Persistence of some additive functionals of Sinai's walk

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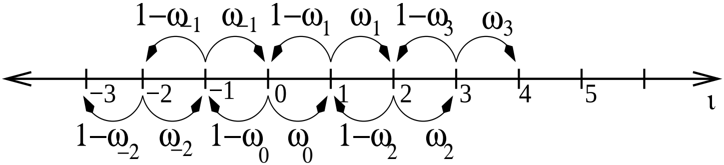
Presentation of the model

- Random Walks in Random Environment (RWRE) introduced by Chernov (1967, biophysicist, DNA) then also physics, metallurgy

- First Step : Construction of the environment ω

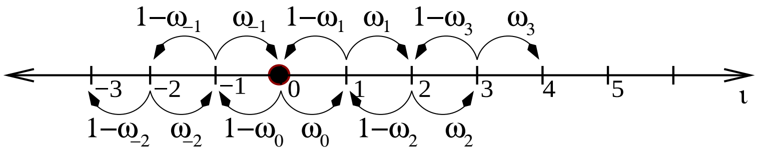
$\omega = (\omega_i)_{i\in\mathbb{Z}}$ i.i.d. $\omega_i \in [\varepsilon_0, 1-\varepsilon_0]$, law η

$\mathbb{V}\text{ar}(\frac{\omega_0}{1-\omega_0}) := \sigma^2 > 0$ (to avoid simple random walks)



Presentation of the model

Second step : Random walk in this environment ω



Quenched law :

$$S_0 = 0 \quad \text{and} \quad P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & \text{if } k = i+1, \\ 1 - \omega_i & \text{if } k = i-1, \\ 0 & \text{else.} \end{cases}$$

Annealed law : $\mathbb{P}(.) = \int P_\omega(.)\eta(d\omega)$

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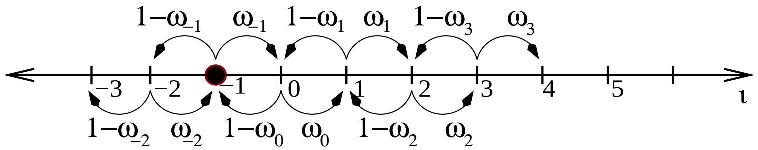

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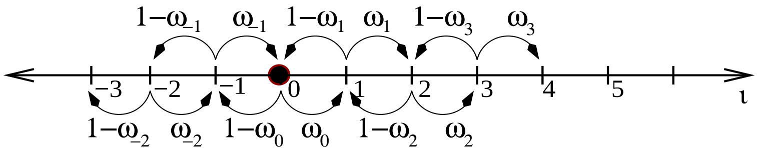

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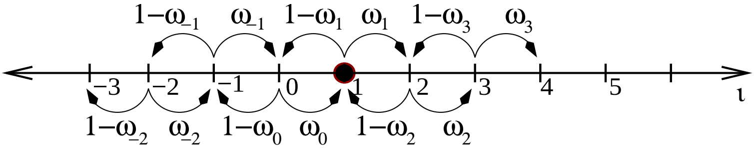

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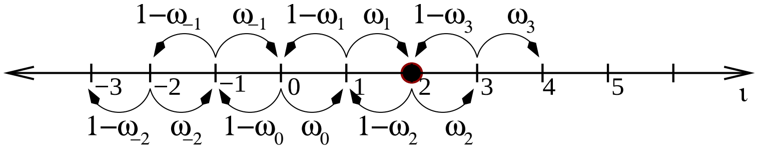

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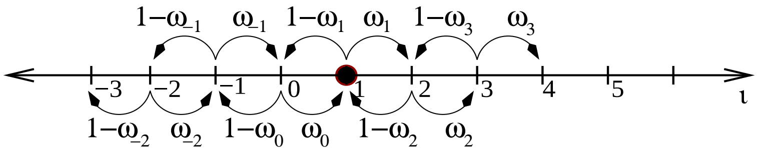

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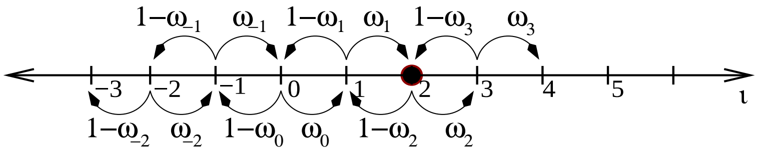

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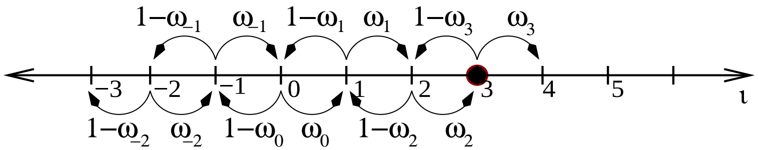

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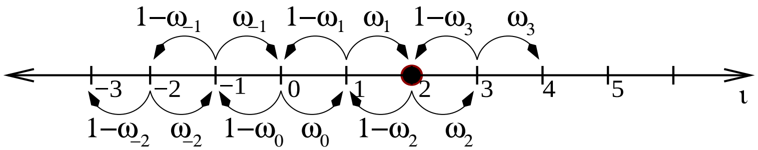

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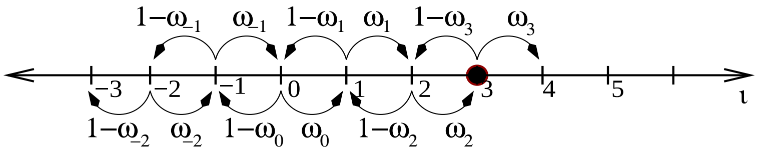

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Classical results

Recurrence criteria (Solomon, 1975)

$(S_n)_{n\in\mathbb{N}}$ recurrent η-a.s. $\Longleftrightarrow \mathbb{E}\log\frac{\omega_0}{1-\omega_0}=0$

Theorem : Sinai (1982)

In the recurrent case

$$\sigma^2\frac{S_n}{(\log n)^2}\xrightarrow[n\to+\infty]{law} b_\infty,$$

b_∞ symmetric, non degenerate, non gaussian

Sinai walk : ellipticity ($\omega_i\in[\varepsilon_0,1-\varepsilon_0]$ a.s.)
$\mathbb{V}\text{ar}(\frac{\omega_0}{1-\omega_0}):=\sigma^2>0$ (to avoid simple random walks)
recurrent ($\mathbb{E}\log\frac{\omega_0}{1-\omega_0}=0$)

Question

Theorem (Sinai, 1992)

For a simple random walk $(R_n)_{n\in\mathbb{N}}$, as $n \to +\infty$,

$$\mathbb{P}\left(\sum_{k=0}^{n} R_k > 0 \quad \forall 1 \leq n \leq N\right) \asymp 1/N^{1/4}.$$

Question : what if we replace Simple random Walk $(R_n)_n$ by Sinai's Walk $(S_n)_n$?

Main result

Theorem (persistence, AD 2014)

Let f be a function $\mathbb{Z} \to \mathbb{R}$, such that

- $f(0) = 0$;
- $\forall x > 0, f(x) \geq 1$; $\forall x < 0, f(x) \leq -1$;
- $|f(x)| \leq \exp(|x|^{o(1)})$ as $x \to \pm\infty$.

We consider $(S_n)_{n\in\mathbb{N}}$ Sinai walk and $u \leq 0$. As $N \to +\infty$,

$$\mathbb{P}\left(\sum_{k=0}^{n} f(S_k) > u \quad \forall 1 \leq n \leq N\right) = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}}.$$

Persistence exponent $\frac{3-\sqrt{5}}{2}$: first appears in Le Doussal, Monthus and Fisher (1999) then Cheliotis (2005)

Main result - some notation

Local time of $(S_n)_{n\in\mathbb{N}}$ at time $n \in \mathbb{N}$:

$$L(A,n) := \sum_{k=0}^{n} \mathbb{1}_{\{S_k\in A\}}, \qquad A \subset \mathbb{Z},$$

$$L(x,n) := L(\{x\},n), \qquad x \in \mathbb{Z}.$$

Useful since for every function g,

$$\sum_{k=0}^{n} g(S_k) = \sum_{x\in\mathbb{Z}} g(x)L(x,n), \qquad n \in \mathbb{N},$$

Main result - examples

Example 1

For $f(x) = \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$, we get as $N \to +\infty$

$$\mathbb{P}\left[L(\mathbb{N}^*, n) > L(\mathbb{Z}^*_-, n) \quad \forall 1 \leq n \leq N\right] = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}}.$$

Main result - examples

Example 2

Let $\alpha > 0$, $\text{sgn}(x) := \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$ for $x \in \mathbb{R}$, and $f(x) = \text{sgn}(x)|x|^\alpha$ for $x \in \mathbb{Z}$. We get for $u \leq 0$, as $N \to +\infty$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=0}^{n}\text{sgn}(S_k)|S_k|^\alpha > 0 \quad \forall 1 \leq n \leq N\right) = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}}.$$

For $\alpha = 1$: persistence of the *temporal average* $\frac{1}{n}\sum_{k=0}^{n} S_k$

Functions increasing more rapidly, such as for $\alpha > 0$,

$$f(x) = \text{sgn}(x)|x|^{|\log(2+|x|)|^\alpha}, \quad x \in \mathbb{Z}$$

Potential : definition

Definition : Potential

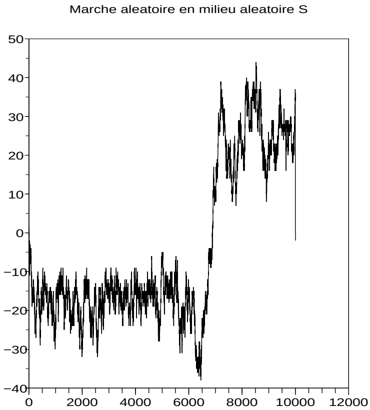
$$V(n) := \begin{cases} \sum_{i=1}^{n} \log \dfrac{1-\omega_i}{\omega_i} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{i=n+1}^{0} \log \dfrac{1-\omega_i}{\omega_i} & \text{si } n < 0 \end{cases}$$

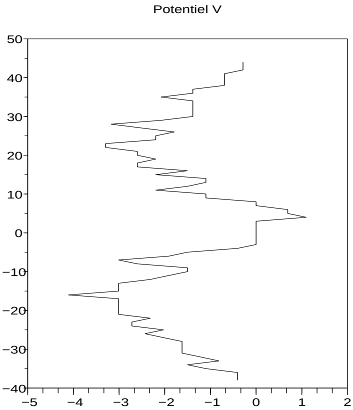
Scale function

$$\tau(p) := \inf\{k \in \mathbb{N},\ S_k = p\}, \qquad p \in \mathbb{Z}.$$

$$P_\omega^q[\tau(r) < \tau(p)] = \left(\sum_{k=p}^{q-1} e^{V(k)}\right)\left(\sum_{k=p}^{r-1} e^{V(k)}\right)^{-1}, \quad p < q < r,$$

Numerical simulation





Some useful classical estimates

If $g < h < i$

$$E_\omega^h[\tau(g) \wedge \tau(i)] \leq \varepsilon_0^{-1}(i-g)^2 \exp\left[\max_{g \leq \ell \leq k \leq i-1}(V(k) - V(\ell))\right].$$

If $p < z < q < r$ or $p < q < z < r$,

$$E_\omega^q[L(z, \tau(p) \wedge \tau(r))] = \frac{P_\omega^q[\tau(z) < \tau(p) \wedge \tau(r)]}{\omega_z P_\omega^{z+1}[\tau(z) > \tau(r)] + (1-\omega_z) P_\omega^{z-1}[\tau(z) > \tau(p)]}.$$

Coupling with Brownian motion

Komlós–Major–Tusnády strong approximation theorem

$\exists C_1 > 0$, $C_2 > 0$, $C_3 > 0$, independent of $K \in \mathbb{N}^*$, such that, possibly in an enlarged probability space, $\exists (W(t),\ t \in \mathbb{R})$, two-sided standard Brownian motion such that

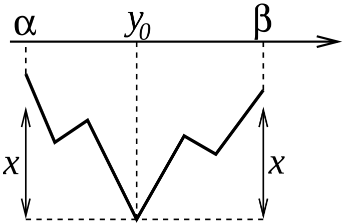
$$\mathbb{P}\left(\sup_{-K \leq i \leq K} \left| V(i) - \sigma W(i) \right| > C_1 \log K \right) \leq C_2 K^{-C_3}.$$

Definition of x-extrema

Definition (Neveu and Pitman, 1989)

For w continuous function $\mathbb{R} \to \mathbb{R}$, $x > 0$, and $y_0 \in \mathbb{R}$:
y_0 is an *x-minimum* for w if $\exists \alpha < y_0 < \beta$, such that

- $w(y_0) = \inf\{w(y),\ y \in [\alpha, \beta]\}$,
- $w(\alpha) \geq w(y_0) + x$,
- $w(\beta) \geq w(y_0) + x$.



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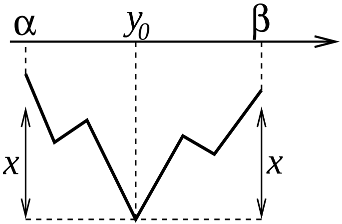
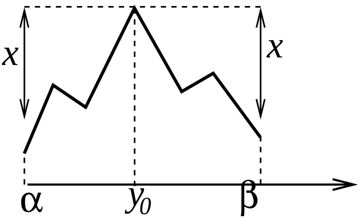




x-extrema of two sided Brownian motion

Two-sided standard Brownian motion $(W(u), u \in \mathbb{R})$.
With probability 1 (Cheliotis) :

- $$\forall x > 0, \quad \{x\text{−extrema of } W\} = \{x_k(W,x),\ k \in \mathbb{Z}\},$$
- $(x_k(W,x))_{k\in\mathbb{Z}}$ strictly increasing,
- unbounded from above and below,
- $x_0(W,x) \leq 0 < x_1(W,x)$;
- $\forall x > 0, \forall k \in \mathbb{Z}$,
  $x_{k+1}(W,x)$ x-maximum $\Leftrightarrow$ $x_k(W,x)$ x-minimum.

x-extrema of two sided Brownian motion

For each $x > 0$,

$$b_W(x) := \begin{cases} x_0(W,x) & \text{if } x_0(W,x) \text{ is an } x\text{-minimum}, \\ x_1(W,x) & \text{otherwise.} \end{cases}$$

Results of Cheliotis

Cheliotis, 2005

$$\mathbb{P}(b_W \text{ does not change its sign in } [1,x]) \sim_{x\to+\infty} \frac{c}{x^{\frac{3-\sqrt{5}}{2}}}.$$

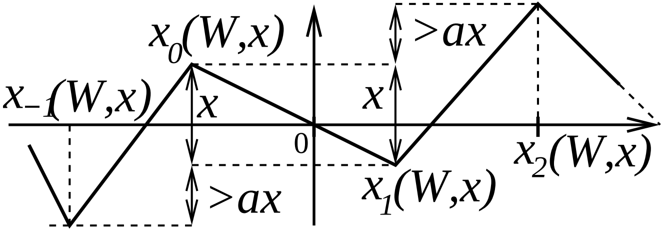
Cheliotis, 2005

Let $\mu_t =$ law of $\frac{1}{t}\sharp\{$changes of sign of b_W in $[1, e^t]\}$.
$(\mu_t)_t$ satisfies a Large Deviation Principle, with speed t, and rate function t.
I is continuous on $\mathbb{R}_+$, $I(0) = \frac{3-\sqrt{5}}{2}$, $I(x) = +\infty$ if $x < 0$.

Sketch of the proof (for $u = 0$, $f(x) = x$) : Upper bound

Definition : strong change of sign

Consider $x > 0$. For $a > 0$, x is an *a-strong change of sign* of b_W $\Leftrightarrow e(T_0(x)) = 0, \quad e(T_{-1}(x)) > ax, \quad \text{and} \quad e(T_1(x)) > ax.$



Lemma

$$\mathbb{P}\left(\begin{array}{c}\text{no } a\text{-strong change of sign in the } k \text{ first} \\ \text{changes of sign of } b_W \text{ in } [x, +\infty)\end{array}\right) \leq (1 - e^{-2a})^k$$

Upper bound : bad environments

Let $\varepsilon > 0$. ω is a **bad environment** if

- $\exists W$ Brownian motion, such that

$$\sup_{-K\le x\le K} \Big| V(\lfloor x \rfloor) - \sigma W(x) \Big| \le C_1 \log K, \qquad K = (\log N)^{\frac{3-\sqrt{5}}{2C_3}+4},$$

- (technical conditions)

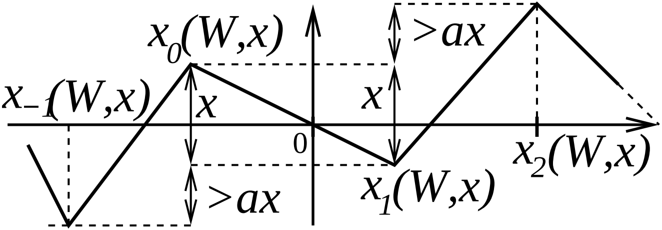
$$d_{\sigma W_+}(5\log N) \le (\log N)^4, \qquad d_{\sigma W_-}(5\log N) \le (\log N)^4,$$

where $d_X(r) := \inf\{t \ge 0,\ X(t) - \inf_{[0,t]} X \ge r\}$,

- The number of sign changes of b_W in $[(\log N)^{\varepsilon}, (\log N)^{1-\varepsilon}]$ is $\ge 2\varepsilon \log\log N + 2$
- Among the $k = 2\varepsilon \log\log n$ first changes of sign of b_W in $[(\log N)^{\varepsilon}, +\infty)$, at least one is a-strong, for $a = \frac{1}{2}\exp\left(\frac{3-\sqrt{5}}{\varepsilon}\right)$.

$$\eta(\text{bad environments}) \geq 1 - \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2} - g(\varepsilon)}}, \qquad \lim_{0^+} g = 0^+.$$

On a bad environment, with x is an a-strong change of sign and $(\log N)^{\varepsilon} \leq x \leq (\log N)^{1-\varepsilon}$,



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On a bad environment ω,

$$P_\omega\left(\exists n \in [1, N], \sum_{k=1}^{n} S_k \leq 0\right) \geq 1 - \frac{4}{(\log N)^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\exists n \in [1, N], \sum_{k=1}^{n} S_k \leq 0\right)$$

$$\geq \quad \int_{\{\text{bad environments}\}} P_\omega\left(\exists n \in [1, N], \sum_{k=1}^{n} S_k \leq 0\right) \eta(\mathrm{d}\omega)$$

$$\geq \quad \left(1 - \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2} - g(\varepsilon)}}\right)\left(1 - \frac{4}{(\log N)^2}\right).$$

Lower bound : good environments

Let $\varepsilon > 0$. ω is a **good environment** if

- $-2\delta k \leq V(k) \leq -\delta k \qquad 0 \leq k \leq \varepsilon \log\log N$
- $h := (\log N)^{\varepsilon}$, $\theta_0 := \inf\{k \geq \varepsilon \log\log N, V(k) \leq -5h\}$,

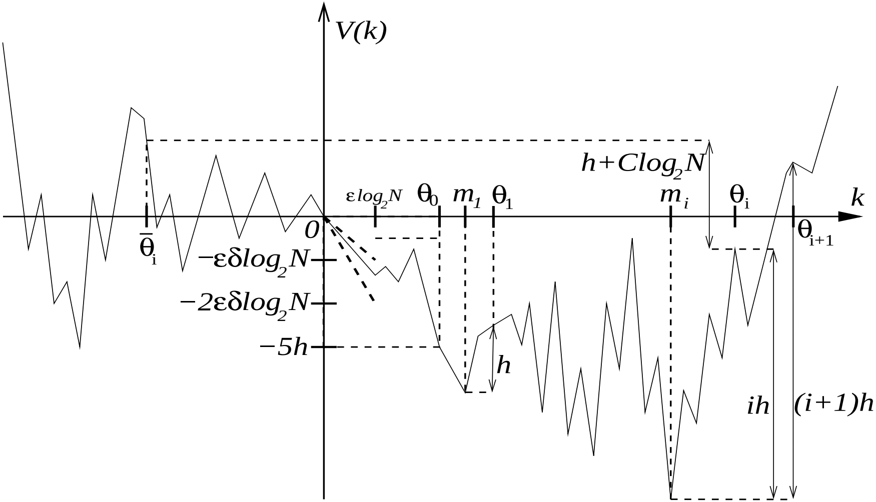
$$\forall \varepsilon \log\log N \leq k \leq \theta_0, \quad V(k) \leq -(\delta/2)\varepsilon \log\log N,$$

- $\theta_0 \leq \varepsilon \log\log N + (\log N)^{\varepsilon\delta/4}$
- $\widehat{V}(k) = V(k), k < 0,\ \widehat{V}(k) = V(k+\theta_0) - V(\theta_0),\ k \geq 0.$
  $\exists W$ Brownian motion, such that

$$\sup_{-K \leq x \leq K} |\widehat{V}(\lfloor x \rfloor) - \sigma W(x)| \leq C_1 \log K, \qquad K = (\log N)^{\frac{3-\sqrt{5}}{2C_3}+4},$$

- $d_{\sigma W_+}(5\log N) \leq (\log N)^4, \qquad d_{\sigma W_-}(5\log N) \leq (\log N)^4,$
- $\forall x \in [1, 5\log N], \qquad b(x) > 0$

Good environment : $\varepsilon > 0$, $h := (\log N)^{\varepsilon}$



$$\eta(\text{good environemnts}) \geq \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+c\varepsilon}}$$

On a good environment ω,

$$\theta_i \quad := \quad \inf\{k > 0,\ V(k) - \inf_{[0,k]} V \geq ih\}, \qquad i \geq 1.$$

By induction, for all $i \geq 1$ such that $ih \leq 4 \log N$,

$$P_\omega[F_i(N)] \geq c - i(\log N)^{-6},$$

where

$$F_i(N) := \left\{ \sum_{k=1}^{n} S_k > 0 \ \forall 1 \leq n \leq \tau(\theta_i) \right\} \cap \left\{ \sum_{k=1}^{\tau(\theta_i)} S_i \geq \frac{\exp(ih)}{2(\log N)^c} \right\}.$$

Lower bound, conclusion

For i_N such that $3 \log N \leq i_N h \leq 4 \log N$,

$$F_{i_N}(N) \subset \{\tau(\theta_{i_N}) \geq N\}$$

$$\Rightarrow P_\omega\left(\sum_{k=0}^{N} S_k > 0 \ \forall 1 < n < N\right) \geq P_\omega[F_{i_N}(N)] \geq c - \frac{i_N}{(\log N)^6} \geq \frac{c}{2}$$

$$\Rightarrow \quad \mathbb{P}\left(\sum_{k=0}^{N} S_k > 0 \ \forall 1 < n < N\right)$$

$$\geq \quad \int_{\{\text{Good environemnts}\}} P_\omega\left(\sum_{k=0}^{N} S_k > 0 \ \forall 1 < n < N\right) \eta(\mathrm{d}\omega)$$

$$\geq \quad (c/2)\eta(\text{good environemnts}) \geq \frac{c/2}{(\log N)^{\frac{3-\sqrt{5}}{2}+c\varepsilon}}$$

Thank you for your attention.