Multifractal formalism for Mandelbrot measures, KPZ formula, and projections of planar Mandelbrot measures

Julien Barral, Darmstadt spring school, session 5/5

If ρ is a fully supported finite positive Borel measure on Σ, its L^q-spectrum is the non decreasing concave function

$$\tau_\rho : q \in \mathbb{R} \mapsto \liminf_{n\to\infty} \Big(\tau_{\rho,n}(q) = -n^{-1} \log \sum_{|u|=n} \rho([u])^q\Big),$$

For $\alpha \in \mathbb{R}$ set

$$\overline{f}_\mu(\alpha) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} n^{-1} \log \#\{u : |u| = n,\ \mu([u]) \in [e^{-n(\alpha+\epsilon)}, e^{-n(\alpha-\epsilon)}]\},$$

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$$E(\rho,\alpha) = \left\{t \in \Sigma : \lim_{n\to\infty} \frac{\log \rho([t_{|}n])}{-n} = \alpha\right\}.$$

One always has:

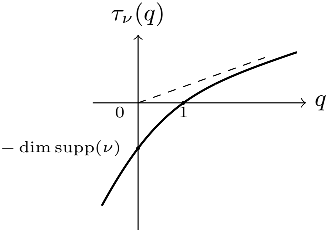
$$\dim E(\rho,\alpha) \le \underline{f}_\mu(\alpha) \le \overline{f}_\mu(\alpha) \le \tau_\rho^*(\alpha) := \inf_{q\in\mathbb{R}} \alpha q - \tau_\rho(q).$$

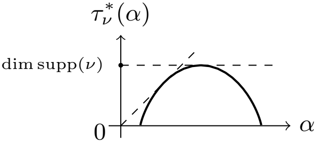
Also, by Gartner-Ellis theorem, $\underline{f}_\mu(\alpha) = \overline{f}_\mu(\alpha) = \tau_\rho^*(\alpha)$ at all α of the form $\tau_\mu'(q)$, and by a result of Ngai, $\tau_\mu'(1^+) \le \underline{\dim}_H(\rho) \le \overline{\dim}_P(\rho) \le \tau_\mu'(1-)$.

Say that the multifractal formalism holds for ρ at α if

$$\dim E(\rho,\alpha) = \tau_\rho^*(\alpha).$$

Multifractal analysis of the Bernoulli product measure $\nu = \nu_p$





$$\tau_\nu(q) = \tau_{\nu,n}(q) = -\log \sum_{i=0}^{b-1} p_i^q,$$

$$\mathrm{dom}(\tau_\nu^*) = [\tau_\nu'(\infty), \tau_\nu'(-\infty)] = [-\log(\max_i p_i), -\log(\min_i p_i)].$$

The multifractal formalism holds at any α.

Set $\tau(q) = \tau_{\widetilde{W}}(q) = -\log(\mathbb{E}\big(\sum_{i=0}^{b-1} \widetilde{W}_i^q\big)$ for all $q \in \mathbb{R}$.
Suppose $\tau'(1-) > 0$ and $\tau(q) > -\infty$ for some $q < 0$ (in particular the components of $\widetilde{W}$ are > 0 and $\mathrm{supp}(\mu) = \Sigma$ a.s.

Theorem (Molchan (1996), B. (2000), Attia-B. (2014))

With probability 1, the Mandelbrot measure μ obeys the multifractal formalism. Specifically, for all $\alpha \in \mathbb{R}$,
$\dim_H E(\mu, \alpha) = \tau_\mu^*(\alpha) := \inf\{\alpha q - \tau_\mu(q) : q \in \mathbb{R}\}$, *where*

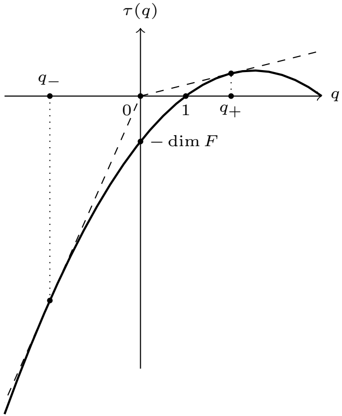
$$\tau_\mu(q) = \lim_{n\to\infty} \Big(\tau_{\mu,n}(q) := -n^{-1} \log \sum_{|u|=n} \mu([u])^q\Big),$$

and setting $J = \{q \in \mathbb{R} : \tau'(q)q - \tau(q) > 0\}$, $q_- = \inf(J)$ and $q_+ = \sup(J)$, one has

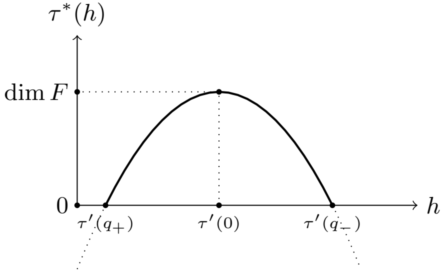
$$\tau_\mu(q) = T(q) := \begin{cases} \tau(q) & \text{if } q \in (q_-, q_+) \\ q\frac{\tau(q_+)}{q_+} & \text{if } q \geq q_+ \\ q\frac{\tau(q_-)}{q_-} & \text{if } q \leq q_- \end{cases}.$$

Note: it is easily seen that for $q \in \{q_-, q_+\}$, $|q| < \infty$ implies $\tau(q) > -\infty$

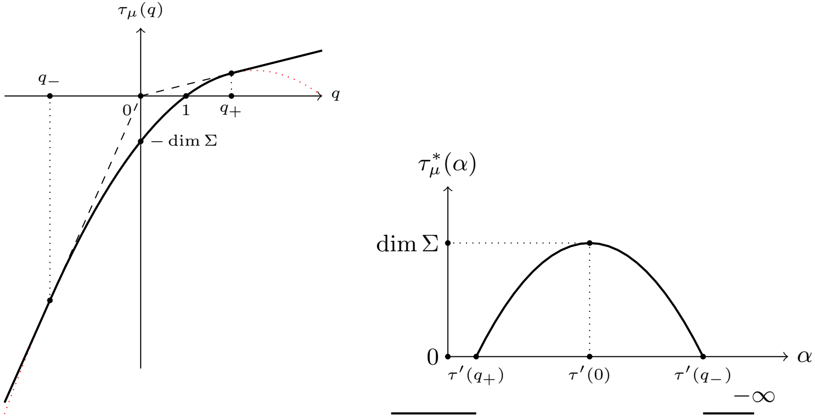
Note: If $q_+ < \infty$, τ_μ presents a second order phase transition or a first order phase transition at q_+ according to whether $\tau'(q_+-) = \frac{\tau(q_+)}{q_+}$ or $\tau'(q_+-) > \frac{\tau(q_+)}{q_+}$ (the same phenomenon occurs at q_- if $q_- > -\infty$.



(a) Function τ.



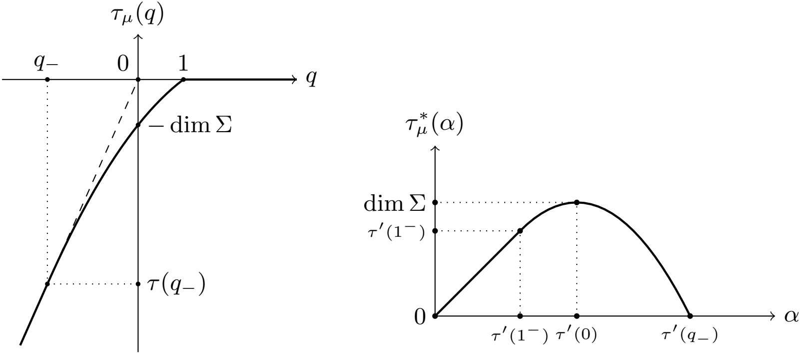
(b) Its Legendre transform τ^*.



(a) The L^q spectrum of μ. (b) Hausdorff spectrum of μ.

Figure: Multifractal nature of a Mandelbrot measure with a second phase transition at both q_- and q_+. This situation occurs, e.g., when the W_i are i.i.d and is lognormal.

Illustration



(a) L^q-spectrum of μ.

(b) Hausdorff spectrum of μ.

Figure: Multifractal nature of a Mandelbrot measure with a second order phase transition at some negative q_- and a first order phase transition at $q_+ = 1$ (i.e. when Kahane's non degeneracy theorem is optimal: $\tau(q) = -\infty$ for all $q > 1$.

Sketch of proof

It sufficies to prove:
(1) for all $q \in \mathbb{R}$, a.s. $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq T(q)$ and use the concavity to get a.s., for all $q \in \mathbb{R}$, $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq T(q)$.

(2) a.s. for all $\alpha \in [T'(\infty), T'(-\infty)]$, $\dim_H E(\mu,\alpha) \geq T^*(\alpha)$, and specifically,

(2) (a) a.s. for all $\alpha = \tau'(q)$, with $q \in (q_-, q_+)$,
$\dim_H E(\mu,\alpha) \geq T^*(\alpha) = \tau'(q)q - \tau(q)$;

(2) (b) a.s. $\dim_H E(\mu,\alpha) \geq T^*(\alpha)$ for $\alpha \in \{\tau'(q_+-), \tau'(q_-+)\}$;

(2) (c) If $q_+ < \infty$ and $\tau'(q_+-) > \frac{\tau(q_+)}{q_+}$, a.s., for all $\alpha \in [\frac{\tau(q_+)}{q_+}, \tau'(q_+-))$,
$\dim_H E(\mu,\alpha) \geq T^*(\alpha) = \alpha q_+ - \tau(q_+)$,
and a similar result if $q_- > -\infty$ and $\tau'(q_-+) > \frac{\tau(q_-)}{q_-}$.

(1) for all $q \in \mathbb{R}$, a.s. $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq T(q)$. Then the concavity yields a.s., for all $q \in \mathbb{R}$, $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq T(q)$.
For $q \in (q_-, q_+)$, one has

$$\begin{aligned}\mathbb{E}(e^{-\tau_{\mu,n}(q)}) = \mathbb{E}\Big(\sum_{|u|=n} \mu([u])^q\Big) &= \mathbb{E}\Big(\sum_{|u|=n} \big(\prod_{k=1}^{n} \widetilde{W}_{u_1\cdots u_k}\big)^q Y_\infty(u)^q\Big)\\ &= e^{-n\tau(q)}\mathbb{E}(Y_\infty^q)\end{aligned}$$

This yields $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq \tau(q) = T(q)$ a.s.

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This yields $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq \tau(q) = T(q)$ a.s.

For $q > q_+$ (if $q_+ < \infty$), take $0 < q' < q_+$, and write

$$\sum_{|u|=n} \mu([u])^q = \sum_{|u|=n} (\mu([u])^{q'})^{q/q'} \leq \Big(\sum_{|u|=n} (\mu([u])^{q'})\Big)^{q/q'},$$

which implies a.s $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq \frac{q}{q'} \cdot \liminf_{n\to\infty} \tau_{\mu,n}(q') \geq \frac{q}{q'}\tau(q')$.
This holds for all $0 < q' < q_+$, hence $\liminf_{n\to\infty} \tau_{\mu,n}(q) \geq \frac{q}{q_+}\tau(q_+) = T(q)$ by letting q' tend to q_+.

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(2) (a) a.s. for all $\alpha = \tau'(q)$, with $q \in (q_-, q_+)$,
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I explained that if we consider $\widetilde{W}_q = (W_{q,i} = \widetilde{W}_i^q e^{\tau(q)})_{0\leq i\leq b-1}$, one gets a new Mandelbrot measure μ_q, as weak limit of the sequence $(\mu_{q,n})_{n\geq 1}$, where

$$\frac{\mathrm{d}\mu_{q,n}}{\mathrm{d}\lambda}(t) = b^n \underbrace{W_{t_1}^q e^{\tau(q)}}_{W_{q,t_1}} \cdots \underbrace{W_{t_1\cdots t_n}^q e^{\tau(q_n)}}_{W_{q,t_1\cdots t_n}},$$

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Also, $\tau'(q) = -\mathbb{E}(\widetilde{W}_{q,i}\log(\widetilde{W}_i))$ and using the Peyrière measure $\mathbb{P}(\mathrm{d}\omega)\mu_{q,\omega}(\mathrm{d}t)$, as well as the fact that $\mathbb{E}(Y_\infty^{-\epsilon}) < \infty$ for some $\epsilon > 0$, one gets (cf exercise 2(d)) that

$$\mu_q(E(\mu,\alpha = \tau'(q))) = \|\mu_q\|.$$

Since the dimension fo μ_q is $\tau'(q)q - \tau(q)$, we get the desired inequality, for a fixed $\alpha = \tau'(q)$, almost surely.

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- To get a result valid a.s. for all $\alpha \in \tau'((q_-, q_+))$, one must modify the approach:
(i) Construct a.s. simultaneously the measures μ_q:uses a complex extension of the martingales in a neighborhood of (q_-, q_+);

(ii) Prove that a.s., for all $q \in (q_-, q_+)$ and $\epsilon > 0$,

$$\sum_{n\geq 1} \mu_q\big(\{t : \mu([t_{|n}]) \notin [e^{-n(\tau'(q)+\epsilon)}, e^{-n(\tau'(q)-\epsilon)}]\}\big) < \infty.$$

The n-th term of the previous sum can be bounded from above (using Markov inequality), for any $\eta > 0$, by

$$f_{n,\epsilon,\eta}(q) = \sum_{n\geq 1} \sum_{|u|=n} \mu_q([u])\big(\mu([u])^\eta e^{n\eta(\tau'(q)-\epsilon)} + \mu([u])^{-\eta} e^{-n\eta(\tau'(q)+\epsilon)}\big).$$

For any $[a, b] \subset J$, one can find η such that

$$\sum_{n\geq 1} \sup_{q\in[a,b]} \mathbb{E}(f_{n,\epsilon,\eta}(q)) + \sup_{q\in[a,b]} \mathbb{E}(|f'_{n,\epsilon,\eta}(q)|) < \infty.$$

This yields the almost sure uniform convergence of $\sum_{n\geq 1} f_{n,\epsilon,\eta}$ on compact subsets of J, and the desired conclusion.

It remains to prove (we consider q_+, the situation is similar for q_-):

(2) (b) a.s. $\dim_H E(\mu,\alpha) \geq T^*(\alpha)$ for $\alpha \in \{\tau'(q_+-), \tau'(q_-+)\}$;
If $q_+ < \infty$ is finite and $\tau'(q_+-) = \frac{\tau(q_+)}{q_+}$, under mild conditions, one can use the so called critical Mandelbrot measure associated with $\widetilde{W}_{q_+}$.

If $q_+ = \infty$, there is no such an adhoc choice.

(2) (c) If $q_+ < \infty$ and $\tau'(q_+-) > \frac{\tau(q_+)}{q_+}$, a.s., for all $\alpha \in [\frac{\tau(q_+)}{q_+}, \tau'(q_+-))$,
$\dim_H E(\mu,\alpha) \geq T^*(\alpha) = \alpha q_+ - \tau(q_+)$,
and a similar result if $\tau'(q_-+) > \frac{\tau(q_-)}{q_-}$,

one uses a method which treats $2(a)$, $2(b)$, and $2(c)$ simultaneously by considering *inhomogenous Mandelbrot measures*, in the sense that one allows the using of vectors of different laws at each generation.

For $A > 1$ and $0 \leq i \leq b-1$, set

$$\widetilde{W}_{A,i} = \frac{\mathbf{1}_{\{A^{-1} \leq W_i \leq A\}} W_i}{\mathbb{E} \sum_{j=0}^{b-1} \mathbf{1}_{\{A^{-1} \leq W_j \leq A\}} W_j}.$$

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There exists a non decreasing sequence $(A_n)_{n\ge 1}$ tending to ∞ and a sequence $(D_n)_{n\ge 1}$ of finite sets such that a.s., for all $p \ge 0$, for all $\boldsymbol{q} = (q_n)_{n\ge 1} \in \boldsymbol{J}_p = \prod_{n\ge 1} D_{n+p}$, the sequence of measures $(\mu_{\boldsymbol{q},n})_{n\ge 1}$ converges weakly to a measure $\mu_{\boldsymbol{q}}$, where

$$\frac{\mathrm{d}\mu_{\boldsymbol{q},n}}{\mathrm{d}\lambda}(t) = b^n \underbrace{W_{A_{p+1},t_1}^{q_1} e^{\tau_{\widetilde{W}_{A_{p+1}}}(q_1)}}_{W_{\boldsymbol{q},t_1}} \cdots \underbrace{W_{A_{p+n},t_1\cdots t_n}^{q_n} e^{\tau_{\widetilde{W}_{A_{p+n}}}(q_n)}}_{W_{\boldsymbol{q},t_1\cdots t_n}};$$

moreover, for all $\alpha \in [T'(\infty), T'(-\infty)]$, there exists $p \ge 0$ and $\boldsymbol{q} = (q_n)_{n\ge 1} \in \boldsymbol{J}_p$ such that

$$\lim_{n\to\infty} \tau'_{\widetilde{W}_{A_{p+n}}}(q_n) = \alpha, \ \lim_{n\to\infty} \tau'_{W_{\boldsymbol{q},t_1\cdots t_n}}(1) = T^*(\alpha),$$

and and both $\mu_{\boldsymbol{q}}(E(\mu,\alpha)) > 0$ and $\underline{\dim}_H(\mu_{\boldsymbol{q}}) \ge T^*(\alpha)$.

Note that if ρ is a continuous and fully supported Borel measure on σ, $d_\rho(x,y) = \rho([x \wedge y])$ defines an ultrametric distance on Σ.

$d_\lambda(x,y) = b^{-|x\wedge y|}$, and we consider the random distance d_μ, associated to a fully supported non degenerate canonical Mandelbrot measure $\mu = Q \cdot \lambda$ (the W_u are i.i.d and Q acts on λ). Here $\tau_{\widetilde{W}}(q) = -\log_b \mathbb{E} \sum_{i=1}^{b-1} \widetilde{W}_i^q$.

Given a deterministic Borel set E, there is a connection between $\dim_H^{d_\lambda}(E)$ and $\dim_H^{d_\mu}(E)$.
Note that the mapping $1 + \tau_{\widetilde{W}}$ is an increasing homeomorphism of $[0,1]$, and that $\dim_H^{d_\lambda}(\Sigma) = 1$.

Theorem (Benjamini-Schram (2009))

Fix a Borel subset E of Σ. Denote $\dim_H^{d_\lambda}(E)$ by s_0 and define s by the relation

$$s_0 = 1 + \tau_{\widetilde{W}}(s).$$

If $s_0 = 0$, or $s_0 > 0$ and $\tau_{\widetilde{W}}(-t) < \infty$ for all $t \in (0,s)$, then $\dim_H^{d_\mu}(E) = s$ almost surely.

Proof of the upper bound: It is clear that $\dim_H^{d_\mu}(\Sigma) = 1$ (use the coverings $(\bigcup_{|u|=n}[u])_{n\geq 1}$ of Σ to get $\mathcal{H}^{d_\mu,1}(\Sigma) \leq \|\mu\|$), so if $\dim_H^{d_\lambda}(E) = s_0 = 1$, as $s = 1$, one has $\dim_H^{d_\mu}(E) \leq \dim_H^{d_\mu}(\Sigma) = 1 = s$.

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If $\dim_H^{d_\lambda}(E) = s_0 < 1$, let $t_0 \in (s_0, 1)$ and for all $j \geq 1$ take a covering $([u_k^j]_{k\geq 1})$ a b^{-j} covering of E such that

$$\sum_{k\geq 1} (\mathrm{diam}^{d_\lambda}([u_k^j]))^{t_0} \leq 1/j^2.$$

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Define t by $t_0=1+\tau_{\widetilde{W}}(t)$. One has

$$\begin{aligned}\mathbb{E}\sum_{k\ge 1}\mathrm{diam}^{d_\mu}([u_k^j]))^t &= \sum_{k\ge 1}\mathbb{E}\mu([u_k^j])^t\\ &= \mathbb{E}(Y_\infty^t)\sum_{k\ge 1}(\mathrm{diam}^{d_\lambda}([u_k^j]))^{t_0}\le\mathbb{E}(Y_\infty^t)/j^2.\end{aligned}$$

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So a.s. $\lim_{j\to\infty}\sum_{k\ge 1}\operatorname{diam}^{d_\mu}([u_k^j]))^t=0$, and the $([u_k^j])_{k\ge 1}$ are δ_j coverings w.r.t. d_μ, where $\delta_j=\sup_{k\ge 1}\mu([u_k^j])$ tends to 0 as $j\to\infty$, since μ is atomless and $\sup_{k\ge 1}\lambda([u_k^j])\le b^{-j}$. It follows that $\dim_H^{d_\mu}(E)\le t$, and this holds for all $t\in(s,1)$.

Lower bound: If $s_0 = 0$ there is nothing to prove. Suppose $s_0 > 0$. Fix $t_0 \in (0, s_0)$. By Frostman's lemma, there exists a Borel probability measure ρ supported on E such that

$$I_{t_0}^{d_\lambda}(\rho) = \iint_{K \times K} \frac{\rho(\mathrm{d}x)\rho(\mathrm{d}y)}{d_\lambda(x,y)^{t_0}} < \infty.$$

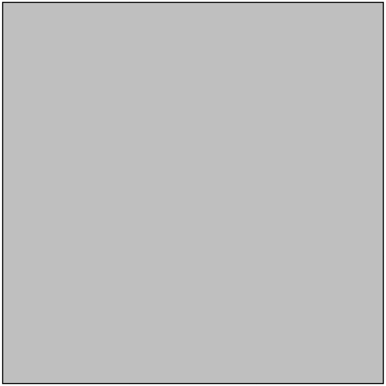
Again, define t by $t_0 = 1 + \tau_{\widetilde{W}}(t)$. Denote by Q_t the multiplicative chaos associated with the weights $W_u^t/\mathbb{E}(W^t)$. The measure $Q_t \cdot \rho$ is non degenerate as shows an application of the criterion for non degeneracy that we established.
Moreover, since the weights are > 0, the limit measure is positive almost surely; and it is supported on E. Also, a calculation (using that $E(Y_\infty^{-t}) < \infty$) shows

$$\mathbb{E} I_t^{d_\mu}(Q_t \cdot \rho) = \iint_{K \times K} \frac{Q_t \cdot \rho(\mathrm{d}x) Q_t \cdot \rho(\mathrm{d}y)}{d_\lambda(x,y)^t} \leq C(W,t) I_{t_0}^{d_\lambda}(\rho).$$

This implies that a.s., $\underline{\dim}_H(Q_t \cdot \rho) \geq t$, hence $\dim_H^{d_\mu}(E) \geq t$.

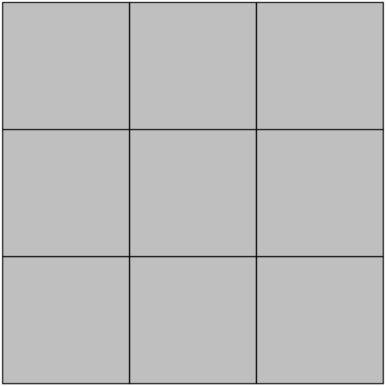
Fractal percolation set

Fix $m \geq 2$. Let $K_0 = [0,1]^2$ be the unit square.



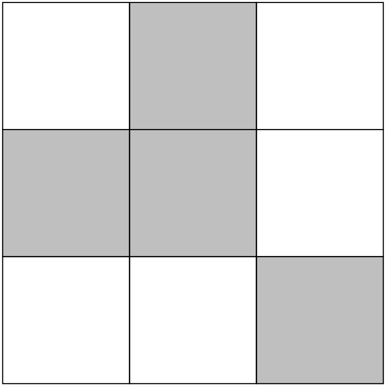
Fractal percolation set

Fix $m \geq 2$. Let $K_0 = [0,1]^2$ be the unit square. Choose a random subcollection $A(\omega)$ of the m^2 subsquares $\{R(i,j) = [im^{-1},(i+1)m^{-1}] \times [jm^{-1},(j+1)m^{-1}]\}_{0 \leq i,j \leq m-1}$ of side m^{-1}, according to some given distribution.



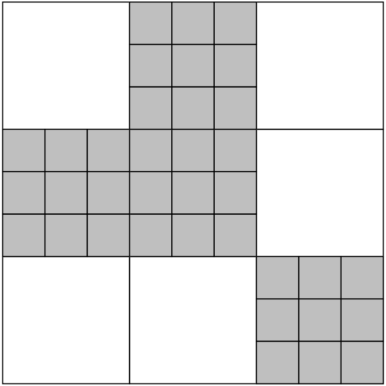
Fractal percolation set

Fix $m \geq 2$. Let $K_0 = [0,1]^2$ be the unit square. Choose a random subcollection of the m^2 subsquares $\{R(i,j) = [im^{-1}, (i+1)m^{-1}] \times [jm^{-1}, (j+1)m^{-1}]\}_{0 \leq i,j \leq m-1}$ of side m^{-1}, according to some given distribution. This yields a set K_1.

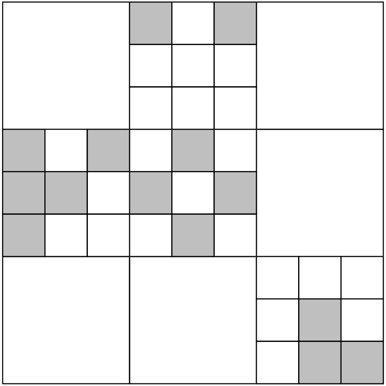


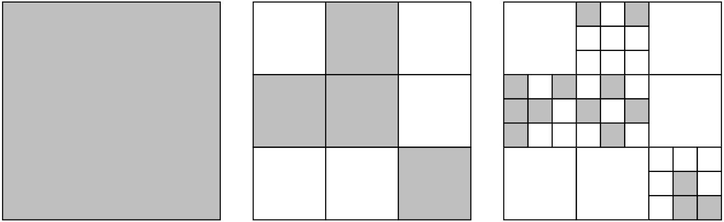
Fractal percolation

Repeat the selection independently and according to the same law in each selected subsquare.



Repeat the selection independently and according to the same law in each selected subsquare. This yields a set K_2.





K_0 K_1 K_2 ...

$$K = \bigcap_{n \geq 0} K_n.$$

Let $N(\omega) = \#A(\omega)$ denote the (random) number of squares kept at generation 1. One has $K \neq \emptyset$ if and only if $\mathbb{E}(N) > 1$ or $N = 1$ almost surely. In the later case K is a singleton.

Theorem

Let N be the number of surviving squares at the first generation. Suppose $\mathbb{E}\, N > 1$. With probability 1, if $K \neq \emptyset$ then

$$\dim_H K = \dim_B K = \log(\mathbb{E}\, N)/\log(m).$$

Let N_j be the number of surviving squares in line j, so that $N = \sum_{j=0}^{m-1} N_j$.
Suppose $\mathbb{E}\, N > 1$.
Denote by π the orthogonal projection on the vertical axis.

Theorem (Dekking-Grimmett (1988), Falconer (1989))

With probability 1, if $K \neq \emptyset$ then

$$\dim_H \pi K = \dim_B \pi K = \inf_{0 \leq \theta \leq 1} \log_m \sum_{i=0}^{m-1} (\mathbb{E}\, N_j)^{\theta}.$$

Moreover, $\dim_H \pi K = \dim K$ *iff the infimum is reached at* 1.

Remark: (1) The difficulty of the question partly comes from the fact that it may happen that $0 < \mathbb{E}\, N_j < 1$ for some j.

(2) The upper bound $\dim_B \pi(K) \leq \inf_{0\leq s\leq 1} \log_m \sum_{i=0}^{m-1} (\mathbb{E}\, N_j)^s$ is easily obtained by using the fact that for all $\theta \in (0,1)$,

$$\begin{aligned}
&\#\{I : |I| = n,\, I \cap \pi(K) \neq \emptyset\} \\
&= \#\{I : |I| = n,\, \#\{J : |J| = n,\, (I \times J) \cap K \neq \emptyset\} \geq 1\} \\
&\leq \sum_{|I|=n} (\#\{J : |J| = n,\, (I \times J) \cap K \neq \emptyset\})^\theta,
\end{aligned}$$

hence (taking expectation and using Jensen's inequality)

$$\begin{aligned}
\mathbb{E}\big(\#\{I : |I| = n,\, I \cap \pi(K) \neq \emptyset\}\big) &\leq \sum_{|I|=n} \Big(\mathbb{E}(\#\{J : |J| = n,\, (I \times J) \cap K \neq \emptyset\})\Big)^\theta, \\
&= \Big(\sum_{j=0}^{m-1} \mathbb{E}(N_j)^\theta\Big)^n,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} n^{-1} \log_m \#\{I : |I| = n,\, I \cap \pi(K) \neq \emptyset\} \leq \log_m \sum_{i=0}^{m-1} (\mathbb{E}\, N_j)^\theta.$$

Before revisiting the previous result, let us mention the result by Rams and Simon.
If $\theta \in (-\pi/2, \pi/2)$, denote by π_θ the orthogonal projection on the line $y = \tan(\theta)x$.

Theorem (Rams-Simon (2014, 2015))

Suppose the squares have been chosen independently and with equal probability $p > m^{-2}$. With probability 1, if $K \neq \emptyset$, for all $\theta \in (-\pi/2, \pi/2)$, the following holds

1. $\dim_H \pi_\theta K = \min(1, \dim_H K)$;
2. *if* $\dim_H K > 1$ *then* $\pi_\theta K$ *contains an interval.*

Take a random non negative vector $W = (W_{i,j})_{0\le i,j\le m-1}$ such that $\mathbb{E}(\sum_{j=0}^{m-1} W_{i,j}) = 1$.

$W_{0,2}$	$W_{1,2}$	$W_{2,2}$
$W_{0,1}$	$W_{1,1}$	$W_{2,1}$
$W_{0,0}$	$W_{1,0}$	$W_{2,0}$

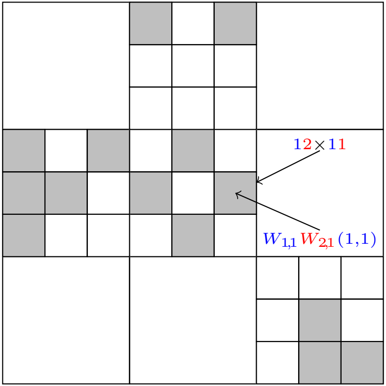
Revisiting the two first results with Mandelbrot measures

Suppose that $\mathbb{E}(N) > 1$. Take a random non negative vector $W = (W_{i,j})_{0 \le i,j \le m-1}$ such that $\mathbb{E}(\sum_{i,j} W_{i,j}) = 1$. Assume that $W_{i,j} = 0$ if (i,j) does not survive, i.e. $(i,j) \notin A(\omega)$.

0	$W_{1,2}$	0
$W_{0,1}$	$W_{1,1}$	0
0	0	$W_{2,0}$

Set $\mu_1(i_1 \times j_1) = W_{i_1,j_1}$

Next independently in each surviving subsquare $i_1 \times j_1$ take a copie $W(i_1, j_1) = (W_{i_2,j_2}(i_1, j_1))_{0 \le i_2, j_2 \le m-1}$ of W and set



$$\mu_2(i_1 i_2 \times j_1 j_2) = W_{i_1,j_1} W_{i_2,j_2}(i_1, j_1)$$

Iterate: for $n \geq 1$ and $I = i_1 \cdots i_n$ and $J = j_1 \cdots j_n$,

$$\mu_n(I \times J) = W_{i_1,j_1} W_{i_2,j_2}(i_1, j_1) \cdots W_{i_n,j_n}(i_1 \cdots i_{n-1}, j_1 \cdots j_{n-1}),$$

the mass being distributed uniformly.
One has

$$\mathrm{supp}(\mu_n) \subset K_n.$$

Set $\mathcal{A} = \{0, \ldots, m-1\}^2$ and

$$\tau(\theta) = -\log \mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j}^{\theta}; \quad \text{note that } \tau'(1^-) = -\mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j} \log W_{i,j}.$$

Theorem (Kahane-Peyrière (1976), Kahane (1987))

With probability 1, conditional on $K \neq \emptyset$, the sequence $(\mu_n)_{n\geq 1}$ weakly converges towards a mesure μ supported on K.
If $\mathbb{P}(\#\{(i,j) : W_{i,j} > 0\} = 1) = 1$, then μ is a Dirac mass almost surely.
Otherwise, $\mathbb{P}(\mu \neq 0 | K \neq \emptyset) > 0$ iff $\tau'(1^-) > 0$, and in this case, conditional on $\mu \neq 0$, then μ is exact dimensional with $\dim(\mu) = \dim_e(\mu)/\log(m)$, where

$$\dim_e(\mu) = \lim_{n\to\infty} n^{-1} \sum_{|I|=|J|=n} -\mu(I \times J) \log \mu(I \times J) = \tau'(1^-).$$

Dimensions of projections of μ

Recall that $\tau(\theta) = -\log \mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j}^{\theta}$.

Theorem (Falconer-Jin, 2014)

Suppose that $\tau(\theta) > -\infty$ for some $\theta > 1$ and $\tau'(1) > 0$. With probability 1, if $\mu \neq 0$, for all θ, the measure $\pi_{\theta *}\mu$ is exact dimentional.

Recall that $\tau(\theta) = -\log \mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j}^{\theta}$.

Theorem (Falconer-Jin, 2014)

Suppose that $\tau(\theta) > -\infty$ for some $\theta > 1$ and $\tau'(1) > 0$. With probability 1, if $\mu \neq 0$, for all θ, the measure $\pi_{\theta *}\mu$ is exact dimentional.

Let

$$\nu = \mathbb{E}(\pi_* \mu).$$

Setting $p_{i,j} = \mathbb{E}(W_{i,j})$, and $q_j = \sum_{i=0}^{m-1} p_{i,j}$ so that $q_0 + q_1 + \ldots + q_{m-1} = 1$, ν is the Bernoulli product measure on $[0,1]$ generated by the probability vector $(q_0, \ldots, q_{m-1})$.

Theorem (B.-Feng, 2018)

Suppose $\tau'(1^-) > 0$. With probability 1, if $\mu \neq 0$:

1. *If* $\dim(\mu) > \dim(\nu)$, *then* $\pi_*\mu \ll \nu$, *hence* $\dim(\pi_*\mu) = \dim(\nu)$.
2. *If* $\dim(\mu) \leq \dim(\nu)$, *then* $\pi_*\mu \perp \nu$.
   If, moreover, $\tau(\theta) > -\infty$ *for some* $\theta > 1$, *then* $\pi_*\mu$ *is exact dimensional and* $\dim(\pi_*\mu) = \dim(\mu)$.

Thus, if $\tau(\theta) > -\infty$ for some $\theta > 1$ and $\tau'(1) > 0$, if $\mu \neq 0$, then

$$\dim(\pi_*\mu) = \min\{\dim(\mu), \dim(\nu)\}, \quad \text{where } \nu = \mathbb{E}(\pi_*\mu).$$

Ingredients of the proof: The structure of π_μ is as follows.

If $y \in [0,1)$ and $J = J_n(y)$ is the semi-open to the right m-adic interval of generation n containing y, then

$$\pi_*\mu(J) = \sum_{|I|=n} \mu(I \times J) = \nu(J) \cdot Z_J \quad \text{where } Z_J = \sum_{|I|=n} \frac{\mu_n(I \times J)}{\nu(J)} Y_\infty(I,J),$$

hence $\pi_*\mu$ is locally essentially the product of its expectation and an inhomogeneous Mandelbrot martingale, more precisely a Mandelbrot martingale in a random environment if one considers $Z_{J_n(y)}$ for ν-almost every y.

To get the dimension of $\pi_*\mu$, one studies its L^q-spectrum and prove that in a neighbourhood of 1,

$$\mathbb{E} \sum_{|J|=n} \pi_*\mu(J)^\theta \leq C_q n \begin{cases} m^{-n \max(\tau_\mu(\theta), \tau_\nu(\theta))} & \text{if } \theta < 1 \\ m^{-n \min(\tau_\mu(\theta), \tau_\nu(\theta))} & \text{if } \theta \geq 1 \end{cases}.$$

This yields

$$\tau'_{\pi_*\mu}(1) = \min(\tau'_\mu(1), \tau'_\nu(1)).$$

Optimizing $\dim(\pi_*\mu)$, one gets

Corollary (B.-Feng (2018))

With probability 1, conditionally on $K \neq \emptyset$, one has

$$\begin{aligned}\dim_H \pi(K) &= \dim_B(\pi(K)) \\ &= \inf_{0\le\theta\le 1} \log_m \sum_{j=0}^{m-1} \mathbb{E}(N_j)^\theta \\ &= \max\{\dim_H(\pi_*\mu) : \mu \text{ is a Mandelbrot measure supported on } K\}.\end{aligned}$$

Moreover, the above maximum is not attained at a unique point if and only if the above infimum is attained at $\theta = 0$ and $\sum_{i=0}^{m-1} \log(\mathbb{E}(N_i)) > 0$.

It is also clear that

$$\dim_H K = \sup\{\dim(\mu) : \mu \text{ is a Mandelbrot measure supported on } K\},$$

and the supremum is uniquely attained at the so called "branching measure", that is the Mandelbrot measure associated to $W_{i,j} = \mathbf{1}_{A_\omega}(i,j)/\mathbb{E}(N)$.