Large deviations probabilities and moderate deviations in random matrix theory

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Menu

- **Objects:** certain statistics in random matrix theory
- **Question:** moderate deviations and large deviations probabilities

- GUE ensembles, WIGNER matrices, invariant ensembles
- local and global statistics

- Four Moment Theorem
- bounds on cumultants, method of orthogonal polynomials

WIGNER matrices

- random Hermitian matrices M_n of size n
- for $i < j$: the real and imaginary parts of $(M_n)_{ij}$ are iid, with mean 0 and variance $1/2$
- $(M_n)_{ii}$ are iid with mean 0 and variance 1

example: entries are Gaussian:
Gaussian Unitary Ensemble (GUE)

condition (C) on M_n

M_n satisfies condition (C) if :

- the real part ξ and the imaginary part $\widetilde{\xi}$ of $(M_n)_{ij}$ are independent
- and have an exponential decay:
  there are two constants C and C' such that

$$P(\xi \geq t^C) \leq e^{-t} \quad \text{and} \quad P(\widetilde{\xi} \geq t^C) \leq e^{-t}$$

  for all $t \geq C'$

can possibly be relaxed (not necessarily identically distributed; finite moment condition: $\mathbb{E}|\xi|^C, \mathbb{E}|\widetilde{\xi}|^C < \infty$ for C suff. large)

known results

GUE: the joint law of the eigenvalues is known

allowing for a lot of descriptions of their limiting behavior both in the global and local regimes

$$W_n := \frac{1}{\sqrt{n}} M_n, \quad A_n := \sqrt{n} M_n \quad \text{coarse/fine-scale}$$

W_n: placing all eigenvalues in a bounded interval ($[-2, 2]$)

A_n: keeping the spacing between adjacent eigenvalues to be roughly of unit size

known results

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$: eigenvalues of W_n

(global) Wigner theorem:
(under substantially more general hypotheses true)

$\frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j}$ converges weakly almost surely as $n \to \infty$ to law

$$\varrho(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \, 1_{[-2,2]}$$

$I \subset \mathbb{R}$:

$$\frac{1}{n} N_I(W_n) := \frac{1}{n} \sum_{j=1}^n 1_{\{\lambda_j \in I\}} \to \varrho(I) \quad \text{a.s.}$$

fluctuation level: global

GUE: M_n', $W_n' := \frac{1}{\sqrt{n}} M_n'$

Theorem (Costin-Lebowitz; 1995)

Let I_n be an intervall and $\mathbb{V}(N_{I_n}(W_n')) \to \infty$ as $n \to \infty$, then

$$\frac{N_{I_n}(W_n') - \mathbb{E}(N_{I_n}(W_n'))}{\sqrt{\mathbb{V}(N_{I_n}(W_n'))}} \to N(0,1)$$

in distribution.

applying orthogonal polynomial techniques and/or the particular determinantal structure of GUE

wave of results

It has been conjectured, since the 1960s, by Wigner, Dyson, Mehta and many others, that the local statistics (the convergence of distribution functions) are universal, in the sense that they hold not only for the GUE, but for any other Wigner random matrix also.

Erdös, Schlein, Yau / Tao, Vu, 2009/2010/2011...

statistics

local:

- distribution of the gaps between consecutive eigenvalues: how many $1 \leq i \leq n$ are there such that $\lambda_{i+1} - \lambda_i \leq s$?
- k-point correlation functions
- distribution of individual λ_i

Gaudin, sin-kernel due to Dyson, Tracy-Widom

What about large and moderate deviations (global/local)?

(joint work with H. Döring, Th. Kriecherbauer, K. Schüler)

large deviations (LDP): i.i.d. summands

$(X_i)_i$ i.i.d.

$$S_n := \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$P\big(S_n \sim x\big) \approx \exp\big(-n\ I(x)\big)$$

with rate function

$$I(x) = \sup_{z \in \mathbb{R}} \{z\,x - \log E e^{zX_1}\}$$

large deviations: CRAMÉR...

moderate deviations (MDP): iid summands

$(X_i)_i$ i.i.d.

$$S_n^{a_n} := \frac{1}{a_n \sqrt{n}} \sum_{i=1}^{n} X_i \qquad \text{with} \qquad 1 \ll a_n \ll \sqrt{n}$$

$$P\big(S_n^{a_n} \sim x\big) \approx \exp\big(-a_n^2 \; I(x)\big)$$

with rate function

$$I(x) = \frac{x^2}{2\mathbb{V}(X_1)}$$

moderate deviations

some universality

(global) deviation results for spectral statistics

- LDP and MDP for empirical measures of eigenvalues in the Gaussian case
- LDP and MDP for empirical measures of eigenvalues for special Gaussian divisible ensembles:

  $$(1-t)^{1/2} M_n' + t^{1/2} V_n$$

  V_n deterministic selfadjoint matrix with convergent spectral measure
  Ben Arous, Guionnet, Dembo, 1997, 2002,...
- LDP for empirical measures for other symmetries
  E., Stolz, 2007, 2011
- traces of powers of Bernoulli random matrices
  Döring, E., 2009

no universal version for Wigner matrices!

first result

Let $\mathbb{V}(N_{I_n}(W_n')) \to \infty$:

Theorem (GUE, H. Döring, P.E., 2013, ALEA)

For any $(a_n)_n$ *with* $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n'))}$

$$\frac{N_{I_n}(W_n') - \mathbb{E}(N_{I_n}(W_n'))}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n'))}}$$

satisfies a MDP with speed a_n^2 *and rate function* $x^2/2$.

consider cumulants or log-LAPLACE transform; use determinantal structure of GUE

also: large deviations probabilities of CRAMÉR type

version with numerics

GUSTAVSSON, 2005, showed:
for $I = [y, \infty)$ with $y \in (-2, 2)$

$$\mathbb{E}[N_I(W_n')] = n\varrho(I) + O(\frac{\log n}{n}) \quad \text{and} \quad \mathbb{V}(N_I(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big) \log n$$

applying strong asymptotics for orthogonal polynomials with respect to exponential weights due to DEIFT et. al.

hence

$$\frac{N_I(W_n') - n\varrho(I)}{a_n \sqrt{\frac{1}{2\pi^2} \log n}}$$

satisfies the same MDP

universal MDP (global)

main result:

Theorem (H. Döring, P.E., 2013, ALEA)

Let M_n be a WIGNER *matrix whose entries satisfy condition (C) and match the corresponding entries of GUE up to order 4. Let $I = I(y) = [y, \infty)$ for $y \in (-2, 2)$. For any $(a_n)_n$ with $1 \ll a_n \ll \sqrt{\mathbb{V}(N_I(W_n))}$*

$$\frac{N_I(W_n) - \mathbb{E}(N_I(W_n))}{a_n \sqrt{\mathbb{V}(N_I(W_n))}}$$

satisfies a MDP with speed a_n^2 and rate function $x^2/2$.

the same is true for $I_n = [y_n, \infty)$ with $y_n \to 2^-$ and $n(2 - y_n)^{3/2} \to \infty$ (result at the edge of the spectrum)

matching 4 moments

complex random variables X and Y match to order k if

$$\mathbb{E}[\mathrm{Re}(X)^m \, \mathrm{Im}(X)^l] = \mathbb{E}[\mathrm{Re}(Y)^m \, \mathrm{Im}(Y)^l]$$

for all $m, l \geq 0$ such that $m + l \leq k$.

matching the corresponding entries of GUE up to order 4:
fix third and fourth moment

due to the famous Four Moment Theorem of TAO and VU

moderate deviations for a local statistic

on the way proving our result:

let $t(x) \in [-2, 2]$ defined for $x \in [0, 1]$ by

$$x = \int_{-2}^{t(x)} d\varrho(t)$$

consider $i = i(n)$ such that $i/n \to a \in (0, 1)$: λ_i is in the bulk

$t(i/n)$: expected location of the i-th eigenvalue

$$\frac{\sqrt{\log n}}{\pi\sqrt{2}} \frac{1}{n\varrho(t(i/n))}$$

standard deviation (mean eigenvalue spacing)

GUE and WIGNER

Theorem

Let $i/n \to a \in (0,1)$, $1 \ll a_n \ll \sqrt{\log n}$. Let W_n be a WIGNER *matrix whose entries satisfy a finite moment condition and match the corresponding entries of GUE up to order 4. Then*

$$\sqrt{\frac{4 - t(i/n)^2}{2}} \, \frac{\lambda_i(W_n) - t(i/n)}{a_n \frac{\sqrt{\log n}}{n}}$$

satisfies a MDP with speed a_n^2 and rate function $x^2/2$.

GUSTAVSSON: CLT, GUE

a similar result follows at the edge of the spectrum: $i/n \to 0$ for $i = i(n) \to \infty$

$$\frac{\lambda_{n-i}(W_n) - \left(2 - \left(\frac{3\pi}{2}\frac{i}{n}\right)^{2/3}\right)}{a_n \left(\frac{\log i}{i^{2/3} n^{4/3}}\right)^{1/2}}$$

proof

(1): GUE: transfer the $N_{I_n}(W_n')$–MDP to $\lambda_i(W_n')$:

use the tight relation: for $I(y) = [y, \infty)$

$$N_{I(y)}(W_n') \leq n - i \quad \Leftrightarrow \quad \lambda_i(W_n') \leq y$$

proof

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$$I_n := \left[t(i/n) + \xi \, a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}, \infty \right)$$

$$\begin{aligned} P_n\left(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \le \xi \right) &= P_n\big(N_{I_n}(W_n') \le n - i\big) \\ &= P_n\left(\frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{a_n \, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \le \frac{n - i - \mathbb{E}[N_{I_n}(W_n')]}{a_n \, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \right) \end{aligned}$$

asymptotics

remember:

$$\mathbb{E}[N_{I_n}(W_n')] = n\,\varrho(I_n) + O\Big(\frac{\log n}{n}\Big)$$

here: I_n depends on a_n and with strong asymptotics for orthogonal polynomials

$$n\,\varrho(I_n) = n - i - \xi a_n (\log n)^{1/2} \frac{1}{\sqrt{2}\pi} + O\Big(\frac{a_n^2 \log n}{n}\Big)$$

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$$n\,\varrho(I_n) = n - i - \xi a_n (\log n)^{1/2} \frac{1}{\sqrt{2\pi}} + O\big(\frac{a_n^2 \log n}{n}\big)$$

moreover we apply

$$\mathbb{V}(N_{I_n}(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big) \log n$$

proof

(2): next transfer the local MDP universally:
apply the Four Moment Theorem:

Theorem (Tao, Vu)

Let M_n be Wigner *whose entries satisfy a moment condition and match the corresponding entries of GUE up to order 4. Then there is a small constant c_0 such that*

$$P\big(\lambda_i(A'_n) \in I_-\big) - n^{-c_0} \le P\big(\lambda_i(A_n) \in I\big) \le P\big(\lambda_i(A'_n) \in I_+\big) + n^{-c_0}$$

and

$$\frac{1}{a_n^2} \log n^{-c_0} \to -\infty$$

by assumption

proof

(3): reverse strategy to go back to $N_I(W_n)$

proof

(3): reverse strategy to go back to $N_I(W_n)$

- deep fact: $\mathbb{E}(N_I(W_n))$ and $\mathbb{V}(N_I(W_n))$ have identical asymptotic behaviour to the ones for GUE matrices!
- Unfortunately, the Four Moment Theorem does not give this!
- Indeed, the Four Moment Theorem deals with a finite number of eigenvalues, whereas the computation of $\mathbb{E}(N_I(W_n))$ and $\mathbb{V}(N_I(W_n))$ involves all the eigenvalues of the matrix
- To achieve the result one can apply recent results by ERDÖS, YAU and YIN providing suitable localization properties of the eigenvalues in the bulk: therefore need assumption (C)

determinant

M_n' GUE:

$$W_n' := \frac{\log|\det M_n'| - \frac{1}{n}\log n + \frac{n}{2}}{\sqrt{\frac{1}{2}\log n}}$$

we are able to bound the jth cumulant κ_j of W_n'

$$\kappa_j \leq const. \frac{j!}{\Delta^{j-2}}$$

with $\Delta = c_j\,\sigma$ for a certain constant c

and apply moderate deviations via culumants (Rudzkid, Saulis, Statuljavicus, 1978), see H. Döring, P.E., 2013, JTP:

determinant

Theorem (H. Döring, P.E., 2013, Götze proceed.)

(1) Cramér–type large deviations probabilities: *There exists two constants C_1 and C_2 such that:*

$$\left| \log \frac{P(W_n' \geq x)}{1 - \Phi(x)} \right| \leq C_2 \frac{1 + x^3}{\sigma}$$

and

$$\left| \log \frac{P(W_n' \leq -x)}{\Phi(-x)} \right| \leq C_2 \frac{1 + x^3}{\sigma}$$

for all $0 \leq x \leq C_1 \sigma$.

(2) Berry-Esseen bounds: *We obtain the following bounds:*

$$\sup_{x \in \mathbb{R}} \left| P(W_n' \leq x) - \Phi(x) \right| \leq C \, (\log n)^{-1/2}.$$

(3) Moderate deviations principle: *For any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll \sigma$ the sequences $\left(\frac{1}{a_n} W_n'\right)_n$ satisfy a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$ (universaly true for W_n; 4 moment match).*

moderate deviations for the largest eigenvalue

$\lambda_{\max}$ largest GUE-eigenvalue

distribution function of $n^{2/3}(\lambda_{\max} - 2)$ converges to F_2 with

$$F_2(s) = \det(I - A_s) :$$

FREDHOLM-determinant of the operator A_s with kernel given in terms of Airy functions Ai by

$$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}$$

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Asymptotics:

$$1 - F_2(x) \sim \frac{e^{-(4/3)x^{3/2}}}{16\pi x^{3/2}} \text{ as } x \to \infty$$

$$\mathrm{Ai}(x) \sim \sin\big(2/3|x|^{3/2} + \frac{\pi}{4}\big) \text{ as } x \to -\infty$$

moderate deviations for the largest eigenvalue

Theorem (Th. Kriecherbauer, K. Schüler, P.E., 2014)

upper-tail moderate deviations for GUE: for any $0 < \alpha < 2/3$ *and* $y > 0$

$$\lim_{n\to\infty} \frac{\log P(\lambda_{\max} \geq 2 + yn^{-\alpha})}{n^{1-(3/2)\alpha}} = -\frac{4}{3} y^{3/2}$$

upper-tail moderate deviations for $\lambda_{\max}$ *universal for invariant matrix ensembles*

$$P_{n,V}(\lambda) = \frac{1}{Z_{n,V}} \prod_{j<k} (\lambda_k - \lambda_j)^2 \prod_{m=1}^{n} e^{-n\,V(\lambda_m)}\, d\lambda$$

for nice V*: real analytic, convex,* $\lim_{|x|\to\infty} V(x) = \infty$*...*

lower-tail moderate deviations, conjecture:

$$\lim_{n\to\infty} \frac{\log P(\lambda_{\max} \leq 2 - yn^{-\alpha})}{n^{2-3\alpha}} = -\frac{1}{12} y^{3}$$

finer asymptotics

by the method of orthogonal polynomials, one can even get finer asymptotics:

Theorem (Th. Kriecherbauer, K. Schüler, P.E., 2014)

upper-tail moderate deviations for $\lambda_{\max}$ *universal for invariant matrix ensembles: for any* $0 < \alpha < 2/3$ *and* $y > 0$

$$\log P(\lambda_{\max} \geq 2 + yn^{-\alpha}) = -\frac{4}{3} y^{3/2}\, n^{1-(3/2)\alpha} + \frac{\log\big(16\pi\, s^{3/2}\, n^{1-(3/2)\alpha}\big)}{s^{3/2}\, n^{1-(3/2)\alpha}} + \mathcal{O}\big(\frac{s}{n^{\alpha}}\big).$$

Thank you for your attention!

concentration of measure enters the 4 Moment Theorem

$P(\lambda_i(A_n) \in I) \approx \mathbb{E}(G(\lambda_i(A_n)))$, bump function

LINDEBERG'S replacement method:

fix all but one entry of A_n, say $z := Z_{pq}$ switch to $z' = Z'_{pq}$

$$\mathbb{E}G(\lambda_i(z)) - \mathbb{E}G(\lambda_i(z')) = \mathbb{E}(F(z)) - \mathbb{E}(F(z'))$$

TAYLOR; first four moments of z and z' match:

$$\sup_x |F^{(v)}| \, \mathbb{E}(|z|^5 + |z'|^5)$$

here: $\mathbb{E}(|z|^5 + |z'|^5) = O(n^{5/2})$

HADAMARD and delocalization

HADAMARD-Variational formulas:

$$|F^k(x)| \leq n^{-k+o(1)}$$

hence

$$|\mathbb{E}G(\lambda_i(z)) - \mathbb{E}G(\lambda(z'))| = \mathcal{O}\left(n^{-5/2+o(1)}\right)$$

swap roughly $n^2/2$ times:

$$O(n^{-1/2+o(1)}) = o(1)$$

universality

WIGNER-matrices:

$$\Big|\mathrm{card}\{\lambda_i \in I\} - \int_I \varrho(x)\, dx\Big| \ll N|I|$$

for all intervals $I \subset \mathbb{R}$ with $|I| \gg 1/n$.
ERDÖS, SCHLEIN, YAU, 2009.

universality

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$$\varrho^{-2} p_n^{(2)}\left(E + \frac{x_1}{n\varrho}, E + \frac{x_2}{n\varrho}\right) \approx \det\left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2$$

for every E in the bulk of the spectrum and $\varrho = \varrho(E)$
ERDÖS, SCHLEIN, YAU and TAO, VU, 2010/11