Persistence Probabilities

Amir Dembo[1]

Stanford University

Darmstadt Workshop; July 2014

[1] Jointly with Jian Ding, Fuchang Gao, Sumit Mukherjee, Bjorn Poonen, Qi-Man Shao and Ofer Zeitouni.

$(X_t, t \geq 0)$ is $\mathbb{R}$-valued $(X_0 = 0)$. First passage time

$$\tau_z = \inf\{t > 0 : X_t > z\} .$$

- Typically $z = 0$.
- Persistence (power law) exponent b if $\mathbb{P}(\tau_z \geq T) = T^{-b+o(1)}$.
- Large deviations problem (hitting probab. of Markov processes).
- In many examples, little gained from general theories (?).
- Focus on few examples of much interest.

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?
- Up to factor 2 same as $\mathbb{P}(Q_n(\cdot)$ has no roots in $J)$.

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?
- $p_{\mathbb{R}}(n) \sim \mathbb{P}(Q_n$ has no real roots$) = \mathbb{P}(N_n = 0)$.

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?

- $p_{\mathbb{R}}(n) \sim \mathbb{P}(Q_n$ has no real roots$) = \mathbb{P}(N_n = 0)$.
- $\mathbb{E}(N_n) \sim c_\alpha \log n$ if ξ_i attracted to $\alpha \in (0, 2]$ stable [Ibragimov-Maslova/Logan-Shepp '68-'71; following Kac '43].

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?
- $p_{\mathbb{R}}(n) \sim \mathbb{P}(Q_n$ has no real roots$) = \mathbb{P}(N_n = 0)$.
- $O(e^{-Cn}) \leq p_{\mathbb{R}}(2n) \leq O(1/\log n)$ if $p > 2$ finite moments [Maslova '74; after Littlewood-Offord '38-'43].

Real roots of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

$$Q_n(x) = \sum_{i=0}^{n} \xi_i x^i \,.$$

- Asymptotics of $p_J(n) = \mathbb{P}(Q_n(x) < 0, \forall x \in J)$?
- $p_{\mathbb{R}}(n) \sim \mathbb{P}(Q_n$ has no real roots$) = \mathbb{P}(N_n = 0)$.
- $O(e^{-Cn}) \leq p_{\mathbb{R}}(2n) \leq O(1/\log n)$ if $p > 2$ finite moments [Maslova '74; after Littlewood-Offord '38-'43].
- Much work on complex roots (e.g. [Edelman-Kostlan '95; Sodin-Tsirelson '04, Hough et al '10, Nishry '11], more).

Persistence for real roots

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbf{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\operatorname{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

$$\mathbb{P}(Y_t^{(\kappa)} \leq 0, \forall t \in [0,T]) = e^{-b_\kappa T(1+o(1))} \approx e^{-b_\kappa T}\,.$$

Persistence for real roots

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbf{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\mathrm{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

$$\mathbb{P}(Y_t^{(\kappa)} \le 0, \forall t \in [0,T]) = e^{-b_\kappa T(1+o(1))} \approx e^{-b_\kappa T}\,.$$

(Slepian's lemma & sub-additive lemma).

Persistence for real roots

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbf{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\mathrm{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

$$\mathbb{P}(Y_t^{(\kappa)} \leq 0, \forall t \in [0,T]) = e^{-b_\kappa T(1+o(1))} \approx e^{-b_\kappa T}\,.$$

(Slepian's lemma $\&$ sub-additive lemma).

$p_J(n) \approx n^{-b_0}$ for $J = \pm[0,1]$, $J = \pm[1,\infty)$
[Dembo-Poonen-Shao-Zeitouni '02].

Persistence for real roots

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbf{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\operatorname{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

$$\mathbb{P}(Y_t^{(\kappa)} \leq 0, \forall t \in [0,T]) = e^{-b_\kappa T(1+o(1))} \approx e^{-b_\kappa T}.$$

(Slepian's lemma $\&$ sub-additive lemma).

$p_J(n) \approx n^{-b_0}$ for $J = \pm[0,1]$, $J = \pm[1,\infty)$
[Dembo-Poonen-Shao-Zeitouni '02].

- $\mathbb{P}(N_{2n} = 0) \approx n^{-4b_0}$.

Persistence for real roots

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbf{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\mathrm{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

$$\mathbb{P}(Y_t^{(\kappa)} \leq 0, \forall t \in [0,T]) = e^{-b_\kappa T(1+o(1))} \approx e^{-b_\kappa T}\,.$$

(Slepian's lemma $\&$ sub-additive lemma).

$p_J(n) \approx n^{-b_0}$ for $J = \pm[0,1]$, $J = \pm[1,\infty)$ [Dembo-Poonen-Shao-Zeitouni '02].

- $\mathbb{P}(N_{2n} = 0) \approx n^{-4b_0}$.

Valid if ξ_i of zero mean and finite moments (by KMT reduce to $\xi_i \sim N(0,1)$).

- Proof in case of $N(0,1)$:

Persistence for real roots, Gaussian

- Proof in case of $N(0,1)$:
  - By symmetry of auto-correlations, just consider $J = [0,1]$.

Persistence for real roots, Gaussian

- Proof in case of $N(0, 1)$:
  - By symmetry of auto-correlations, just consider $J = [0, 1]$.
  - $x \in [0, 1 - n^{-o(1)}]$ done by $\xi_i \geq 1$ for $i \leq o(T_n)$, $T_n := \log n$.

Persistence for real roots, Gaussian

- Proof in case of $N(0,1)$:
  - By symmetry of auto-correlations, just consider $J = [0,1]$.
  - $x \in [0, 1 - n^{-o(1)}]$ done by $\xi_i \geq 1$ for $i \leq o(T_n)$, $T_n := \log n$.
  - $\mathbf{E}[Q_n'(x)^2] \leq n^2 \mathbf{E}[Q_n(x)^2]$ so ignore $x \in [1 - n^{-1}, 1]$.

Persistence for real roots, Gaussian

- Proof in case of $N(0,1)$:
  - By symmetry of auto-correlations, just consider $J = [0,1]$.
  - $x \in [0, 1 - n^{-o(1)}]$ done by $\xi_i \geq 1$ for $i \leq o(T_n)$, $T_n := \log n$.
  - $\mathbf{E}[Q_n'(x)^2] \leq n^2 \mathbf{E}[Q_n(x)^2]$ so ignore $x \in [1 - n^{-1}, 1]$.
  - $x = e^{-e^{-s}} \in [1 - n^{-o(1)}, 1 - n^{-1}]$ mapped to $s \in [o(T_n), T_n]$.

Persistence for real roots, Gaussian

- Proof in case of $N(0,1)$:
  - By symmetry of auto-correlations, just consider $J = [0,1]$.
  - $x \in [0, 1 - n^{-o(1)}]$ done by $\xi_i \geq 1$ for $i \leq o(T_n)$, $T_n := \log n$.
  - $\mathbf{E}[Q'_n(x)^2] \leq n^2 \mathbf{E}[Q_n(x)^2]$ so ignore $x \in [1 - n^{-1}, 1]$.
  - $x = e^{-e^{-s}} \in [1 - n^{-o(1)}, 1 - n^{-1}]$ mapped to $s \in [o(T_n), T_n]$.
  - Auto-correlation of $Q_n(e^{-e^{-s}})$ explicit $\&$ close to that of $Y_s^{(0)}$.

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
  - Formal argument for $\kappa > -1$ [Schehr-Majumdar '07-'08].

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
  - Formal argument for $\kappa > -1$ [Schehr-Majumdar '07-'08].
  - Extended and proved in [Dembo-Mukherjee '12], for $\xi_i \sim N(0, L(i))$, $i \mapsto L(i)$ regularly varying of order $\kappa \in \mathbb{R}$ (i.e. $L(\lambda i)/L(i) \to \lambda^\kappa$ as $i \to \infty$).

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
  - Formal argument for $\kappa > -1$ [Schehr-Majumdar '07-'08].
  - Extended and proved in [Dembo-Mukherjee '12], for $\xi_i \sim N(0, L(i))$, $i \mapsto L(i)$ regularly varying of order $\kappa \in \mathbb{R}$ (i.e. $L(\lambda i)/L(i) \to \lambda^\kappa$ as $i \to \infty$).
  - For general $L(\cdot)$ arbitrary convergence rate ϵ_n for auto-correlations of $Q_n(e^{-e^{-s}})$ to those of $Y_s^{(\kappa)}$.

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
  - Formal argument for $\kappa > -1$ [Schehr-Majumdar '07-'08].
  - Extended and proved in [Dembo-Mukherjee '12], for $\xi_i \sim N(0, L(i))$, $i \mapsto L(i)$ regularly varying of order $\kappa \in \mathbb{R}$ (i.e. $L(\lambda i)/L(i) \to \lambda^\kappa$ as $i \to \infty$).
  - For general $L(\cdot)$ arbitrary convergence rate ϵ_n for auto-correlations of $Q_n(e^{-e^{-s}})$ to those of $Y_s^{(\kappa)}$.
  - When $T_n \epsilon_n \to \infty$ persistence power exponent can be discontinuous (e.g. $\sqrt{1-\epsilon_n} Y_s + \sqrt{\epsilon_n} Z$).

Persistence for real roots, $\kappa \in \mathbb{R}$

Consider independent $\xi_i \sim N(0, i^\kappa)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
  - Formal argument for $\kappa > -1$ [Schehr-Majumdar '07-'08].
  - Extended and proved in [Dembo-Mukherjee '12], for $\xi_i \sim N(0, L(i))$, $i \mapsto L(i)$ regularly varying of order $\kappa \in \mathbb{R}$ (i.e. $L(\lambda i)/L(i) \to \lambda^\kappa$ as $i \to \infty$).
  - For general $L(\cdot)$ arbitrary convergence rate ϵ_n for auto-correlations of $Q_n(e^{-e^{-s}})$ to those of $Y_s^{(\kappa)}$.
  - When $T_n \epsilon_n \to \infty$ persistence power exponent can be discontinuous (e.g. $\sqrt{1-\epsilon_n} Y_s + \sqrt{\epsilon_n} Z$).
  - Resolved in [Dembo-Mukherjee '12] by new theorem about continuity of persistence exponents for Gaussian processes.

Continuity of persistence exponents

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \le k \le \infty$, of **non-negative** covariance.
Persistence exponents:

$$\mathbb{P}(Z_k(t) \le 0, \forall t \in [0, T]) \approx e^{-\bar{b}_k T} .$$

Then, $\bar{b}_k \to \bar{b}_\infty$ [Dembo-Mukherjee '12], provided:

Continuity of persistence exponents

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \le k \le \infty$, of **non-negative** covariance. Persistence exponents:

$$\mathbb{P}(Z_k(t) \le 0, \forall t \in [0,T]) \approx e^{-\bar{b}_k T}\,.$$

Then, $\bar{b}_k \to \bar{b}_\infty$ [Dembo-Mukherjee '12], provided:

- $M_k = \sup_{t \le T} Z_k(t)$ converges weakly to M_∞ (per fixed T).

Continuity of persistence exponents

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \le k \le \infty$, of **non-negative** covariance.
Persistence exponents:

$$\mathbb{P}(Z_k(t) \le 0, \forall t \in [0,T]) \approx e^{-\bar{b}_k T}\,.$$

Then, $\bar{b}_k \to \bar{b}_\infty$ [Dembo-Mukherjee '12], provided:

- $M_k = \sup_{t \le T} Z_k(t)$ converges weakly to M_∞ (per fixed T).
- $\bar{b}_\infty$ unchanged for threshold $T^{-\eta}$, some $\eta > 0$.

Continuity of persistence exponents

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \le k \le \infty$, of **non-negative** covariance.
Persistence exponents:

$$\mathbb{P}(Z_k(t) \le 0, \forall t \in [0,T]) \approx e^{-\bar{b}_k T}\,.$$

Then, $\bar{b}_k \to \bar{b}_\infty$ [Dembo-Mukherjee '12], provided:

- $M_k = \sup_{t \le T} Z_k(t)$ converges weakly to M_∞ (per fixed T).
- $\bar{b}_\infty$ unchanged for threshold $T^{-\eta}$, some $\eta > 0$.
- $\sup_k \, t^{1+\eta}\, \mathbf{E}[Z_k(0) Z_k(t)] \to 0$ as $t \to \infty$ (for some $\eta > 0$).

Continuity of persistence exponents

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \leq k \leq \infty$, of **non-negative** covariance.
Persistence exponents:

$$\mathbb{P}(Z_k(t) \leq 0, \forall t \in [0,T]) \approx e^{-\bar{b}_k T}\,.$$

Then, $\bar{b}_k \to \bar{b}_\infty$ [Dembo-Mukherjee '12], provided:

- $M_k = \sup_{t \leq T} Z_k(t)$ converges weakly to M_∞ (per fixed T).
- $\bar{b}_\infty$ unchanged for threshold $T^{-\eta}$, some $\eta > 0$.
- $\sup_k \, t^{1+\eta}\,\mathbf{E}[Z_k(0)Z_k(t)] \to 0$ as $t \to \infty$ (for some $\eta > 0$).

[Dembo-Mukherjee '14] $\mathbf{E}[Z_k(0)Z_k(t)]$ reg. var. index $\kappa \in [-1, 0)$.
[Feldheim-Feldheim '13] which cov. yield $\bar{b} > 0$?

Persistence for real roots

- Open problems:

Persistence for real roots

- Open problems:
  - Evaluate b_0?
    Numerically near $3/16$, best rigorous bounds $(1/(4\sqrt{3}), 1/4]$
    [Molchan '12; Li-Shao '02].

- Open problems:
  - Evaluate b_0?
    Numerically near $3/16$, best rigorous bounds $(1/(4\sqrt{3}), 1/4]$
    [Molchan '12; Li-Shao '02].
  - α-stable ξ_i no result/method;
    For $\alpha = 1$, numerically persistence exponent near $0.86 > 4b_0$.

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x},t)}{\partial t} = \Delta \phi_d(\mathbf{x},t)$$

with $\phi_d(\mathbf{x},0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x}-\mathbf{y})$ (white noise).

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x},t)}{\partial t} = \Delta \phi_d(\mathbf{x},t)$$

with $\phi_d(\mathbf{x},0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x}-\mathbf{y})$ (white noise).

- ϕ_d well-defined on $\mathbb{R}^d \times (0,\infty)$ [Bramson-Lebowitz '01].
$d \leq 3$ controls asymptotics of two-particle reactions.

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x},t)}{\partial t} = \Delta \phi_d(\mathbf{x},t)$$

with $\phi_d(\mathbf{x},0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x}-\mathbf{y})$ (white noise).

- ϕ_d well-defined on $\mathbb{R}^d \times (0,\infty)$ [Bramson-Lebowitz '01].
$d \leq 3$ controls asymptotics of two-particle reactions.

- From [Schehr-Majumdar '08]:

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x}, t)}{\partial t} = \Delta \phi_d(\mathbf{x}, t)$$

with $\phi_d(\mathbf{x}, 0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x} - \mathbf{y})$ (white noise).

- ϕ_d well-defined on $\mathbb{R}^d \times (0, \infty)$ [Bramson-Lebowitz '01]. $d \leq 3$ controls asymptotics of two-particle reactions.

- From [Schehr-Majumdar '08]:
  - $\phi_d(\mathbf{0}, e^t) \stackrel{\mathcal{L}}{=} c_t Y_t^{(\kappa)}$ for $\kappa = d/2 - 1$ ($c_t > 0$ non-random).

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x},t)}{\partial t} = \Delta \phi_d(\mathbf{x},t)$$

with $\phi_d(\mathbf{x},0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x}-\mathbf{y})$ (white noise).

- ϕ_d well-defined on $\mathbb{R}^d \times (0,\infty)$ [Bramson-Lebowitz '01]. $d \le 3$ controls asymptotics of two-particle reactions.

- From [Schehr-Majumdar '08]:
  - $\phi_d(\mathbf{0}, e^t) \stackrel{\mathcal{L}}{=} c_t Y_t^{(\kappa)}$ for $\kappa = d/2 - 1$ ($c_t > 0$ non-random).
  - $\mathbb{P}(\phi_d(\mathbf{0},t) > 0, \forall t \in [1,T]) \approx T^{-b_\kappa}$.

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x},t)}{\partial t} = \Delta \phi_d(\mathbf{x},t)$$

with $\phi_d(\mathbf{x},0) = \psi(\mathbf{x})$ zero-mean, Gaussian field of covariance $\delta_d(\mathbf{x}-\mathbf{y})$ (white noise).

- ϕ_d well-defined on $\mathbb{R}^d \times (0,\infty)$ [Bramson-Lebowitz '01]. $d \le 3$ controls asymptotics of two-particle reactions.

- From [Schehr-Majumdar '08]:
  - $\phi_d(\mathbf{0}, e^t) \stackrel{\mathcal{L}}{=} c_t Y_t^{(\kappa)}$ for $\kappa = d/2 - 1$ ($c_t > 0$ non-random).
  - $\mathbb{P}(\phi_d(\mathbf{0},t) > 0, \forall t \in [1,T]) \approx T^{-b_\kappa}$.

- No intuitive explanation of these connections!

Gaussian interface models

- $\nabla\phi$-interface model in $(d+1)$ dimensions solves SDS:

$$d\phi_t(\mathbf{x}) = -\frac{\partial H}{\partial \phi(\mathbf{x})}(\phi_t(\mathbf{x})) + \sqrt{2} dB_t(\mathbf{x})$$

with $\phi_0(\mathbf{x}) = 0$, $B_t(\mathbf{x})$ independent Brownian motions,

$$H(\phi) = \frac{1}{4} \sum_{\mathbf{x}} \sum_{\mathbf{y}} q(\mathbf{x}-\mathbf{y})(\phi(\mathbf{x}) - \phi(\mathbf{y}))^2$$

- q symmetric law of bounded support $\Longleftrightarrow$ jumps for $\boldsymbol{Z}^d$-irreducible RW S_t of expected local time $L(\cdot)$ at $\mathbf{0}$.

Gaussian interface models

- $\nabla\phi$-interface model in $(d+1)$ dimensions solves SDS:

$$d\phi_t(\mathbf{x}) = -\frac{\partial H}{\partial \phi(\mathbf{x})}\big(\phi_t(\mathbf{x})\big) + \sqrt{2} dB_t(\mathbf{x})$$

with $\phi_0(\mathbf{x}) = 0$, $B_t(\mathbf{x})$ independent Brownian motions,

$$H(\phi) = \frac{1}{4}\sum_{\mathbf{x}}\sum_{\mathbf{y}} q(\mathbf{x}-\mathbf{y})(\phi(\mathbf{x}) - \phi(\mathbf{y}))^2$$

- q symmetric law of bounded support $\Longleftrightarrow$ jumps for $\boldsymbol{Z}^d$-irreducible $\textsc{rw}$ S_t of expected local time $L(\cdot)$ at $\mathbf{0}$.

- $G_t = \phi_t(\mathbf{0})$, zero-mean, continuous Gaussian process.

$\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1,T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1, T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

  - $P_1(T) \approx T^{-c_1}$, universal $0 < c_1 < \infty$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1, T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

  - $P_1(T) \approx T^{-c_1}$, universal $0 < c_1 < \infty$.
  - $P_3(T) \approx e^{-O(\sqrt{T}\log T)}$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1, T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

  - $P_1(T) \approx T^{-c_1}$, universal $0 < c_1 < \infty$.
  - $P_3(T) \approx e^{-O(\sqrt{T}\log T)}$.
  - $P_4(T) \approx e^{-O(T \log\log T / \log T)}$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log |t+s|}{\log |t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1, T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

  - $P_1(T) \approx T^{-c_1}$, universal $0 < c_1 < \infty$.
  - $P_3(T) \approx e^{-O(\sqrt{T} \log T)}$.
  - $P_4(T) \approx e^{-O(T \log \log T / \log T)}$.
  - $P_d(T) \approx e^{-c_d(q)T}$, for any $d \geq 5$, $c_d(q) \in (0, \infty)$.

Persistence for Gaussian interface models

- $\mathbf{E}[G_s G_t] = L(t+s) - L(|t-s|)$.

$\mathbf{E}[G_s G_t] \approx |t-s|^{1-d/2}$, $d \geq 3$.

$\mathbf{E}[G_s G_t] \approx \frac{\log|t+s|}{\log|t-s|}$ for $d = 2$.

- $P_d(T) := \mathbb{P}(G_t > 0, \forall t \in [1, T])$.

[Dembo-Mukherjee '14], building on [Sakagawa '12]:

  - $P_1(T) \approx T^{-c_1}$, universal $0 < c_1 < \infty$.
  - $P_3(T) \approx e^{-O(\sqrt{T}\log T)}$.
  - $P_4(T) \approx e^{-O(T \log\log T / \log T)}$.
  - $P_d(T) \approx e^{-c_d(q)T}$, for any $d \geq 5$, $c_d(q) \in (0, \infty)$.

- Open: Find decay rates for $P_2(T)$.

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)}\,.$$

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

$$q_\ell(n) := \mathbb{P}(S_k^{(\ell)} < 0, \ \forall k \leq n) \approx n^{-r_\ell}, \quad \forall \ell \geq 1 .$$

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

$$q_\ell(n) := \mathbb{P}(S_k^{(\ell)} < 0, \; \forall k \leq n) \approx n^{-r_\ell}, \quad \forall \ell \geq 1 .$$

  - Reduce by KMT to $\xi_i \sim N(0,1)$ or

$$\bar{q}_\ell(T) := \mathbb{P}(Z_t^{(\ell)} \leq 1, \forall t \in [0,T]) \approx T^{-r_\ell} .$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

$$q_\ell(n) := \mathbb{P}(S_k^{(\ell)} < 0, \ \forall k \leq n) \approx n^{-r_\ell}, \quad \forall \ell \geq 1 .$$

- Reduce by KMT to $\xi_i \sim N(0,1)$ or

$$\bar{q}_\ell(T) := \mathbb{P}(Z_t^{(\ell)} \leq 1, \forall t \in [0,T]) \approx T^{-r_\ell} .$$

  $Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

- $r_1 = 1/2$ [Feller '71; Spitzer's formula].

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

$$q_\ell(n) := \mathbb{P}(S_k^{(\ell)} < 0, \ \forall k \leq n) \approx n^{-r_\ell}, \quad \forall \ell \geq 1 .$$

- Reduce by KMT to $\xi_i \sim N(0,1)$ or

$$\bar{q}_\ell(T) := \mathbb{P}(Z_t^{(\ell)} \leq 1, \forall t \in [0,T]) \approx T^{-r_\ell} .$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

- $r_1 = 1/2$ [Feller '71; Spitzer's formula].
- $r_2 = 1/4$ [Aurzada-Dereich '11, following McKean '63].

ℓ-iterated sums & integrals of Brownian motion

$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

$$S_k^{(\ell)} = S_{k-1}^{(\ell)} + S_k^{(\ell-1)} .$$

- $S_n^{(1)} = S_n$ are partial sums of $\{\xi_i\}$, $S_n^{(2)} = \sum_{k=1}^n S_k$, etc.

$$q_\ell(n) := \mathbb{P}(S_k^{(\ell)} < 0, \ \forall k \leq n) \approx n^{-r_\ell}, \quad \forall \ell \geq 1 .$$

- Reduce by KMT to $\xi_i \sim N(0,1)$ or

$$\bar{q}_\ell(T) := \mathbb{P}(Z_t^{(\ell)} \leq 1, \forall t \in [0,T]) \approx T^{-r_\ell} .$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

- $r_1 = 1/2$ [Feller '71; Spitzer's formula].
- $r_2 = 1/4$ [Aurzada-Dereich '11, following McKean '63].
- $r_\ell \downarrow b_0$ for $\ell \uparrow \infty$ [Li-Shao '05].

ℓ-iterated integrals of Brownian motion

- Open problems:

$$\mathbb{P}(Z_t^{(\ell)} \le 1,\ \forall t \in [0,T]) \approx T^{-r_\ell}\,.$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

ℓ-iterated integrals of Brownian motion

- Open problems:

$$\mathbb{P}(Z_t^{(\ell)} \le 1, \ \forall t \in [0,T]) \approx T^{-r_\ell} \,.$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

  - Find r_ℓ for any $\ell \ge 3$.

- Open problems:

$$\mathbb{P}(Z_t^{(\ell)} \leq 1, \; \forall t \in [0,T]) \approx T^{-r_\ell} \, .$$

$Z_t^{(1)} = B_t$ Brownian motion, $Z_t^{(2)} = \int_0^t B_s ds$, etc.

  - Find r_ℓ for any $\ell \geq 3$.
  - Fully handle $\ell = 2$, stable case ([Profeta-Simon '14]).

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].
- $\ell = 2$ and Brownian motion [from Mckean '63].

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].
- $\ell = 2$ and Brownian motion [from Mckean '63].
- $\ell = 2$ and ± 1-valued ξ_i [Sinai '92; extended by Vysotsky '10].

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].
- $\ell = 2$ and Brownian motion [from Mckean '63].
- $\ell = 2$ and ± 1-valued ξ_i [Sinai '92; extended by Vysotsky '10].
- $\ell = 2$ any ξ_i of finite second moment [Dembo-Ding-Gao '13].

Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].
- $\ell = 2$ and Brownian motion [from Mckean '63].
- $\ell = 2$ and ± 1-valued ξ_i [Sinai '92; extended by Vysotsky '10].
- $\ell = 2$ any ξ_i of finite second moment [Dembo-Ding-Gao '13].

- Loss of $\log n$ factors by strong approximation (at small k).
- Relevant for conditional law [Denisov-Wachtel '12].

Gaussian AR sequences

$\underline{a} := (a_1, a_2, \dots, a_L)$ non-random $\mathbb{R}^L$-valued.

For $\xi_i \sim N(0,1)$ i.i.d.

$$X_k = 0, \forall k \le 0, \quad \& \quad X_k = \sum_{i=1}^{L} a_i X_{k-i} + \xi_k, \qquad \forall k \ge 1$$

a Gaussian auto-regressive sequence ($\underline{a} = (1)$ for $X_k = S_k$; $\underline{a} = (2,-1)$ for $S_k^{(2)}$, $\underline{a} = (3,-3,1)$ for $S_k^{(3)}$, etc).

Gaussian AR sequences

$\underline{a} := (a_1, a_2, \ldots, a_L)$ non-random $\mathbb{R}^L$-valued.

For $\xi_i \sim N(0,1)$ i.i.d.

$$X_k = 0, \forall k \leq 0, \quad \& \quad X_k = \sum_{i=1}^{L} a_i X_{k-i} + \xi_k, \qquad \forall k \geq 1$$

a Gaussian auto-regressive sequence ($\underline{a} = (1)$ for $X_k = S_k$; $\underline{a} = (2, -1)$ for $S_k^{(2)}$, $\underline{a} = (3, -3, 1)$ for $S_k^{(3)}$, etc).

Stability of $k \mapsto X_k$ according to $\rho := \max\{|z| : z \in \Lambda\}$,

$$\Lambda := \{z \in \mathbb{C} : z^L - \sum_{i=1}^{L} a_i z^{L-i} = 0\}.$$

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1, n]) \,.$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min \left\{ \frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho \right\} \in [0, 1] \,.$$

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1, n]) .$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min \left\{ \frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho \right\} \in [0, 1] .$$

  - $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1, n]) \,.$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min\left\{\frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho\right\} \in [0, 1] \,.$$

  - $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.
  - $q(n) \approx e^{-O(n)}$ for $\rho < 1$ or $\beta = 0$.

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1,n])\,.$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min\left\{\frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho\right\} \in [0,1]\,.$$

- $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.
- $q(n) \approx e^{-O(n)}$ for $\rho < 1$ or $\beta = 0$.
- $q(n) = e^{-n^{1-\beta+o(1)}}$ if $\rho = 1$ and $\beta \in (0,1)$.

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1,n]) \,.$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min\left\{\frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho\right\} \in [0,1] \,.$$

- $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.
- $q(n) \approx e^{-O(n)}$ for $\rho < 1$ or $\beta = 0$.
- $q(n) = e^{-n^{1-\beta+o(1)}}$ if $\rho = 1$ and $\beta \in (0,1)$.
- $q(n) \approx n^{-r_{\underline{a}}}$, $r_{\underline{a}} \in (0,\infty)$
  if $\rho = \beta = 1$ or $\rho > 1$ and $\beta \in (0,1)$.

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1, n]) \,.$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min\left\{\frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho\right\} \in [0, 1] \,.$$

  - $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.
  - $q(n) \approx e^{-O(n)}$ for $\rho < 1$ or $\beta = 0$.
  - $q(n) = e^{-n^{1-\beta+o(1)}}$ if $\rho = 1$ and $\beta \in (0, 1)$.
  - $q(n) \approx n^{-r_{\underline{a}}}$, $r_{\underline{a}} \in (0, \infty)$
    if $\rho = \beta = 1$ or $\rho > 1$ and $\beta \in (0, 1)$.
  - $r_{\underline{a}}$ related to persistence power exponent of d-dimensional Brownian motion staying in certain cones ([Banuelos-Smits '97]).

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1,n]) .$$

$m(z)$ is multiplicity of z in Λ and

$$\beta := \min \left\{ \frac{m(\rho)}{m(z)} : z \in \Lambda, |z| = \rho \right\} \in [0,1] .$$

  - $q(n) = O(1)$ for $\rho > 1$ and $\beta = 1$.
  - $q(n) \approx e^{-O(n)}$ for $\rho < 1$ or $\beta = 0$.
  - $q(n) = e^{-n^{1-\beta+o(1)}}$ if $\rho = 1$ and $\beta \in (0,1)$.
  - $q(n) \approx n^{-r_{\underline{a}}}$, $r_{\underline{a}} \in (0,\infty)$
    if $\rho = \beta = 1$ or $\rho > 1$ and $\beta \in (0,1)$.
  - $r_{\underline{a}}$ related to persistence power exponent of d-dimensional Brownian motion staying in certain cones ([Banuelos-Smits '97]).

- $\theta \mapsto r_{\underline{a}}$ discontinuous at $\theta \in \mathbb{Q}$ when $\Lambda = \{1, e^{2\pi i\theta}, e^{-2\pi i\theta}\}$
  (i.e. $\underline{a} = (\eta, -\eta, 1)$, $\eta = 1 + 2\cos(2\pi\theta)$).

Thank you!