Rigidity Phenomena in random point sets and Applications

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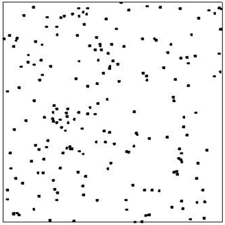
Background



Poisson Process

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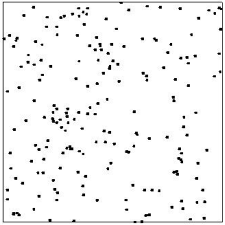

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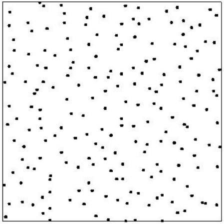

Ginibre Ensemble

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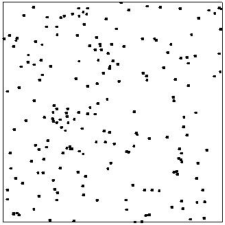

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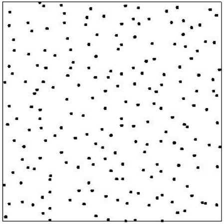

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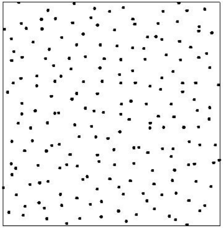

Gaussian Zeroes

Ginibre Ensemble

- Finite n: μ_n = Eigenvalues of $G_n = ((\xi_{ij}))_{1 \leq i,j \leq n}$, ξ_{ij} i.i.d $N_{\mathbb{C}}(0,1)$ (NO normalization by $\sqrt{n}$)

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Theorem (Sodin rigidity)

$f(z)$ is the unique (up to a deterministic multiplier) Gaussian entire function with a translation invariant zero process of intensity 1.

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- In Poisson point process, the points inside and outside $\mathbb{D}$ are independent of each other

Our Results: Ginibre Ensemble

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In the Ginibre ensemble,
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$M(\omega)$ and $m(\omega)$ positive constants
$\Delta(\zeta) = \prod_{i<j}(\zeta_i - \zeta_j)$ (Vandermonde)

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Limit as $n \to \infty$ is Poisson : Not Rigid !

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$\Sigma_{S(\omega)}$: constant sum hypersurface $\sum_{i=1}^{N(\omega)} \zeta_i = S(\omega)$ inside $\mathbb{D}^{N(\omega)}$

$M(\omega)$, $m(\omega)$ and $\Delta(\zeta)$ are as before.

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- Know outside zeroes $\Rightarrow$ Know $\int_{\mathbb{D}_L \setminus \mathbb{D}} \varphi_L d\nu \Rightarrow$ Compute $n(\mathbb{D})$ approximately, now let $L \to \infty$

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Uniqueness involves understanding of rigidity and tolerance.

Some Interesting Results en route

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The zeroes of the GAF determine the function a.s. (up to a multiplicative factor of modulus 1). In other words, if ν denotes the zeroes of the GAF f, then $\exists$ an analytic function $g(z) = \sum_{k=0}^{\infty} a_k(\nu) z^k$ such that $f(z) = \gamma . g(z)$
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Theorem (Stability of Inverse Sums, G.-Peres)

Let z denote the points of Ginibre ensemble. Then the random series $\sum_{|z| \uparrow} \frac{1}{z}$ converges almost surely, and in fact, has finite first moment.

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- A conjecture of Lyons-Steif
- Studying mixtures of point processes

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- Classical results are parametrized by some sort of asymptotic density of Λ; complete if supercritical, incomplete if subcritical.

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- What if pathological configurations are eliminated by choosing a "generic" point configuration - i.e., a point process ?

Theorem (G.)

Let Π be a determinantal point process with kernel K and background measure μ, such that K is the integral kernel corresponding to a projection on to a subspace $\mathcal{H}$ of $L_2(\mu)$. Clearly, $\mathcal{E}_\Pi = \{K(x,\cdot) : x \in \Pi\} \subset \mathcal{H}$.

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- Theorem implies positive answer to analogous completeness question in 2-d for random exponentials (sampled from the Ginibre ensemble) inside the Fock Bargmann space.

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- Natural point processes for Levels $k \geq 3$??

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- Rigidity for dpp If a determinantal point process has a kernel which is a proper contraction, then it must be insertion and deletion tolerant.

Thank you !!