Hyperbolic Hole Probabilities.

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Joint work with Alon Nishry, Ron Peled and Mikhail Sodin.

The hyperbolic GAF

Fix $L > 0$ and let $(\zeta_n)_n$ be iid $N_{\mathbb{C}}(0,1)$.

$$f_L(z) = \sum_{n=0}^{\infty} \zeta_n \binom{n+L-1}{n}^{1/2} z^n, \qquad z \in \mathbb{D},$$

where

$$\binom{n+L-1}{n} = \frac{(n+L-1)(n+L-2)\dots(L+1)L}{n!} = \frac{\Gamma(n+L)}{\Gamma(L)\Gamma(n)}.$$

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The distribution of the zero set is invariant under automorphisms of the disc.
f_L the only GAF with this property (up to trivialities).

Hyperbolic geometry

$$K_L(z, w) = \mathbb{E}[f_L(z)\overline{f_L(w)}] = \frac{1}{(1 - z\overline{w})^L}$$

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The average number of zeroes of this GAF is given by

$$\frac{1}{4\pi}\Delta \log K_L(z,z) dm(z) = \frac{L}{\pi}\frac{dm(z)}{(1-|z|^2)^2}$$

Different L give genuinely different processes

We now focus on the random variable

$$n_L(r) = \text{number of zeroes of } f_L \text{ in } D(0, r)$$

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$$\mathbb{E}[n_L(r)] = \frac{Lr^2}{1 - r^2}.$$

No scaling!

The case $L = 1$

Theorem (Peres-Virág 05)

The point process given by the zero set of the hyperbolic GAF forms a deteminantal point process for $L = 1$. The random variable $n_1(r)$ has the same distribution as

$$\sum_{n=0}^{\infty} B_n$$

where B_n are independent Bernoulli random variables with $\mathbb{P}[B_n = 1] = r^{2n}$.

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Corollary

- $\mathbb{V}[n_1(r)] = \frac{r^2}{1-r^4}$
- $\mathbb{P}[n_1(r) = 0] = \exp\left(-\frac{\pi^2}{12}\frac{1}{1-r}(1 + o(1))\right)$ *as* $r \to 1$

The variance

Theorem (Sodin, unpublished)

$$\mathbb{V}[n_L(r)] \sim \frac{\zeta(3/2)}{8\pi^{3/2}} \sqrt{L}\, \mathrm{length}_{\mathrm{hyp}}(r\mathbb{T}) \qquad \textit{as } L \to \infty.$$

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Theorem

As $r \to 1^-$

$$\mathbb{V}[n_L(r)] \sim \begin{cases} c_L \frac{1}{1-r} & \textit{for } L > 1/2 \\ \frac{1}{8\pi} \frac{1}{1-r} \log \frac{1}{1-r} & \textit{for } L = 1/2 \\ c_L \frac{1}{(1-r)^{2-2L}} & \textit{for } L < 1/2. \end{cases}$$

Hole Probability

Theorem

As $L \to \infty$

$$\mathbb{P}[n_L(r) = 0] \begin{cases} \approx \exp\left(-c(r)L^2\right) & \textit{for fixed } r \\ = \exp\left(-\frac{e^2}{4}L^2 r^4 (1 + o(1))\right) & \textit{for } r \to 0,\ Lr^2 \to \infty \end{cases}$$

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Theorem (B., Nishry, Peled, Sodin)

As $r \to 1^-$

$$\mathbb{P}[n_L(r) = 0] \begin{cases} = \exp\left(-\frac{(L-1)^2}{4}\frac{1}{1-r}(\log\frac{1}{1-r})^2(1+o(1))\right) & \textit{for } L > 1 \\ \approx \exp\left(-c_L \frac{1}{(1-r)^L} \log \frac{1}{1-r}\right) & \textit{for } L < 1 \end{cases}$$

What changes at $L = 1$?

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$$\binom{n+L-1}{n} = \frac{\Gamma(n+L)}{\Gamma(L)\Gamma(n)} \sim \frac{n^{L-1}}{\Gamma(L)} \text{ for large } n.$$

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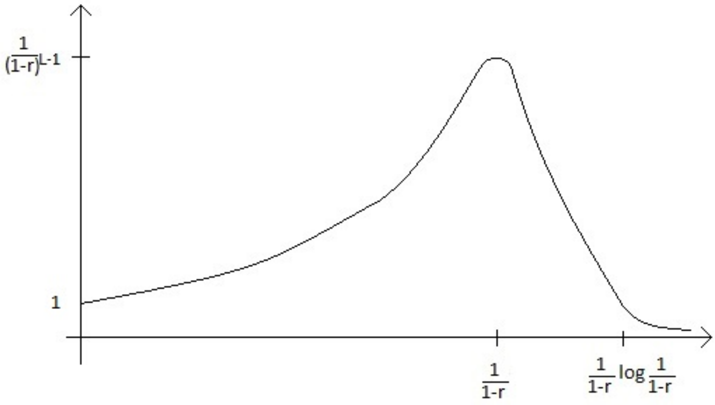

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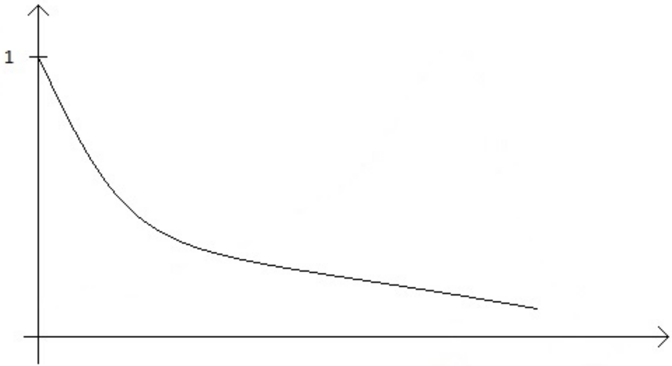

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Condition on $f_L(0)$

Upper bound heuristics

Define $I = \int_0^{2\pi} \log|f_L(re^{i\theta})| \frac{d\theta}{2\pi}$ and write $f_L(z) = f_L(0) + g_L(z)$.

$$\mathbb{E}[I|f_L(0)] = \log|f_L(0)| + \mathbb{E}\left[\log\left|1 + \frac{g_L(r)}{f_L(0)}\right| |f_L(0)\right]$$

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Maybe, for $L < 1$,

$$\mathbb{P}[n_L(r) = 0] = \exp\left(-c_L \frac{1}{(1-r)^L} \log \frac{1}{1-r}(1 + o(1))\right)$$

with

$$c_L \simeq \begin{cases} 1 - L & \text{for } L \text{ near } 1 \\ L & \text{for } L \text{ near } 0. \end{cases}$$

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Wild speculation:

$$c_L = \begin{cases} \frac{1-L}{2^L} & \text{for } L \leq 1/2 \\ \frac{L}{2^L} & \text{for } L \geq 1/2. \end{cases}$$

Go raibh maith agaibh as éisteacht liom