Random walk in a Poisson system of moving traps

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Contents

Model

- particle moving as SRW X on $\mathbb{Z}^d$ (jump rate κ);
- traps $(Y_j^y)_{y\in\mathbb{Z}^d,1\le j\le N_y}$ SRWs (jump rate ρ) in Poisson-equilibrium (i.e., the N_y are i.i.d. $\mathrm{Pois}(\nu)$);
- $$\xi(t,x) := \sum_{y\in\mathbb{Z}^d,1\le j\le N_y} \delta_x(Y_j^y(t))$$

  (# of traps at x at time t);

Definition

For coupling constant $\gamma\in\mathbb{R}$, *quenched survival probability of X*:

$$Z_{t,\xi}^{\gamma} := \mathbb{E}_0^X \exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,ds\Big\}.$$

Annealed survival probability of X:

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Annealed asymptotics

Theorem (Asymptotic annealed survival probability)

Let $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$. Then

$$\mathbb{E}^{\xi} Z^{\gamma}_{t,\xi} = \begin{cases} \exp\Big\{ -\nu\sqrt{\frac{8\rho t}{\pi}}(1+o(1))\Big\}, & d = 1, \\ \exp\Big\{ -\nu\pi\rho\frac{t}{\log t}(1+o(1))\Big\}, & d = 2, \\ \exp\Big\{ -\lambda_{d,\gamma,\kappa,\rho,\nu}\, t(1+o(1))\Big\}, & d \geq 3, \end{cases}$$

some $\lambda_{d,\gamma,\kappa,\rho,\nu}$ (*annealed Lyapunov exponent*). Furthermore, $\lambda_{d,\gamma,\kappa,\rho,\nu} \geq \lambda_{d,\gamma,0,\rho,\nu} = \nu\gamma/(1+\frac{\gamma G_d(0)}{\rho})$, where $G_d(0) := \int_0^\infty p_t(0)\, dt$ is the Green function of a jump rate 1 SRW.

Quenched asymptotics

Theorem (Asymptotic quenched survival probability)

With same parameters as before, there exists deterministic $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$ (*quenched Lyapunov exponent*) such that $\mathbb{P}^\xi$-a.s.,

$$Z_{t,\xi}^{\gamma} = \exp\left\{ -\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}\, t(1+o(1))\right\} \quad \text{as } t \to \infty.$$

Furthermore, $0 < \tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} \leq \gamma\nu + \kappa$ for all $d \geq 1, \gamma > 0, \kappa \geq 0, \rho > 0$ and $\nu > 0$.

Remark

Quenched survival probability always decays exponentially, annealed one only for $d \geq 3$;

Connections to PAM

Parabolic Anderson model

$$\frac{\partial u}{\partial t}(t,x) = \kappa \Delta u(t,x) - \gamma \xi(t,x) u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d,$$
$$u(0,x) = 1, \qquad \forall x \in \mathbb{Z}^d,$$

$\kappa \geq 0$ diffusion constant, Δ the discrete Laplace operator, γ and ξ as before.

Feynman-Kac formula yields

$$u(t,0) = \mathbb{E}_0^X \exp\left\{ -\gamma \int_0^t \xi(t-s, X(s))\, ds \right\},$$

where as before X is SRW with jump rate κ.

Observe: only difference between $u(t,0)$ and survival probability $\mathbb{E}^\xi Z^\gamma_{t,\xi}$ is time reversal $\xi(t-s, X(s))$ instead of $\xi(s, X(s))$;

initial condition for the Y traps reversible $\rightsquigarrow$ $u(t,0)$ has same annealed asymptotics as $\mathbb{E}^\xi Z^\gamma_{t,\xi}$ for $t \to \infty$.

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Quenched Lyapunov exponent for PAM

Theorem

Let $d \geq 1$, $\gamma > 0$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$ and $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} > 0$ as before. Then $\mathbb{P}^\xi$-a.s.,

$$u(t,0) = \exp\left\{ -\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}\, t(1+o(1))\right\} \quad \text{as } t \to \infty.$$

Remark

- As mentioned before, distributions of $u(t,0)$ and $Z^\gamma_{t,\xi}$ coincide, hence convergence in distribution for $\frac{1}{t}\log u(t,0)$ follows for free from $\mathbb{P}^\xi$-a.s. convergence of $\frac{1}{t}\log Z^\gamma_{t,\xi}$ (above result). Then deduce existence of $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$ via bounding variances of $\frac{1}{t}\log u(t,0)$.
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Literature: Immobile traps

Continuum version is BM among Poissonian obstacles:

Asymptotics of survival probabilities

- $\exp\{-C_{d,\gamma} t^{\frac{d}{d+2}}(1+o(1))\}$ (annealed, [DV75] via LDP for the volume of the Wiener sausage),
- $\exp\{-\bar{C}_{d,\gamma} \frac{t}{(\log t)^{2/d}}(1+o(1))\}$ (quenched, [Szn98] via renormalization scheme *Enlargement of obstacles*, also obtained information about path behavior).

Related results for SRW among Bernoulli traps [DV79, Bol94, Ant95];

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Literature: Mobile traps

- [Red94] established exponentially decaying upper bound for annealed survival probability for ξ generated by a reversible Markov process;
- Annihilating two-type random walks [BL91];
- Random walk among moving catalysts (i.e., the case $\gamma < 0$ in our notation) [KS03], [GdH06];

Some ideas of the proofs

Integrating out Poisson initial distribution ("Campbell's formula") yields

$$\mathbb{E}^{\xi} Z^{\xi}_{t,\gamma} = \mathbb{E}^{X}_{0} \exp\Big\{ \nu \sum_{y\in\mathbb{Z}^d} \Big(\underbrace{\mathbb{E}^{Y}_{y} \exp\Big\{ -\gamma \int_0^t \delta_0(Y(s) - X(t-s))\, ds \Big\}}_{=: v_X(t,y)} - 1 \Big) \Big\}$$

where we also used reversibility of Y.

Next observe that $v_X(t,y)$ is the Feynman-Kac representation of the solution to

$$\frac{\partial v_X}{\partial t}(t,y) = \rho \Delta v_X(t,y) - \gamma \delta_{X(t)}(y) v_X(t,y),$$
$$v_X(0,\cdot) \equiv 1.$$

Summing above over y and setting $X \equiv 0$ (i.e, considering the case $\kappa = 0$) one can use Tauberian theorems to obtain strong logarithmic asymptotics for $\mathbb{E}^{\xi} Z^{\xi}_{t,\gamma}$.

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Upper bound

Lemma (Pascal's principle)

For all trajectories $X : [0,t] \to \mathbb{Z}^d$ *well-behaved (i.e. piecewise constant, finite number of discontinuities)*

$$\sum_{y\in\mathbb{Z}^d} 1 - \mathbb{E}_y^Y \exp\Big\{ -\gamma \int_0^t \delta_0(Y(s) - X(s))\, ds \Big\}$$

$$\geq \sum_{y\in\mathbb{Z}^d} 1 - \mathbb{E}_y^Y \exp\Big\{ -\gamma \int_0^t \delta_0(Y(s))\, ds \Big\}.$$

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Idea of proof of Pascal's principle

Pascal's principle had been known in the physics literature (without proof).

Proof.

Intuitive in discrete time for $t = 1$ and ρ small (i.e. Y remains at its current site with high probability);

more formally: induction argument, afterwards discretization of continuous time case. □

Lower bound in $d = 1, 2$

Strategy (for fixed t) the same as for immobile traps: Enforce

1. $\xi(s, x) = 0$ for all times $s \in [0, t]$ and x in ball B_{R_t}
2. $X(s) \in B_{R_t}$ for all $s \in [0, t]$ $(:= G_t)$

First requirement is equivalent to:

- $N_y = 0$ for all $y \in B_{R_t}$ $(:= E_t)$ and
- none of the Y_j^y, $y \notin B_{R_t}$, $1 \le j \le N_y$, enters B_{R_t} up to time t $(:= F_t)$.

independence $\rightsquigarrow$

$$\mathbb{E}^\xi Z_{t,\gamma}^\xi \ge \mathbb{P}(E_t)\mathbb{P}(F_t)\mathbb{P}(G_t)$$

Optimize R_t such that RHS is maximized — possible choices: $R_t = \sqrt{t/\log t}$ in $d = 1$ and $R_t = \ln t$ in $d = 2$.

In $d = 1, 2$ above computations for upper and lower bound on annealed survival probability even give same constant in the exponential term $\rightsquigarrow$ annealed Lyapunov exponent exists.

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Annealed Lyapunov exponent for general dimensions

In higher dimensions: Use subadditivity argument and subadditive ergodic theorem to deduce existence of Lyapunov exponent.

Short glimpse on quenched proof

Set

$$a(s,t,x,y,\xi) := -\log \mathbb{E}^X_{x,s} \exp\Big\{-\gamma\int_s^t \xi(u,X(u))\,du\Big\} 1_{X(t)=y}.$$

Main step (works for any bd. ergodic ξ):

Theorem (Shape theorem)

$\exists\, \alpha : \mathbb{R}^d \to [0,\infty)$ deterministic, convex (*shape function*) s.t. $\mathbb{P}^\xi$-a.s., for any $K \subset \mathbb{R}^d$ compact:

$$\lim_{t\to\infty} \sup_{y\in tK\cap\mathbb{Z}^d} |t^{-1} a(0,t,0,y,\xi) - \alpha(y/t)| = 0.$$

Combining this with the fact that for any M, $\exists K \subset \mathbb{R}^d$ compact s.t.

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0^X \exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,ds \cdot 1_{X(t)\notin tK}\Big\} \le -M.$$

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Short glimpse on quenched proof

Still to show: positivity of $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$.

No details, proof goes via complicated so-called *renormalization scheme* using *good* space-time boxes where at least a close to normal number of traps occur.

Corollary to Pascal's principle

Corollary

The expected cardinality of the range of a continuous time symmetric random walk increases under perturbation by a deterministic path: Y *symmetric irreducible RW on* $\mathbb{Z}^d$, $X : [0, t] \to \mathbb{Z}^d$ *piecewise constant with finite number of discontinuities. Then* $\forall t \geq 0$:

$$\mathbb{E}_0^Y \, |R_t(Y - X)| \geq \mathbb{E}_0^Y \, |R_t(Y)|,$$

with $R_t := \{y \in \mathbb{Z}^d : \exists s \in [0, t] \text{ with } Y(s) = y\}$.

Future goal

Understand path behavior of X with respect to the induced Gibbs-measure.

So far: Can show that X behaves subdiffusively in $d = 1$ for $\gamma = \infty$.

Peter Antal.
Enlargement of obstacles for the simple random walk.
Ann. Probab., 23(3):1061–1101, 1995.

Maury Bramson and Joel L. Lebowitz.
Asymptotic behavior of densities for two-particle annihilating random walks.
J. Statist. Phys., 62(1-2):297–372, 1991.

Erwin Bolthausen.
Localization of a two-dimensional random walk with an attractive path interaction.
Ann. Probab., 22(2):875–918, 1994.

M. D. Donsker and S. R. S. Varadhan.
Asymptotics for the Wiener sausage.
Comm. Pure Appl. Math., 28(4):525–565, 1975.

M. D. Donsker and S. R. S. Varadhan.
On the number of distinct sites visited by a random walk.
Comm. Pure Appl. Math., 32(6):721–747, 1979.

J. Gärtner and F. den Hollander.
Intermittency in a catalytic random medium.
Ann. Probab., 34(6):2219–2287, 2006.

Harry Kesten and Vladas Sidoravicius.
Branching random walk with catalysts.
Electron. J. Probab., 8:no. 5, 51 pp. (electronic), 2003.

F. Redig.
An exponential upper bound for the survival probability in a dynamic random trap model.
J. Statist. Phys., 74(3-4):815–827, 1994.

Alain-Sol Sznitman.
Brownian motion, obstacles and random media.
Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.