Continuity and estimates for the Liouville heat kernel

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Liouville Quantum Gravity

- Construct a random metric on a 2d-manifold $D \subseteq \mathbb{R}^2$ of the form

$$e^{\gamma X(x)}\, dx^2, \qquad \gamma \in (0,2), \tag{1}$$

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where $\bar{B}$ is a BM on D independent of X.
- Dambis-Dubins-Schwarz Theorem: Try to define $\mathcal{B}$ as a time-change of a planar Brownian motion B.

Gaussian free field

Let D be bounded. The **(continuum) GFF** X is the **Gaussian Hilbert space** associated with the Sobolev space $H_0^1(D)$, which is the closure of $C_0^\infty(D)$ w.r.t. the Dirichlet inner product

$$(f, g)_\nabla = \int_D \nabla f \cdot \nabla g \, dx,$$

i.e. $\{(X, f)_\nabla\}_{f \in H_0^1(D)}$ is a family of centered Gaussian random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$\operatorname{cov}((X, f)_\nabla, (X, g)_\nabla) = (f, g)_\nabla = \big(f, (-\Delta) g\big)_{L^2}.$$

The covariance function is given by the standard Green function on D.

Irregularity of the GFF

Formally,

$$X = \sum_n \alpha_n f_n,$$

where (α_n) i.i.d. $\mathcal{N}(0,1)$ and (f_n) ONB of $H_0^1(D)$. Since

$$\Big\| \sum_{n=1}^N \alpha_n f_n \Big\|_\nabla^2 = \sum_{n=1}^N \alpha_n^2 \to \infty \qquad \text{as } N \to \infty,$$

the GFF cannot be defined pointwise as a random element in $H_0^1(D)$, but it can be identified as **random distribution**.

Massive Gaussian free field

From now on: $D = \mathbb{R}^2$.

The **massive GFF** X is the **Gaussian Hilbert space** associated with the Sobolev space $\mathcal{H}^1_m$, which is the closure of $C_0^\infty(\mathbb{R}^2)$ w.r.t. the Dirichlet inner product

$$(f, g)_m = m^2 (f, g)_{L^2} + (f, g)_\nabla, \qquad m > 0 \textbf{ mass},$$

i.e. $\{(X, f)_m\}_{f \in \mathcal{H}^1_m}$ is a family of centered Gaussian random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$\operatorname{cov}\big((X, f)_m, (X, g)_m\big) = (f, g)_m = \big(f, (m^2 - \Delta) g\big)_{L^2}.$$

The covariance function is given by the massive Green function associated with $m^2 - \Delta$.

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- Let $M_{n,\gamma}$ be the volume measure

$$M_{n,\gamma}(dx) = \exp\big(\gamma X_n(x) - \tfrac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]\big)\, dx,$$

associated with the metric

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- By the theory of **Gaussian multiplicative chaos** (Kahane '85), $\mathbb{P}$-a.s.

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for some Radon measure M_γ called **Liouville measure**.
- For $\gamma \in (0,2)$, $M_\gamma \perp dx$ and M_γ is concentrated on the set of γ-thick points of the field X.

n-regularized Liouville Brownian motion

- The n-regularized LBM $\mathcal{B}^n$ is the solution of

$$d\mathcal{B}_t^n = e^{-\frac{\gamma}{2} X_n(\mathcal{B}_t^n) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}_t^n)^2]} d\bar{B}_t,$$

where $\bar{B}$ is a standard BM on $\mathbb{R}^2$.

n-regularized Liouville Brownian motion

- The n-regularized LBM $\mathcal{B}^n$ is the solution of

$$d\mathcal{B}^n_t = e^{-\frac{\gamma}{2}X_n(\mathcal{B}^n_t)+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^n_t)^2]}d\bar{B}_t,$$

  where $\bar{B}$ is a standard BM on $\mathbb{R}^2$.
- By the Dambis-Dubins-Schwarz theorem

$$\mathcal{B}^n_t = B_{(F^n)^{-1}_t}$$

  is a time-change of a standard BM $(B, \{P_x\}_{x\in\mathbb{R}^2})$, where

$$F^n_t := \int_0^t e^{\gamma X_n(B_s)-\frac{\gamma^2}{2}\mathbb{E}[X_n(B_s)^2]}\, ds, \qquad t \geq 0,$$

  which is strictly increasing.

Definition of Liouville Brownian motion

Theorem (Garban, Rhodes, Vargas (2013))

Let $\gamma \in [0, 2)$. Then, $\mathbb{P}$-a.s. the following hold:

i) *For all $x \in \mathbb{R}^2$, F^n converges to some F in P_x-probability in $C([0, \infty), \mathbb{R})$.*

ii) *For all $x \in \mathbb{R}^2$, P_x-a.s., F is strictly increasing and satisfies $\lim_{t \to \infty} F_t = \infty$.*

iii) *The functional F is the unique PCAF with Revuz measure M_γ.*

The process $(\mathcal{B}, \{P_x\}_{x \in \mathbb{R}^2})$, $\mathbb{P}$-a.s. defined by

$$\mathcal{B}_t := B_{F_t^{-1}}, \qquad t \geq 0,$$

is called the (massive) **Liouville Brownian motion** (LBM).

A similar result has been obtained by N. Berestycki (2013).

Properties of the LBM

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- The LBM is symmetric (reversible) w.r.t. M_γ.
- For $t \geq 0$, P_t is absolutely continuous w.r.t. M_γ, so there exists the **Liouville heat kernel** $p_t(\cdot, \cdot)$ such that

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- The intrinsic metric $d_{\mathcal{B}}$ associated with the LBM is identically zero, which indicates that

$$\lim_{t\to 0} t \log p_t(x,y) = -\frac{d_{\mathcal{B}}(x,y)^2}{2} = 0.$$

That is, irregular off-diagonal behaviour of $p_t(\cdot,\cdot)$ is expected.

Continuity and on-diagonal upper bounds on $p_t(x, y)$

Theorem (A., Kajino (2014))

i) *The Liouville heat kernel* $p_t(x, y)$ *is jointly continuous on* $(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$.

ii) *For any bounded* $U \subset \mathbb{R}^2$ *and* $\beta > (\gamma + 2)^2/2 > 2$, $\mathbb{P}$*-a.s., there exists* $c_i = c_i(X, \gamma, U, \beta)$ *such that for all* $t < \frac{1}{2}$, $x, y \in U$,

$$p_t(x, y) \leq c_1\, t^{-1} \log(t^{-1}) \exp\Big(- c_2(|x - y|^{\beta}/t)^{1/(\beta-1)}\Big).$$

Related previous result by Maillard, Rhodes, Vargas, Zeitouni (2014):

- Continuity of the Liouville heat kernel on the torus $\mathbb{T}$.
- For any $\delta > 0$ there exists $\beta_\delta > 0$ such that for $t > 0$ and $x, y \in \mathbb{T}$.

$$p_t(x, y) \leq c_1\, t^{-(1+\delta)} \exp\Big(-c_2(|x - y|^{\beta_\delta}/t)^{1/(\beta_\delta-1)}\Big),$$

$$p_t(x, y) \geq \exp\Big(- t^{-\frac{1}{1+\gamma^2/4-\delta}}\Big), \quad \forall t \in (0, t_0] \text{ with } t_0 = t_0(X, \gamma, x, y).$$

On-diagonal lower bound

Theorem (A., Kajino (2014))

Let $\gamma \in [0,2)$. *Then,* $\mathbb{P}$*-a.s., for* M_γ*-a.e.* $x \in \mathbb{R}^2$ *there exist* $c_3 = c_3(X,\gamma)$ *and* $t_0 = t_0(X,\gamma,x)$ *such that*

$$p_t(x,x) \geq c_3\, t^{-1} \left(\log(t^{-1}) \right)^{-\eta}, \qquad \forall t \in (0, t_0],$$

for some explicit constant $\eta > 0$ *(*$\eta = 34$ *would be enough).*

Corollary (A., Kajino (2014))

Let $\gamma \in [0,2)$. *Then,* $\mathbb{P}$*-a.s., for* M_γ*-a.e.* $x \in \mathbb{R}^2$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t(x,x)}{-\log t} = 2.$$

Rhodes, Vargas (2013):

$$\lim_{y \to x} \int_0^\infty e^{-\lambda t} t^\alpha p_t(x,y)\, dt \begin{cases} < \infty, & \text{if } \alpha > 0, \\ = \infty, & \text{if } \alpha = 0. \end{cases}$$

Volume and exit time estimates

Lemma

Let $\varepsilon > 0$, $\alpha_1 := \frac{1}{2}(\gamma+2)^2$ *and* $\alpha_2 := 2(1-\frac{\gamma}{2})^2$*. Then,* $\mathbb{P}$*-a.s., for any* $R > 0$ *there exists* $c_i = c_i(X, \gamma, R, \varepsilon) > 0$*, such that*

$$c_4\, r^{\alpha_1+\varepsilon} \leq M_\gamma(B(x,r)) \leq c_5\, r^{\alpha_2-\varepsilon}, \qquad \forall x \in B(0,R),\ r \in (0,1).$$

Let $\tau_U := \inf\{s \geq 0 : \mathcal{B}_s \notin U\}$ be the first exit time from an open set U.

Proposition

For any $\beta > \alpha_1$ *and* $R \geq 1$, $\mathbb{P}$*-a.s., there exist* $c_i = c_i(X, \gamma, R, \beta)$ *such that*

$$P_x[\tau_{B(x,r)} \leq t] \leq c_6 \exp\left(-c_7 (r^\beta/t)^{1/(\beta-1)}\right),$$

for all $t > 0$, $x \in B(0,R)$ *and* $r \in (0,1]$.

The Dirichlet heat kernel

Consider the heat kernel $p_t^U(x,y)$ of the LBM killed upon exiting a bounded open set U.

- Strong Feller property of the resolvent.
- Faber-Krahn inequality, which implies a Nash inequality and on-diagonal bounds (ultracontractivity).
- By a general result by Davies ('89)

$$p_t^U(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y)$$

  converges uniformly on $[s_0, \infty) \times U \times U$ for all $s_0 > 0$, which implies continuity of $p_t^U(x,y)$.
- In combination with the exit time estimate a general result by Grigor'yan, Hu and Lau (2010) gives the off-diagonal bound.