceed slightly formally and ignore this). We define $A \subset \mathbf{R}^2_+$ by setting

$$A = \{(x,y) \in \mathbf{R}^2_+ \mid |x| < y/2 \wedge 1/2\},$$

which looks like an infinitely long vertical pencil of width 1 whose sharpened end makes angle $2\pi/6$ and touches the origin. Define (slightly formally) the Gaussian centered process X on the real axis by setting

$$X(x) := W(x + A),$$

where $x + A$ is the set A translated to right by amount x. Prove that X is a log-correlated field whose covariance has the simple (locally purely logaritmic) form

$$\mathbf{E}\, X(x)X(y) = \log^+(1/|x-y|) \qquad \text{for all} \quad x, y \in \mathbf{R}.$$

2. Consider the log-correlated field X of exercise 1. Let X_ε be a standard mollification of X, and let μ_β be the chaos defined by the field X, where $\beta > 0$.

(i) Let $0 < a < 1$. Prove that for $x \in (0,1)$ and small enough ε one has $X_\varepsilon(ax) \sim X_{a^{-1}\varepsilon}(x) + G_a$, where G_a is a centered Gaussian that is independent of the process $X_{a^{-1}\varepsilon}$ on $(0,1)$. What is the covariance of G_a?

(ii) Use part (i) to prove (at least heuristically) the *exact scaling law* for μ_β: if $A \subset [0,1]$, then

$$\mu_\beta(aA) \sim a e^{G_{a,\beta}} \mu_\beta(A)$$

where $G_{a,\beta}$ is a Gaussian independent of the chaos μ_β on the right hand side. What are the parameters of $G_{a,\beta}$?

NOTE: the chaos in exercises 3-5, 8 below is the μ_β of this exercise.

3. (i) Prove that the random measure μ_β satisfies a non-trivial scaling law (intermittency as described at the lectures), i.e. for $q < \sqrt{2}$ and for small r

$$\mathbf{E}\, \mu_\beta([0,r])^q \approx C_q r^{\varphi(q)},$$

where φ is a non-linear function.

(ii) Prove that measures that have density of the form $\exp(X(x))dx$, where X is a Gaussian field with a Hölder-continuous covariance cannot have the 'intermittency' property of part (i)

4. Assume that $\beta \in (0,\sqrt{2})$ so that μ_β is nontrivial. Show that

$$\mathbf{E}\,\mu_\beta([0,1])^p = \infty \quad \text{if} \quad p > 2/\beta^2.$$

[Hint: Use the exact scaling and the identity $\mu_\beta([0,1]) = \mu_\beta([0,1/2]) + \mu_\beta([1/2,1|)$.]

5. Assume that one knows the fact $\mathbf{E}\,(\mu([0,1])^{-\varepsilon} < \infty$ for some $\varepsilon > 0$. Prove that then $\mathbf{E}\,(\mu([0,1])^{-q} < \infty$ for all $q > 0$.

6. Try to use the exact scaling of the chaos to show that the dimension of the chaos measure μ_β is less that one.

7. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain and let $\Omega' \subset \Omega$ be a subdomain. Prove that GFF_Ω has the Markov property: on the domain Ω' one may write (at least in the sense of distribution)

$$\mathrm{GFF}_\Omega = \mathrm{GFF}_{\Omega'} + Y,$$

where $Y \perp \mathrm{GFF}_{\Omega'}$ and every realization of the field Y is harmonic in Ω'.

Note that, slightly imprecisely, this means that we may say that we obtain GFF_Ω in Ω' by extending its boundary values on $\partial\Omega'$ harmonically to Ω' and adding an independent copy of $\mathrm{GFF}_{\Omega'}$.

[Hint: recall that GFF on a bounded domain can be written as $\sum_{n=1}^\infty A_n f_n$, where f_n:s are any orthonormal basis in Ω and A_n:s are independent standard Gaussians.]

8. An often used approximation for $X := \mathrm{GFF}_\Omega$ is to consider X_ε which is defined via the circle averages

$$X_\varepsilon(z) = \int \frac{1}{2\pi}\int_0^{2\pi} X(z + re^{i\theta} d\theta.$$

(i) Prove that these yield a standard martingale-like approximation of X (at some distance from the boundary of the domain).

(ii) Especially, verify that for fixed z the process $t \mapsto X_{e^{-t}}(z) - X_1(z)$ is a standard Brownian motion.

[Hint: It might be useful to recall that $z \to \log(1/[z|)$ is hamonic and satisfies the mean value principle in the domain $\mathbf{C} \setminus \{0\}$.

9. (i) Find an explicit formula for small $\beta > 0$ of fourth moment $\mathbf{E}\,(\mu_\beta([0,1])^4$. Try to read of from the result that the fourth moment is finite if and only if $\beta < 1/\sqrt{2}$.

(ii) Do the same for the imaginary chaos $\mathbf{E}\,(\mu_{\beta i}([0,1])^4$. Show that in this case the fourth moment is finite for all $\beta \in (0,1)$,

10. (i) What is the scaling law we get for the critical chaos $\mu_{\sqrt{2}}$? Show that critical chaos does not have finite expectation.

(ii) Try to show that the dimension of the critical chaos measure is zero.

11. Define the analytic function

$$Z(z) = \sum_{n=1}^{\infty} \frac{A_n}{\sqrt{n}} z^n$$

on the (complex) unit disc $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$. Here A_n:s are (complex) $N(0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$ -Gaussians. Denote $Z = U + iV$ on $\mathbf{D}$. Write $\mathbf{T} = \partial\mathbf{D}$. Show that the boundary fields U, V are log-correlated on the boundary $\mathbf{T} = \partial\mathbf{D}$ and their mutual covariance is bounded but not continuous.

12. (Continuation of the previous exercise) Prove that almost surely the boundary distribution of the analytic function $z \mapsto \exp(Z(z))$ is not given by a measure.