Persistence probabilities for integrated random walks

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Persistence probabilities @ Darmstadt

16/07/2014

Outline

Statement of the problem and relations to other questions

Known results

Main results [Aurzada, D '13]

Idea of the proofs

Pinned bridges [Aurzada, D, Lifshits '14]

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Given: $(A_t)_{t\geq 0}$ stochastic process with $A_0 = 0$.

Goal: Find asymptotics of

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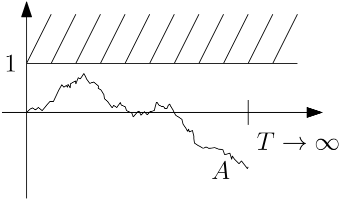

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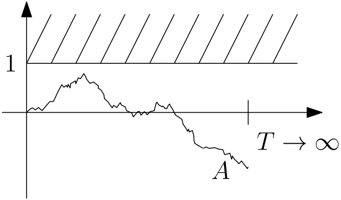


Typically, one expects

$$\mathbb{P}\left[\sup_{0\leq t\leq T} A_t \leq 1\right] = T^{-\theta+o(1)}, \qquad \text{as } T \to \infty$$

with $\theta > 0$, called survival exponent

Relations to other questions

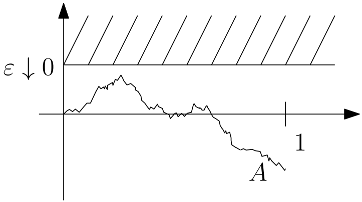
$$\mathbb{P}\left[\sup_{0\le t\le T} A_t \le 1\right] = T^{-\theta+o(1)}, \qquad \text{as } T\to\infty.$$

- statistical mechanics: Burgers' equation – a PDE considered with random initial condition (Sinaĭ'92, Bertoin'98, Molchan'99, Simon'08)
- Entropic repulsion/wetting models – discrete case (Caravenna/Deuschel'08)
- pursuit problems – 'random prisoner is followed by a random policeman' (Li/Shao'02)
- zeros of random polynomials (Dembo/Poonen/Shao/Zeitouni'02, Li/Shao'04)

Relations to other questions

For an H-self-similar processes, the question is the same as

$$\mathbb{P}\left[\sup_{0\le t\le 1} A_t \le \varepsilon\right] = \varepsilon^{\theta/H+o(1)}, \qquad \text{as } \varepsilon \to 0.$$



that is, the lower tail of $A_1^* := \sup_{t\in[0,1]} A_t$

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Known results: continuous-time processes

$$\mathbb{P}\left[\sup_{0\leq t\leq T} A_t \leq 1\right] = T^{-\theta+o(1)}, \qquad \text{as } T\to\infty.$$

- Brownian motion: $\theta = 1/2$ (reflection principle gives even the law)
- $A_t = \int_0^t B_s \,\mathrm{d}s$ integrated Brownian motion: $\theta = 1/4$ (McKean '63, Goldman '71, Sinaĭ '92)
- fractional Brownian motion: $\theta = 1 - H$ (Molchan '99)
- Lévy processes (LP) (classical results of fluctuation theory)
- many Gaussian processes: polynomial scale (Li/Shao '04)
- integrated stable LP with no negative jumps (Simon '07)

Known results: discrete case

$$\mathbb{P}\left[\sup_{1\le n\le T} A_n \le 1\right] = T^{-\theta+o(1)}, \qquad \text{as } T\to\infty.$$

Let $A_n = \sum_{i=1}^n X_i$ be integrated random walk.

- *X simple* RW: $\theta = 1/4$ (Sinaĭ '92)

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- X *simple* RW: $\theta = 1/4$ (Sinaĭ '92)
- X with finite exp. moments: $\theta \le 1/2$ and logarithmic upper bound (Caravenna/Deuschel '08)
- X with Gaussian increments: polynomial scale (Li/Shao '04)
- X (lattice valued, other special cases): $\theta = 1/4$ (Vysotsky '10)

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- Our work: true for general X with finite exp. moments

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- X (lattice valued, other special cases): $\theta = 1/4$ (Vysotsky '10)
- Our work: true for general X with finite exp. moments
- Strong asymptotics with $\theta = 1/4$, if $2+\varepsilon$-moment finite (Dembo/Ding/Gao 13, Denisov/Wachtel '14+)

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Main results: overview

Three main results:

- (1) universality of the asymptotics
- (2) existence of the survival exponent

(1) Universality result

- X: either a LP or RW with $\exists \beta > 0$ s.th. $\mathbb{E}e^{\beta |X_1|} < \infty$ and $\mathbb{E}X_1 = 0$

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Theorem: For two processes X and Y as above we have

$$\mathbb{P}\left[\sup_{0\leq t\leq T} \mathcal{I}(X)_t \leq 1\right] \asymp_{\log} \mathbb{P}\left[\sup_{0\leq t\leq T} \mathcal{I}(Y)_t \leq 1\right]$$

Here: $f \asymp_{\log} g$ means that there exists $c, \delta > 0$ such that for large T

$$(c\log T)^{-\delta} f(T) \leq g(T) \leq (c\log T)^{\delta} f(T)$$

(1) Universality result: Main example

Fractional integration operator:

$$\mathcal{I}_\alpha(X)_t := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X_s \, \mathrm{d}s, \qquad t \geq 0$$

for some $\alpha > 0$ (recall: $\mathcal{I}_\alpha = (\mathcal{I}_1)^\alpha$ for integer α).

Corollary: For two processes X and Y as above we have

$$\mathbb{P}\left[\sup_{0 \leq t \leq T} \mathcal{I}_\alpha(X)_t \leq 1\right] \asymp_{\log} \mathbb{P}\left[\sup_{0 \leq t \leq T} \mathcal{I}_\alpha(Y)_t \leq 1\right]$$

In particular, the asymptotics are equivalent w.r.t. $\asymp_{\log}$!

(1) Universality result: Integrated random walk

Usual integration operator:

$$\mathcal{I}_1(X)_t = \int_0^t X_s \, \mathrm{d}s, \qquad t \geq 0$$

Corollary: For any LP X with $\exists \beta > 0$: $\mathbb{E} e^{\beta |X_1|} < \infty$ and $\mathbb{E} X_1 = 0$.

$$\mathbb{P}\left[\sup_{0 \leq n \leq T} \sum_{i=1}^{n} X_i \leq 1\right] \asymp_{\log} \mathbb{P}\left[\sup_{0 \leq t \leq T} \int_0^t X_s \, \mathrm{d}s \leq 1\right]$$

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(1) Universality result: More precise formulation

- X: either a LP or RW with $\exists \beta > 0$ s.th. $\mathbb{E}e^{\beta|X_1|} < \infty$ and $\mathbb{E}X_1 = 0$.
- Integration operator:

$$\mathcal{I}(X)_t = \int_0^t K(t-s)\, X_s \,\mathrm{d}s, \qquad t \geq 0$$

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- with $K : (0, \infty) \to [0, \infty)$ such that $K(s) \leq k(s^{\beta-1} + s^{\alpha-1})$, $\alpha \geq \beta$ (and some unimportant regularity condition)

Theorem: For a process X as above and a Brownian motion B

$$(c \log T)^{-2(1+\alpha)} \leq \frac{\mathbb{P}\left[\sup_{0 \leq t \leq T} \mathcal{I}(X)_t \leq 1\right]}{\mathbb{P}\left[\sup_{0 \leq t \leq T} \mathcal{I}(B)_t \leq 1\right]} \leq (c \log T)^{2(1+\alpha)}$$

(2) Existence of the survival exponent

- X: either a LP or RW with $\exists \beta > 0$ s.th. $\mathbb{E} e^{\beta |X_1|} < \infty$ and $\mathbb{E} X_1 = 0$.
- fractional integration operator ($\mathcal{I}_0 := \mathrm{Id}$):

$$\mathcal{I}_\alpha(X)_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X_s \, \mathrm{d}s, \qquad t \geq 0$$

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Theorem: There is a non-increasing function $\theta : [0,\infty) \to (0, 1/2]$ such that for any process X as above

$$\mathbb{P}\left[\sup_{0\leq t\leq T} \mathcal{I}_\alpha(X)_t \leq 1\right] = T^{-\theta(\alpha)+o(1)}.$$

In particular, $\theta(0) = 1/2$ and $\theta(1) = 1/4$.

(2) Existence of the survival exponent: boundedness

$$\mathcal{I}_\alpha(B)_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_s \, \mathrm{d}s, \qquad t \geq 0$$

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The constant b actually has a relation to the question of random polynomials having no real zeros (studied by Dembo et al. '02):

$$\mathbb{P}\left[\sum_{i=0}^{2n} \xi_i x^i < 0 \;\; \forall x \in \mathbb{R}\right] = n^{-4b+o(1)}, \qquad n \to \infty$$

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Except $\theta(0) = 1/2$ and $\theta(1) = 1/4$, no other values are known. Even $\theta(2)$ is unknown:

$$\mathbb{P}\left[\sup_{0\leq t\leq T} \int_0^t \int_0^s B_u \, \mathrm{d}u \, \mathrm{d}s \leq 1\right] = T^{-\theta(2)+o(1)}.$$

(2) Survival exponent: comparison to FBM

Recall that $\mathcal{I}_\alpha(B)$ and FBM B^H with $H = \alpha + 1/2$, $\alpha \in [0, 1/2]$ are closely related: with an independent, very smooth process M^H,

$$B^H = \mathcal{I}_\alpha(B) + M^H$$

Theorem: [Molchan '99, Aurzada '11] For fractional Brownian motion we have, for some $c > 0$,

$$(\log T)^{-c}\, T^{-(1-H)} \leq \mathbb{P}[\sup_{0\leq t\leq T} B_t^H \leq 1] \leq (\log T)^c\, T^{-(1-H)},$$

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Corollary: The survival exponents of α-times integrated BM $\mathcal{I}_\alpha(B)$ and FBM B^H with $H = \alpha + 1/2$ do not coincide, at least for $\alpha > 1/4$, i.e. $H \in (3/4, 1]$.

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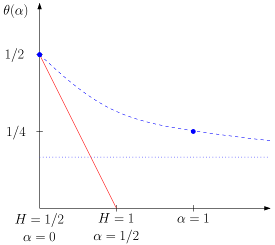

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Coupling of the LP/RW with BM via KMT

- Komlós/Major/Tusnády '75: one can couple a LP/RW X with a BM B such that

$$|X_s - B_s| \leq c \log T \qquad \text{for all } 0 \leq s \leq T$$

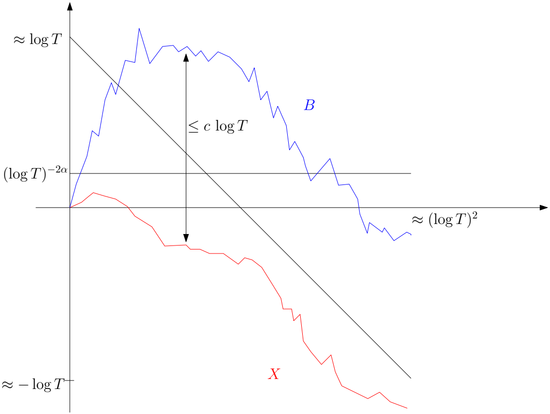
  with very high probability.
- Problem: $|X_s - B_s| \approx \log T$ may happen at the beginning (for small s), which adds up too much error when integrating:

$$\int_0^t X_s \, ds \leq \int_0^t B_s \, ds + ct \log T \leq 1 + t \log T$$

$$\mathbb{P}\left[\int_0^t B_s \, ds \leq 1, \forall t \leq T\right] \leq \mathbb{P}\left[\int_0^t X_s \, ds \leq 1 + t \log T, \forall t \leq T\right]$$

Main idea for the universality result

Make the process X behave as follows:



- Behaviour of X costs only a logarithmic probability... (one has to use a decoupling argument, FKG-type inequality)
- After $\approx (\log T)^2$, the process can be estimated by B using the coupling.

Main idea for the universality result

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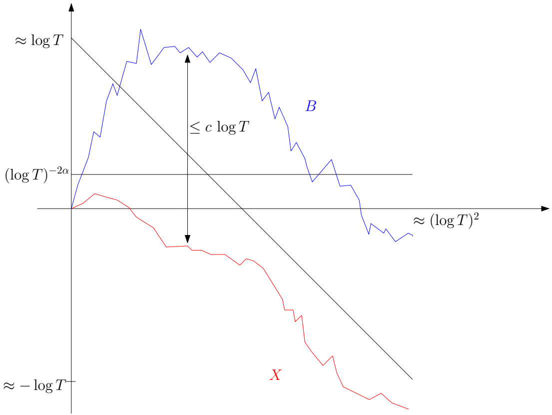

$$\mathbb{P}\left[\int_0^t X_s\,ds \leq 1, \forall t \leq T\right] \geq \mathbb{P}\left[\text{construction}\right]\mathbb{P}\left[\int_0^t B_s\,ds \leq 1, \forall t \leq T\right]$$

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Problem

Now:

- $(X_n)_{n\in\mathbb{N}}$ simple random walk
- $(A_n)_{n\in\mathbb{N}}$ given by $A_n = \mathcal{I}_1(X) = \sum_{k=1}^n X_k$

Question: For $T \in 4\mathbb{N}$

$$\mathbb{P}(\min_{n=1,\dots,T} A_n \geq 0 | X_T = A_T = 0) \approx T^{-?}.$$

Origin of the problem: Caravenna and Deuschel '09

Result

Theorem: One has for $T \in 4\mathbb{N}$

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Proof based on:

- Local central limit theorem for $(n^{-1/2}X_n, n^{-3/2}A_n)$ as $n \to \infty$ $\rightsquigarrow$ functional central limit theorem for pinned bridges

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- Local central limit theorem for $(n^{-1/2}X_n, n^{-3/2}A_n)$ as $n \to \infty$ $\rightsquigarrow$ functional central limit theorem for pinned bridges
- The conditioned process has the same fluctuations as the unconditional one:
  - $\mathbb{E}\left[|X_n| \middle| \min_{k=1,\dots,n} A_k \geq 0\right] \leq \text{const}\, n^{1/2}$
  - $\mathbb{E}\left[A_n \middle| \min_{k=1,\dots,n} A_k \geq 0\right] \leq \text{const}\, n^{3/2}$

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  - $\mathbb{E}\left[A_n \middle| \min_{k=1,\dots,n} A_k \geq 0\right] \leq \text{const}\, n^{3/2}$
- The conditional process escapes with sufficiently large probability from the origin: for $\forall c_1, c_2 > 0 \exists \kappa > 0$ s.th. for large n

$$\mathbb{P}(X_n \geq c_1 n^{1/2}, A_n \geq c_2 n^{3/2} | \min_{k=1,\dots,n} A_k \geq 0) \geq \kappa$$

Sketch of the proof

- $\mathbb{P}(\min_{k=1,\dots,T/4} A_k \geq 0,\ X_{T/4} \in [c_1 T^{1/2}, c_2 T^{1/2}],$
  $A_{T/4} \in [c_1 T^{3/2}, c_2 T^{3/2}]) \approx T^{-1/4}$

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- $\mathbb{P}(\text{same property for reversed process}) \approx T^{-1/4}$

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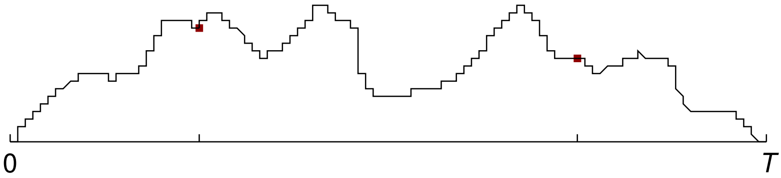

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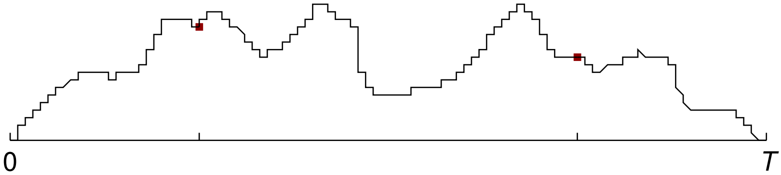
  $A_{T/4} \in [c_1 T^{3/2}, c_2 T^{3/2}]) \approx T^{-1/4}$
- $\mathbb{P}$(same property for reversed process) $\approx T^{-1/4}$
- $\inf_{\text{bdy}} \mathbb{P}^{\text{lbdy}}$(bridge stays pos. and ends in rbdy) $\geq \kappa T^{-2}$

Sketch of the proof



- $\mathbb{P}(\min_{k=1,\dots,T/4} A_k \geq 0,\ X_{T/4} \in [c_1 T^{1/2}, c_2 T^{1/2}],$
  $$A_{T/4} \in [c_1 T^{3/2}, c_2 T^{3/2}]) \approx T^{-1/4}$$
- $\mathbb{P}(\text{same property for reversed process}) \approx T^{-1/4}$
- $\inf_{\text{bdy}} \mathbb{P}^{\text{lbdy}}(\text{bridge stays pos. and ends in rbdy}) \geq \kappa T^{-2}$
- $\mathbb{P}((X_T, A_T) = (0,0)) \approx T^{-2}$

Thank you for your attention!

References:

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