Persistence of Gaussian stationary processes

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Real Gaussian Stationary Processes (GSP)

Let $T \in \{\mathbb{Z}, \mathbb{R}\}$. A **GSP** is a random function $f : T \to \mathbb{R}$ s.t.

- It has Gaussian marginals:
  $\forall n \in \mathbb{N},\ x_1, \dots, x_n \in T$: $(f(x_1), \dots, f(x_n)) \sim \mathcal{N}_{\mathbb{R}^n}(0, \Sigma)$
- It is Stationary:
  $\forall n \in \mathbb{N},\ x_1, \dots, x_n \in T$ and $\forall t \in T$:
  $(f(x_1 + t), \dots, f(x_n + t)) \overset{d}{\sim} (f(x_1), \dots, f(x_n))$

If $T = \mathbb{Z}$ we call it a **GSS** (Gaussian Stationary Sequence).
If $T = \mathbb{R}$ we call it a **GSF** (Gaussian Stationary Function). We assume GSFs are a.s. continuous.

Covariance kernel

For a GSP $f : T \to \mathbb{R}$ the **covariance kernel** $r : T \to \mathbb{R}$ is defined by:

$$r(x) = \mathbb{E}\left[f(0)f(x)\right] = \mathbb{E}\left[f(t)f(x+t)\right].$$

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- positive-definite: $\sum_{1\le i,j\le n} c_i c_j r(x_i - x_j) \ge 0$.

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- symmetric: $r(-x) = r(x)$.

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- symmetric: $r(-x) = r(x)$.
- continuous.

Spectral measure

Bochner's Theorem

Write $\mathbb{Z}^* = [-\pi, \pi]$, $\mathbb{R}^* = \mathbb{R}$. Then

$$r(x) = \widehat{\rho}(x) = \int_{T^*} e^{-ix\lambda} d\rho(\lambda),$$

where ρ is a finite, symmetric, non-negative measure on T^*.

We call ρ the **spectral measure** of f.

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We assume:

$$\exists \delta > 0 : \int |\lambda|^\delta d\rho(\delta) < \infty.$$

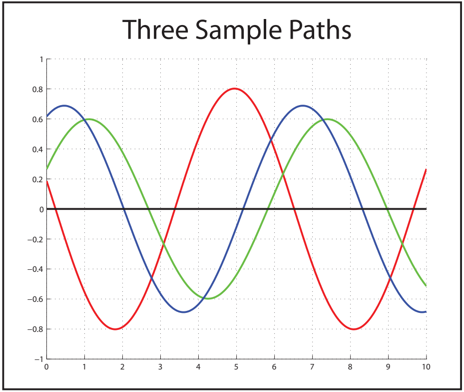
Toy-Example Ia - Gaussian wave

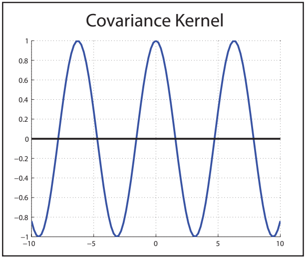
ζ_j i.i.d. $\mathcal{N}(0,1)$

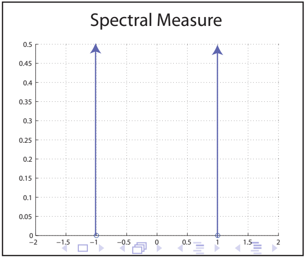
$f(x) = \zeta_0 \sin(x) + \zeta_1 \cos(x)$

$r(x) = \cos(x)$

$\rho = \frac{1}{2}(\delta_1 + \delta_{-1})$





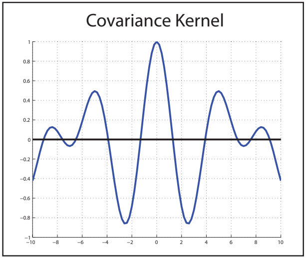


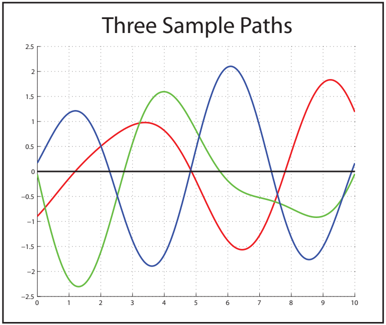
Toy-Example Ib - Almost periodic wave

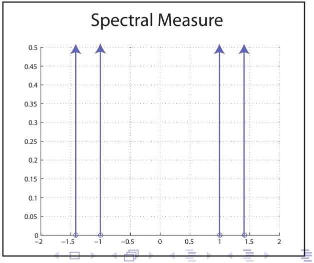
$$f(x) = \zeta_0 \sin(x) + \zeta_1 \cos(x) + \zeta_2 \sin(\sqrt{2}x) + \zeta_3 \cos(\sqrt{2}x)$$

$$r(x) = \cos(x) + \cos(\sqrt{2}x)$$

$$\rho = \frac{1}{2}\left(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}}\right)$$

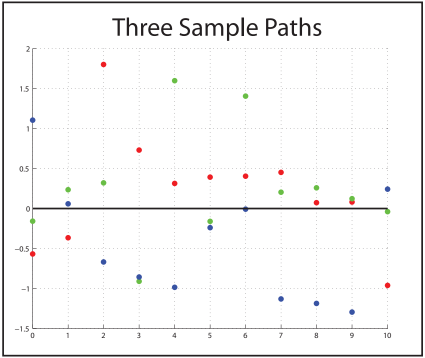


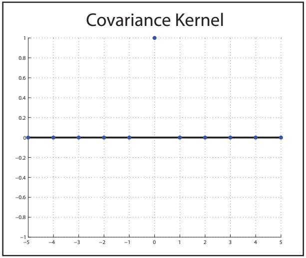


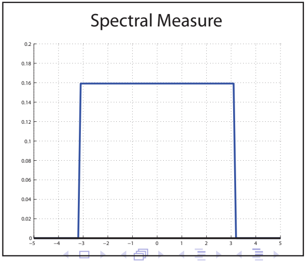


Example II - i.i.d. sequence

$$f(n) = \zeta_n$$
$$r(n) = \delta_{n,0}$$
$$d\rho(\lambda) = \frac{1}{2\pi}\mathbb{1}_{[-\pi,\pi]}(\lambda)d\lambda$$

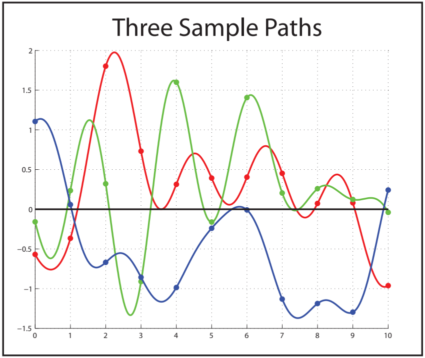


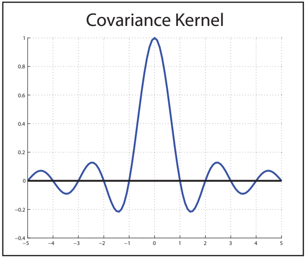


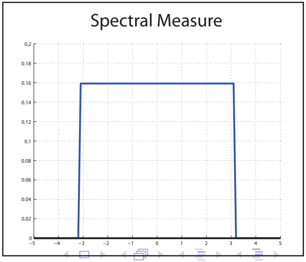


Example IIb - Sinc Kernel

$$f(x) = \sum_{n\in\mathbb{N}} \zeta_n \operatorname{sinc}(x-n)$$
$$r(x) = \frac{\sin(\pi x)}{\pi x} = \operatorname{sinc}(x)$$
$$d\rho(\lambda) = \frac{1}{2\pi} \mathbb{1}_{[-\pi,\pi]}(\lambda) d\lambda$$





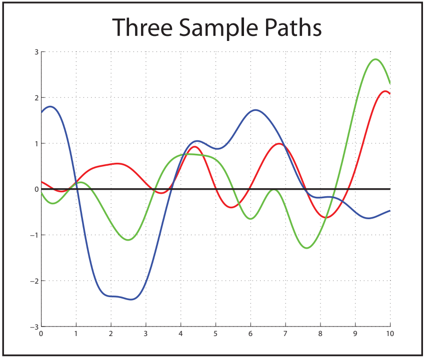


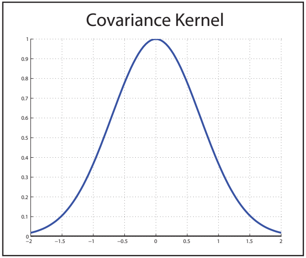
Example III - Gaussian Covariance

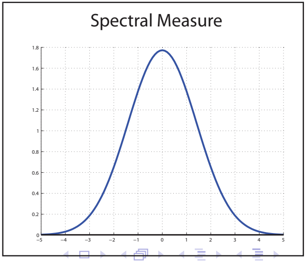
$$f(x) = \sum_{n \in \mathbb{N}} \zeta_n \frac{x^n}{\sqrt{n!}} e^{-\frac{x^2}{2}}$$

$$r(x) = e^{-\frac{x^2}{2}}$$

$$d\rho(\lambda) = \sqrt{\pi} e^{-\frac{\lambda^2}{2}} d\lambda$$

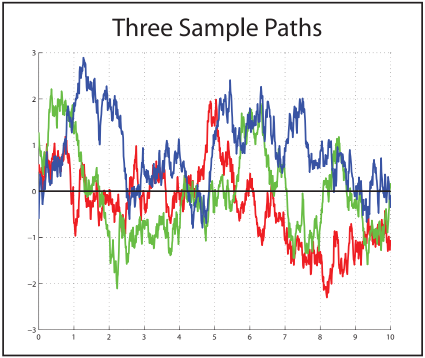


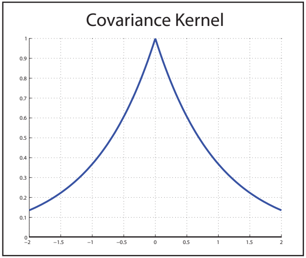


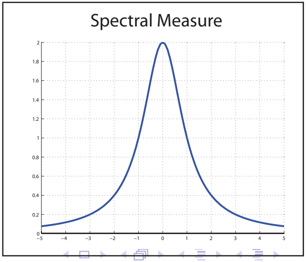


Example IV - Exponential Covariance

$$r(x) = e^{-|x|}$$
$$d\rho(\lambda) = \frac{2}{\lambda^2+1} d\lambda$$







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ρ - a finite, symmetric, non-negative measure on T^*
$\{\psi_n\}_n$ - ONB of $\mathcal{L}^2_\rho(T^*)$

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$$f(t) \stackrel{d}{=} \sum_n \zeta_n\varphi_n(t), \quad \text{where } \zeta_n \text{ are i.i.d. } \mathcal{N}(0,1).$$

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- make sure that φ_n are $\mathbb{R}$-valued.

Persistence Probability

Definition

Let f be a GSP on T. The **persistence probability** of f up to time $t \in T$ is

$$P_f(t) := \mathbb{P}\Big(f(x) > 0,\ \forall x \in (0, t]\Big).$$

a.k.a. *gap* or *hole* probability (referring to gap between zeroes or sign-changes).

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$(X_n)_{n\in\mathbb{Z}}$ i.i.d. $\Rightarrow P_X(N) = 2^{-N}$

$Y_n = X_{n+1} - X_n \Rightarrow P_Y(N) = \frac{1}{(N+1)!} \asymp e^{-N \log N}$

$Z_n \equiv Z_0 \Rightarrow P_Z(N) = \mathbb{P}(Z_0 > 0) = \frac{1}{2}$

Engineering and Applied Mathematics

40's - 60's

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Mathematical Analysis of Random Noise

By S. O. RICE

INTRODUCTION

THIS paper deals with the mathematical analysis of noise obtained by passing random noise through physical devices. **The random noise considered is that which arises from shot effect in vacuum tubes or from thermal agitation of electrons in resistors.** Our main interest is in the statistical properties of such noise and we leave to one side many physical results of which Nyquist's law may be given as an example.[1]

Engineering and Applied Mathematics

40's - 60's

- 1944 Rice - "Mathematical Analysis of Random Noise".
  - Mean number of level-crossings (Rice formula)
  - Behavior of $P(t)$ for $t \ll 1$ (**short range**).

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- 1962 Longuet-Higgins
  - generalized short-range results to gaps between nearly consecutive zeroes.

Engineering and Applied Mathematics

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- 1962 Newell & Rosenblatt
  - If $r(x) \to 0$ as $x \to \infty$, then $P(t) = o(t^{-\alpha})$ for any $\alpha > 0$.
  - If $|r(x)| < ax^{-\alpha}$ then $P(t) \leq \begin{cases} e^{-Ct} & \text{if } \alpha > 1 \\ e^{-Ct/\log t} & \text{if } \alpha = 1 \\ e^{-Ct^{\alpha}} & \text{if } 0 < \alpha < 1 \end{cases}$
  - examples for $P(t) > e^{-C\sqrt{t}\log t} \gg e^{-Ct}$ $(r(x) \asymp x^{-1/2})$.

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There are parallel independent results in the Soviet industry and Academia (e.g., by Piterbarg, Kolmogorov)

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- New motivation from physics:
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- 1998-2004 Bray, Ehrhardt, Majumdar (and others).
    - "independent interval approximation"
    - "correlator expansion method": a series expansion for the persistence exponent
    - numerical simulations

Probability and Analysis

00's-

- 2005-14 Hole probability for Gaussian analytic functions
  - in the plane (Sodin-Tsirelson 2005, Nishry 2010)
  - in the hyperbolic disc (Buckley, Nishry, Peled, Sodin - 2014)

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- 2013 Dembo & Mukherjee:
  - no zeroes for random polynomials $\leftrightarrow$ persistence of GSP
  - If $r(x) \geq 0$, then exists $\lim_{t\to\infty} \frac{-\log P(t)}{t} \in [0, \infty)$ (uses Slepian).

Probability and Analysis

Bounds for the sinc kernel

Theorem (Antezana, Buckley, Marzo, Olsen 2012)

For the sinc-kernel process ($r(t) = \text{sinc}(t)$), there is a constant $c > 0$ such that

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- Upper bound: notice $(f(n))_{n\in\mathbb{Z}}$ are i.i.d., so

$$\mathbb{P}(f > 0, \text{ on } (0, N] \cap \mathbb{R}) \le \mathbb{P}(f > 0, \text{ on } (0, N] \cap \mathbb{Z}) = \frac{1}{2^N}.$$

- Lower bound: an explicit construction + computation.

Main Result

Theorem (F. & Feldheim, 2013)

Let f be a GSP (on $\mathbb{Z}$ or $\mathbb{R}$) with spectral measure ρ. Suppose that $\exists a, m, M > 0$ such that ρ has density in $[-a, a]$, denoted by $\rho'(x)$, and

$$\forall x \in (-a, a): \; m \leq \rho'(x) \leq M.$$

Then $\exists c_1, c_2 > 0$ s.t. for all large enough N:

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- Main tool: "spectral decomposition"

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$Y_n = X_{n+1} - X_n \Rightarrow P_Y(N) \asymp e^{-N \log N}$ $\qquad \rho' = 2(1-\cos\lambda)1\!\!1_{[-\pi,\pi]}$

$Z_n \equiv Z_0 \Rightarrow P_Z(N) = \frac{1}{2}$ $\qquad \rho = \delta_0$

Ideas from the proof.

Spectral decomposition

Key Observation

$$\rho = \rho_1 + \rho_2 \Rightarrow f \overset{d}{=} f_1 \oplus f_2,$$

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Proof:

$$\begin{aligned}
&\operatorname{cov}((f_1 + f_2)(0), (f_1 + f_2)(x)) \\
&= \operatorname{cov}(f_1(0), f_1(x)) + \operatorname{cov}(f_2(0), f_2(x)) \\
&= \widehat{\rho_1}(x) + \widehat{\rho_2}(x) = \widehat{\rho_1 + \rho_2}(x) = \operatorname{cov}(f(0), f(x)). \quad \square
\end{aligned}$$

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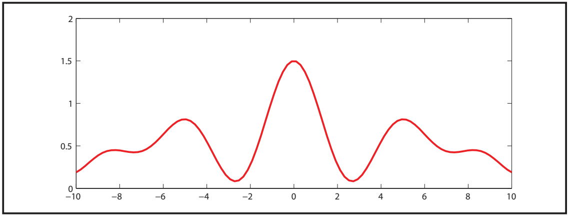
Application:

$$\rho = m\mathbb{1}[-\frac{\pi}{k}, \frac{\pi}{k}] + \mu \Rightarrow f = S \oplus g$$

where $r_S(x) = c \ \mathrm{sinc}(\frac{x}{k})$, and g is some GSP.



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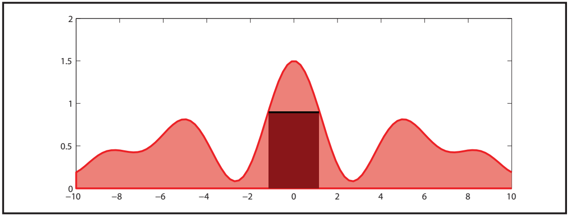

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Application:

$$\rho = m\mathbb{1}[-\frac{\pi}{k}, \frac{\pi}{k}] + \mu \Rightarrow f = S \oplus g$$

where $r_S(x) = c \,\mathrm{sinc}(\frac{x}{k})$, and g is some GSP.

Observation.

$(S(nk))_{n\in\mathbb{Z}}$ are i.i.d.

Proof: $\mathbb{E}[S(nk)S(mk)] = r_S((m-n)k) = 0.$

Upper Bound

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Let us use this observation to obtain an upper bound on $P_f(N)$.

$$P_f(N) \le \mathbb{P}\left(S \oplus g > 0 \text{ on } (0, N] \,\middle|\, \frac{1}{N}\sum_{n=1}^{N} g(n) < 1 \right) + \mathbb{P}\left(\frac{1}{N}\sum_{n=1}^{N} g(n) \ge 1 \right)$$

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Lemma 1.

$\frac{1}{N}\sum_{n=1}^{N} g(n) \sim \mathcal{N}_{\mathbb{R}}(0,\sigma_N^2)$, where $\sigma_N^2 \le \frac{C_0}{N}$.

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- Lemma 1 $\Rightarrow \mathbb{P}(\frac{1}{N}\sum_{n=1}^{N} g(n) \ge 1) \le e^{-c_1 N}$.

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Lemma 2.

Let $X_1, \ldots, X_N$ be i.i.d $\mathcal{N}(0,1)$, and $b_1, \ldots, b_N \in \mathbb{R}$ such that $\frac{1}{N}\sum_{j=1}^{N} b_j < 1$. Then $\exists C > 0$ so that

$$\mathbb{P}\left(X_j + b_j > 0,\ 1 \le j \le N\right) \le e^{-CN}.$$

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Proof:

$$\log \mathbb{P}(X_j \ge -b_j,\ 1 \le j \le N) = \log \prod_{j=1}^{N} \Phi(b_j)$$

$$= \sum_{j=1}^{N} \log \Phi(b_j) \le N \log \Phi\left(\frac{1}{N}\sum b_j\right) \le N \log \Phi(1).$$

Lower bound

Reduction to functions: $(f(j))_{j\in\mathbb{Z}} \rightsquigarrow \rho \in \mathcal{M}([-\pi,\pi]) \subset \mathcal{M}(\mathbb{R})$, so may "extend" to $(f(t))_{t\in\mathbb{R}}$ with the same ρ. Now,

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$$\begin{aligned}&\mathbb{P}(S \oplus g > 0 \text{ on } (0,N])\\ &\geq \mathbb{P}(S > 1 \text{ on } (0,N])\, \mathbb{P}(|g| \leq \frac{1}{2} \text{ on } (0,N])\end{aligned}$$

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$$\geq \underbrace{\mathbb{P}(S > 1 \text{ on } (0,N])}_{\text{ABMO}} \; \underbrace{\mathbb{P}(|g| \leq \tfrac{1}{2} \text{ on } (0,N])}_{\text{small ball prob.}}$$

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Talagrand, Shao-Wang (1994), Ledoux (1996)

Suppose $(X(t))_{t\in I}$ is a centered Gaussian process on an interval I, and

$$\mathbb{E}|X(s)-X(t)|^2 \le C|t-s|^\gamma$$

for all $s,t\in I$ and some $0<\gamma\le 2$, $C>0$. Then

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For stationary processes, the moment condition is enough.

$$\exists\delta>0: \int|\lambda|^\delta d\rho(\delta)<\infty.$$

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- compute it.

Thank you.

"Persistence can grind an iron beam down into a needle."
– – Chinese Proverb.