Potential Theory, Self-Similarity & Statistical Mechanics

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Outline

1. Review of some Stochastic Potential Theory
 - Example: Random Interlacements
2. Self-Similar Markov Processes
 - Example: Hausdorff Dimensions of “Self-Similar Sets”

Part 1: Review of some Stochastic Potential Theory

Principal Objects

- locally compact statespace $(E, \mathcal{E})$,
- Markov process (X_t) with transition semigroup (P_t)
- excessive measure on $(E, \mathcal{E})$ for (P_t), i.e. σ-finite and

$$P^\eta(X_t \in \cdot) =: \eta P_t(\cdot) \leq \eta(\cdot)$$

  Special cases:

  - invariant measure: $\eta P_t = \eta$
  - purely excessive: $\eta P_t(A) \downarrow 0$ for $t \to \infty$

Examples:

- $\eta(dx) = dx$ for Brownian motion
- potential measures $\eta_\nu(A) = \mathbb{E}^\nu[\int_0^\infty 1_A(X_t)\, dt]$ are purely excessive

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Research Directions

- Structure of excessive measures
  Example: Riesz decomposition of η into invariant and purely excessive parts: $\eta = \eta_I + \eta_P$.
- Relation of excessive measures and path-behavior (entrance boundaries)
- Martin boundary
- ...

Trajectories with Birth & Death

Add to E a cemetary state ∂. Set

$$D = \left\{ Y : \mathbb{R} \to E \cup \{\partial\} \text{ RCLL} \mid Y_t \in E \text{ for } t \in (\alpha(w), \beta(w)) \right\}$$

and call

- $\alpha(Y)$ time of birth of Y,
- $\beta(Y)$ time of death of Y,
- $\zeta(Y) = \beta(Y) - \alpha(Y)$ life-time of Y.

Processes with Birth & Death - Kuznetsov Measures

Measure $\mathcal{Q}_\eta$ on $(D, \mathcal{D})$ is called Kuznetsov measure for

- Markov process (P_t)
- excessive measure η

if $\mathcal{Q}_\eta$ is a Markov process with birth & death and marginals η.

That is, for $-\infty < t_1 < \cdots < t_n < +\infty$

$$\mathcal{Q}_\eta\big(\alpha(Y) < t_1, Y_{t_1} \in dx_1, \cdots, Y_{t_n} \in dx_n, t_n < \beta(Y)\big)$$
$$= \eta(dx_1) P_{t_2 - t_1}(x_1, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n).$$

Theorem (Kuznetsov): Existence and Uniqueness.

Processes with Birth & Death - Quasi Processes

Measure $\mathcal{P}_\eta$ on $(D, \mathcal{D})$ - modulus time translation - is called quasi process for

- Markov process (P_t)
- excessive measure η

if

(a) for all shift-invariant stopping times S

$$(Y_{S+t})_{t \geq 0}$$

is Markov with transitions (P_t) under $\mathcal{P}_{|_{S \in \mathbb{R}}}$.

(b) $\eta(A) = \mathcal{P}_\eta(\int_\alpha^\beta 1_A(Y_s)\, ds)$

Theorem (Weil, Hunt): Existence and Uniqueness.

Palm Measure Relation of $\mathcal{Q}_\eta$ and $\mathcal{P}_\eta$

Theorem (Fitzsimmons): Fix any finite shift-invariant time S, then

$$\mathcal{P}_\eta(A) = \mathcal{Q}_\eta(A, S \in [0, 1])$$

for all shift-invariant events A.

Riesz Decomposition - Fitzsimmons/Maisonneuve

Theorem: Let η excessive and define

$$\mathcal{P}_P(\cdot) := \mathcal{P}_\eta(\cdot, \alpha > -\infty), \qquad \mathcal{P}_I(\cdot) := \mathcal{P}_\eta(\cdot, \alpha = -\infty).$$

Then $\eta = \eta_I + \eta_P$ with

$$\eta_P(A) = \mathcal{P}_P\Big(\int_\alpha^\beta 1_A(Y_s)\,ds\Big) \qquad \eta_I(A) = \mathcal{P}_I\Big(\int_{-\infty}^\beta 1_A(Y_s)\,ds\Big).$$

Remark: Analogous theorem for Kuznetsov measures.

Riesz and a "Way of starting stochastic processes"

starting process (X_t) $\iff$ purely excessive measure

$'' \Rightarrow''$ occupation measure
$'' \Leftarrow''$ take quasi process and shift to $\alpha = 0$ (finite measure?)

"take" is abstract, often means "take Kuznetsov, then Palm measure"

starting process (X_t) at time $-\infty$ $\iff$ invariant measure

Problem: Only know processes forwards in time, not backwards.

Example: Lévy processes.

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Kuznetsov Measures and Duality

Definition: Two Markov processes (X_t) and $(\hat{X}_t)$ are in duality with respect to a measure m if

$$P_t(x, dy)m(dx) = \hat{P}_t(y, dx)m(dy).$$

Examples:

- Brownian motion is self-dual with $m(dx) = dx$
- Lévy process (ξ_t) is dual to $(\hat{\xi}_t) = (-\xi_t)$ with $m(dx) = dx$

Mitro's Theorem: If m is invariant for (P_t) and $(\hat{P}_t)$ is a dual process, then $\mathcal{Q}_m$ is constructed as follows:

- sample m at time 0
- run (P_t) in positive time direction
- run $(\hat{P}_t)$ in negative time direction

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Example: Random Interlacements

Definition: Brownian interlacement is the (infinite) measure ν on $(D, \mathcal{D})$ - modulus time translation - so that restricted to trajectories hitting balls $B \subset \mathbb{R}^n$, $n \geq 3$,

- first hitting distribution at B is harmonic measure
- forwards in time paths are Brownian motion
- backwards in time paths are conditioned Brownian motion

Brownian random interlacement is Poisson random measure (loop soup) with intensity ν.

Theorem (D., Dereich '14, Rosen '14):

Interlacement is quasi process of BM with invariant measure $m(dx) = dx$.

Corollary:

Interlacement is two-sided BM started in Lebesgue measure restricted to be closest to the origin at time $S \in [0, 1]$ modulus time translations.

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Part 2: Self-Similar Markov Processes

Self-Similarity

A real-valued Markov process is called self-similar if it scales in time and space:

$$\text{the law of } (cX_{c^{-1/\gamma}t})_{t\geq 0} \text{ under } \mathbb{P}_z \text{ is } \mathbb{P}_{cz}$$

γ is called the index of self-similarity.

Goals

(1) Understand the general form of a self-similar process.

(2) Deduce non-trivial consequences.

Seeking magic arguments of type

- know self-similarity for a model
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For simplicity now restrict ourselves to positive processes.

This is NOT needed, but enough for application in the end!

Lamperti's Representation

Behavior up to hitting zero governed by Lévy processes:

Theorem (Lamperti '72)

If (X_t) is positive self-similar there is a Lévy process (ξ_t) s.t. for $z > 0$

$$X_t^{(z)} := z \exp\left(\xi_{A(tz^{-\gamma})}\right), \quad t \leq T_0,$$

where

$$A(t) := \left(\int_0^t \exp(\gamma \xi_s)\, ds\right)^{-1}.$$

Lamperti's Dichotomy

1. ξ drifts to $-\infty$ if and only if

$$\mathbb{P}_z(T_0 < \infty) = 1$$

$\rightarrow$ "recurrent case"

2. ξ does not drift to $-\infty$ if and only if

$$\mathbb{P}_z(T_0 = \infty) = 1$$

$\rightarrow$ "transient case"

Example

Squared-Bessel processes of dimension $\delta \in \mathbb{R}$

$$dX_t = 2\sqrt{X_t}dB_t + \delta dt, \quad t \leq T_0,$$

self-similar of index 1 with corresponding Lévy process

$$\xi_t = 2B_t + (\delta - 2)t.$$

Lamperti's dichotomy:

- $\delta < 2 \Rightarrow$ squared-Bessel processes hit zero
- $\delta \geq 2 \Rightarrow$ squared-Bessel processes do not hit zero

Two Natural Tasks

(R) Recurrent case: describe (if exists) the positive self-similar process after T_0.

(T) Transient case: construct (if $\lim_{z\to 0}\mathbb{P}_z=\mathbb{P}_0$ exists) the limit $\mathbb{P}_0$!

Recurrent case via potential theory by Rivero '06, Fitzsimmons '05. Lamperti SDE approach in Barczy/D. '13.

Transient case gradually by Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, Yor ('02-13')

Extensions for real processes in D. '13, Dereich, D., Kyprianou '14+

Now: Transient case + application

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Definition: We say ξ has stationary overshoots if

$$O := \lim_{x\to\infty} (\xi_{T_x} - x) \qquad \text{exists non-trivially.}$$

Remark: If $\xi \uparrow +\infty$, equivalent to $\mathbb{E}[|\xi_1|] < \infty$.

Theorem:

$\lim_{z\to 0} \mathbb{P}_z =: \mathbb{P}_0$ exists if and only if ξ has stationary overshoots.

Proof (Necessity): For $a < b$

$$\begin{aligned} &\lim_{z\to 0} \mathbb{P}_z(X_{T_a} \le b) \\ &= \lim_{z\to 0} P(z \exp(\xi_{T_{\log(a/z)}}) \le b) \\ &= \lim_{z\to 0} P(\xi_{T_{\log(a/z)}} \le \log(b/z)) \\ &= \lim_{z\to 0} P(\xi_{T_{\log(a/z)}} - \log(a/z) \le \log(b/a)) \end{aligned} \tag{1}$$

Proof (Sufficiency): Needs a construction of $\mathbb{P}_0$, harder.

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$\mathbb{P}_0$ through Kuznetsov Measures (Dereich, D., Kyprianou '14+)

Idea: Take "the good" purely excessive measure η_0 and set

$$\mathbb{P}_0 = \frac{\mathcal{P}_{\eta_0}}{P_{\eta_0}(\mathcal{D})}$$

shifted to $\alpha = 0$. Need $\mathcal{P}_{\eta_0}(\mathcal{D}) < \infty$ and left-limits 0!

Question: Which excessive measure η_0?

Examples: $\eta_\beta(dx) = x^{1/\gamma - 1 + \beta}\, dx$ are purely excessive for many $\beta \geq 0$.

Answer: $\eta_0(dx) = x^{1/\gamma - 1} dx$ is the good one.

Problem: No idea how to prove this directly! No duality $\rightarrow$ left limits?

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A Theorem of Kaspi

Setting:

- Markov processes with $\tilde{X}_t = X_{A_t^{-1}}$ with $A_t = \int_0^t f(X_s)\,ds$.
- Kuznetsov measure $\mathcal{Q}_m$ for X with invariant measure m
- $B_t := \int_{-\infty}^t f(Y_s)\,ds < \infty$ under $\mathcal{Q}_m$.

Theorem (Kaspi): There is a purely excessive measure $\tilde{\eta}$ for $\tilde{X}$ so that the quasi process $\tilde{\mathcal{P}}_{\tilde{\eta}}$ shifted to $\alpha = 0$ is obtained from $\mathcal{Q}_m(\cdot, B_0^{-1} \in [0,1])$ as $\tilde{Y}_t = Y_{B_t^{-1}}$.

Important: Know left limits for $\mathcal{Q}_m \Rightarrow$ know left limits for $\tilde{\mathcal{P}}\eta$.

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Construction of $\mathbb{P}_0$ (for $\xi \uparrow +\infty$):

- Set $m(dx) = dx$ for ξ and take Kuznetsov measure $\mathcal{Q}_m$ - Mitro's two-sided construction implies left limits $\lim_{t \downarrow -\infty} Y_t = -\infty$.
- Define $B_t = \int_{-\infty}^{t} \exp(\gamma Y_s)\, ds$ and show $B_t < \infty$.
- Use Kaspi's theorem, left limits remain $-\infty$.
- Take exponential, left limits are 0.
- Normalize to probability law $\mathbb{P}_0$.

Convergence $\lim_{z \to 0} \mathbb{P}_z =: \mathbb{P}_0$:

Consequence of Prokhorov metric for Skorokhod topology and the Kuznetsov measure construction.

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- Define $B_t = \int_{-\infty}^{t} \exp(\gamma Y_s)\, ds$ and show $B_t < \infty$.
- Use Kaspi's theorem, left limits remain $-\infty$.
- Take exponential, left limits are 0.
- Normalize to probability law $\mathbb{P}_0$.

Convergence $\lim_{z\to 0} \mathbb{P}_z =: \mathbb{P}_0$:

Consequence of Prokhorov metric for Skorokhod topology and the Kuznetsov measure construction.

An Application to Random Sets

Proposition: Suppose $M \subset [0, \infty)$ is a random set with

(a) M is the range of some increasing self-similar Markov process issued from 0.

(b) For some $\kappa \in (4, 8)$ and all $a < b$

$$\mathbb{P}(M \cap [a, b] \neq \emptyset) = C \int_0^{\frac{b-a}{a}} \frac{1}{u^{2-8/\kappa}(1-u)^{4/\kappa}}\, du.$$

Then

$$dim_H(M) = 2 - \frac{8}{\kappa}$$

almost surely.

Example: If γ is $SLE(\kappa)$ and $M = \gamma \cap [0, \infty)$, then (b) holds. Schramm/Zhou '08 and Alberts/Sheffield '08 proved $dim_H(M) = 2 - \frac{8}{\kappa}$.

Remark: If (a) holds for SLE, then our approach gives much more than only Hausdorff dimension.

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Proof of Proposition

From self-similarity: Recall the construction of $\mathbb{P}_0$. Only need to know Hausdorff dimension of $Range(\xi)$.

From subordinator theory:

$$dim_H(Range(\xi)) = \sup\left\{\gamma \leq 1 : \lim_{b\downarrow 0} b^{\gamma-1}\int_0^b \Pi(r,\infty)\,dr = +\infty\right\}.$$

From fluctuation theory: Get Π from overshoot distributions as

$$C\int_0^b \Pi(r,\infty)\,dr = \lim_{x\uparrow+\infty} P(\xi_{T_x} - x \leq b) \tag{2}$$

Proof of Proposition

Lemma: $\int_0^b \Pi(r,\infty)\,dr \sim Cb^{8/\kappa-1}$ as $b \downarrow 0$.
Proof:

$$\begin{aligned}
\int_0^{\frac{b-a}{a}} \frac{1}{u^{2-8/\kappa}(1-u)^{4/\kappa}}\,du &= \mathbb{P}(M \cap [a,b] \neq \emptyset) \\
&= \mathbb{P}_0(X_{T_a} \le b) \\
&= \lim_{z\downarrow 0} \mathbb{P}_z(X_{T_a} \le b) \\
&\overset{\text{above}}{=} \lim_{z\to 0} P(\xi_{T_{\log(a/z)}} - \log(a/z) \le \log(b/a)) \\
&\overset{\text{above}}{=} C\int_0^{\log(b/a)} \Pi(r,\infty)\,dr
\end{aligned}$$

This gives $\Pi(r,\infty)$. Only need $a = 1$ to get

$$C\int_0^b \Pi(r,\infty)\,dr = \int_0^{e^b-1} \frac{1}{u^{2-8/\kappa}(1-u)^{4/\kappa}}\,du \sim Cb^{8/\kappa-1}.$$

Proof of Proposition

Lemma: $dim_H(Range(\xi)) = 2 - \frac{8}{\kappa}$ almost surely.
Proof:

$$
\begin{aligned}
dim_H(M) &\overset{\text{above}}{=} dim_H(Range(\xi)) \\
&= \sup\left\{\gamma \le 1 : \lim_{b \downarrow 0} b^{\gamma-1} \int_0^b \Pi(r,\infty)\, dr = +\infty\right\} \\
&\overset{\text{above}}{=} \sup\left\{\gamma \le 1 : \lim_{b \downarrow 0} b^{\gamma-1} b^{8/\kappa-1} = +\infty\right\} \\
&= 2 - \frac{8}{\kappa}.
\end{aligned}
$$

Remark: The calculations above also show that

$$\Pi(r,\infty) = C(1-e^{-r})^{8/\kappa-2} e^{(1-4/\kappa)r}.$$

Hence, any result for subordinators that involves the Lévy measure only gives a result for M.

Proof of Proposition

Lemma: $dim_H(Range(\xi)) = 2 - \frac{8}{\kappa}$ almost surely.
Proof:

$$\begin{aligned} dim_H(M) &\overset{\text{above}}{=} dim_H(Range(\xi)) \\ &= \sup\left\{\gamma \leq 1 : \lim_{b\downarrow 0} b^{\gamma-1}\int_0^b \Pi(r,\infty)\,dr = +\infty\right\} \\ &\overset{\text{above}}{=} \sup\left\{\gamma \leq 1 : \lim_{b\downarrow 0} b^{\gamma-1} b^{8/\kappa-1} = +\infty\right\} \\ &= 2 - \frac{8}{\kappa}. \end{aligned}$$

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Hence, any result for subordinators that involves the Lévy measure only gives a result for M.