Persistence exponent for random processes in Brownian scenery

Nadine Guillotin-Plantard

joint work with F. Castell (I2M) and F. Watbled (LMBA)

Persistence probabilities and related fields – Darmstadt

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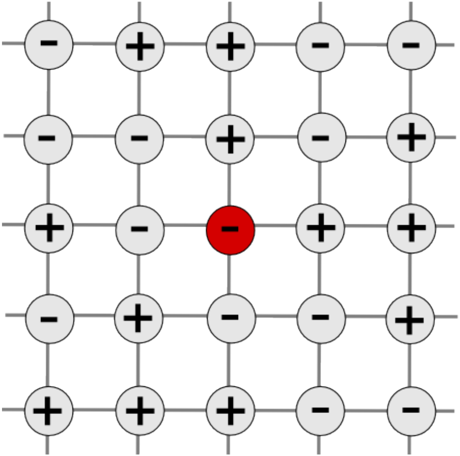
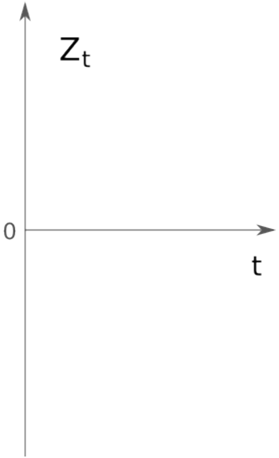
$\mathbb{P}$ the product law of ξ, S.

An example

$d = 2$, $\xi_x \in \{+1, -1\}$, S nearest-neighbour.





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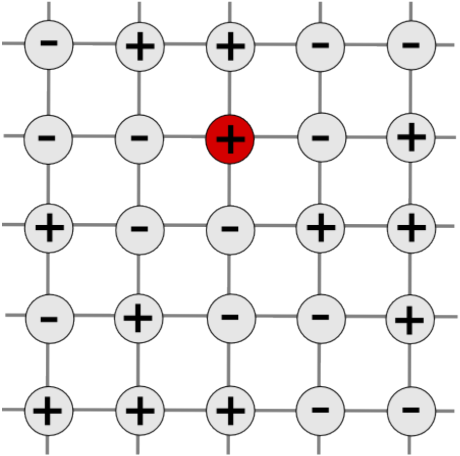

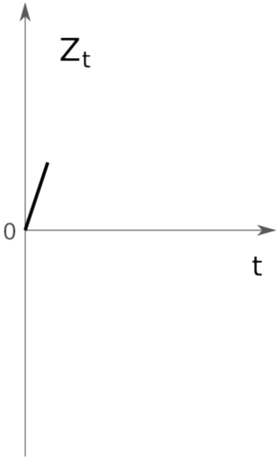

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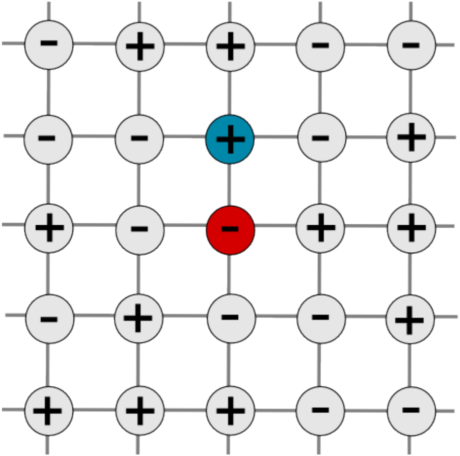

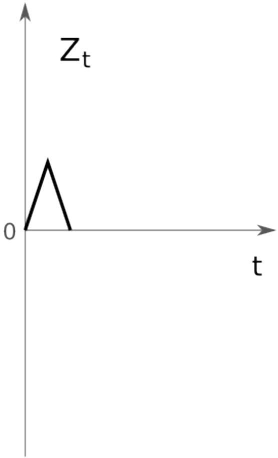

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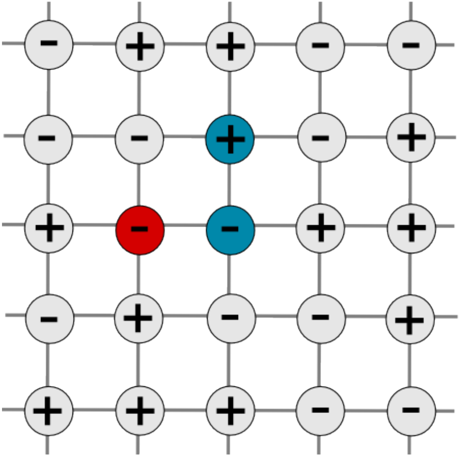

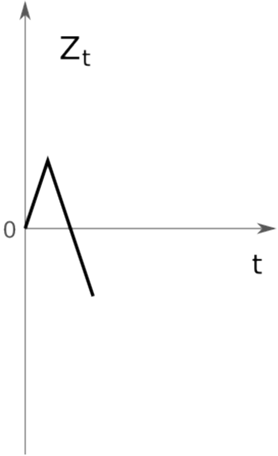

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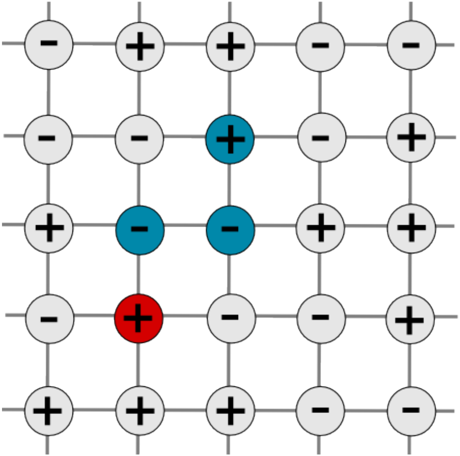

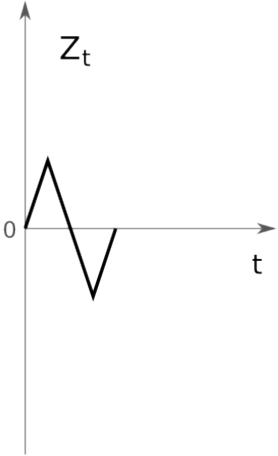

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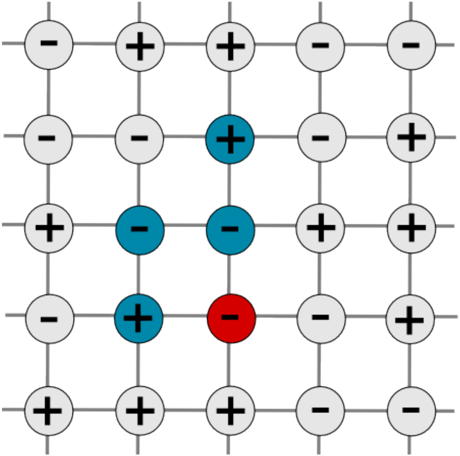

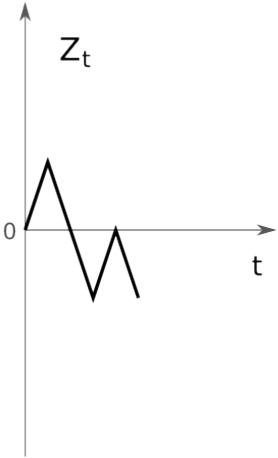

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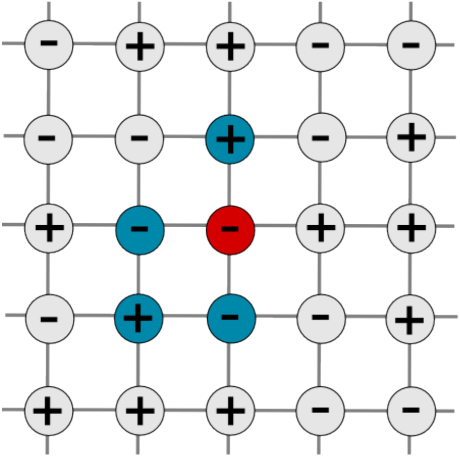

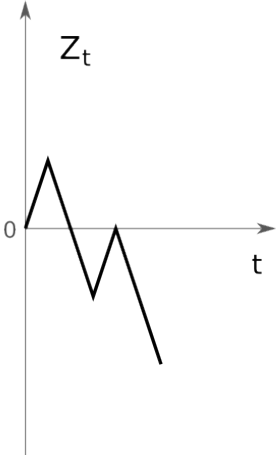

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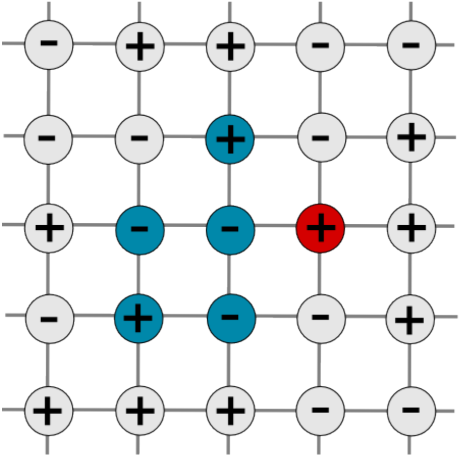

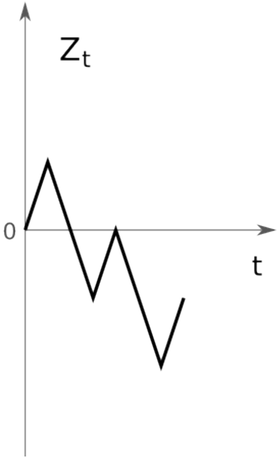

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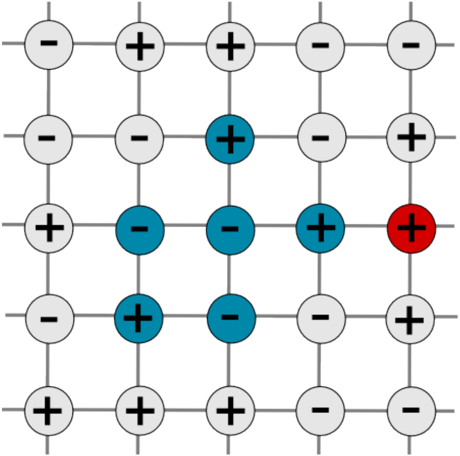

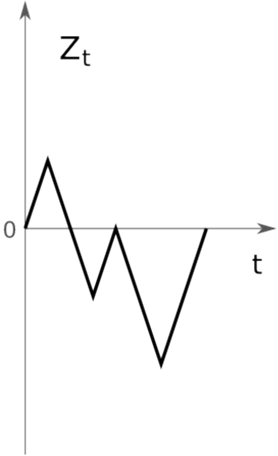

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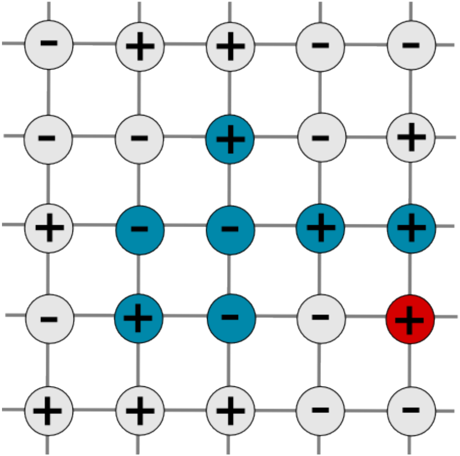

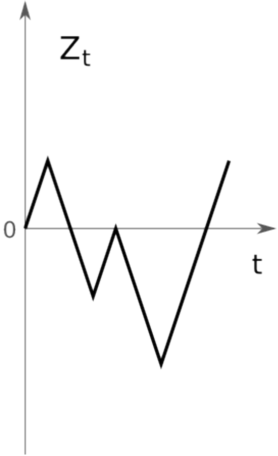

Motivation

Links to other problems and models:

1. Energy function of a polymer in a random medium;
2. Random walks in randomly directed lattices;
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Question:

What is the limiting distribution of $(Z_n)_{n\geq 1}$? Or what is its continuous counterpart?

Assumptions

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Assumptions on the random walk:

$$S \text{ is the simple random walk in } \mathbb{Z}^d.$$

Connection to the self-intersection local time of S

$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{S_k=x\}}$ the local time of S at x up to time n.

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$$\mathrm{Var}(Z_n) = \mathbb{E}\left[I_n\right] \sim \begin{cases} Cn^{3/2} & d=1, \\ Cn\log n & d=2, \\ Cn & d\ge 3. \end{cases}$$

Limit theorems

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$d \geq 3$: $a_n = \sqrt{n}$, Δ Brownian motion (Spitzer's Book '76).

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The proof heavily depends on the increments independence of the process Y.
Is it possible to compute the persistence exponent without it? for more general processes Y?

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The process Δ is itself a h-self-similar process with stationary increments, with

$$h := 1 - \frac{\gamma}{2}.$$

Our result

Let

$$V_1 := \int_{\mathbb{R}} L_1^2(x)\,\mathrm{d}x \text{ " } = \text{ " } \int_0^1 \int_0^1 \delta_0(Y_t - Y_s)\,\mathrm{dsdt}$$

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Theorem (F. Castell, N. G-P, F. Watbled, 2014)

Assume (H1) and (H2) hold. $\exists c > 0$, s.t. for T large enough,

$$T^{-\gamma/2}(\ln T)^{-c} \leq \mathbb{P}\Big[\sup_{t\in[0,T]} \Delta_t \leq 1\Big] \leq T^{-\gamma/2}(\ln T)^{+c}.$$

Examples of Processes Y satisfying our assumptions

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3. Iterated Brownian motion with $\gamma = 1/4$, $\alpha = \beta = 4/3$.

Aurzada and Molchan's approach for FBM

Theorem (Molchan, 1999 + Aurzada, 2011)
Let $\{X_t; t \geq 0\}$ be a continuous process, self-similar with index $h > 0$, with stationary increments s.t. for every $\theta > 0$,

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Then, there exists $c > 0$ s.t. for T large enough,

$$T^{-(1-h)}(\ln T)^{-c} \leq \mathbb{P}\left[\sup_{t\in[0,T]} X_t \leq 1\right].$$

Given Y, the process $\{\Delta_t, t \geq 0\}$ is a centered Gaussian process with covariance matrix

$$\mathbb{E}[\Delta_s \Delta_t | Y] = \int_{\mathbb{R}} L_s(x) L_t(x) \, \mathrm{d}x \geq 0.$$

Moreover, for any $0 < s < t$,

$$\mathbb{E}[\Delta_s(\Delta_t - \Delta_s) | Y] = \int_{\mathbb{R}} L_s(x)(L_t(x) - L_s(x)) \, \mathrm{d}x \geq 0.$$

Use Slepian's Lemma conditionally to Y and then integrate with respect to the law of Y.

Open problems

- Persistence exponent of the discrete model

$$Z_n = \sum_{k=1}^{n} \xi_{S_k}, \;\; n \geq 1$$

with $\xi_x \in \{\pm 1\}$, $\{S_n, n \geq 0\}$ the simple random walk in $\mathbb{Z}$?

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Conjecture 2: The persistence exponent should be equal to

$$h = \frac{\beta - 1}{\alpha \beta}.$$

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Thank you for your attention!