A solution selection problem with small stable perturbations

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joint work with

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Well-posed ordinary differential equations in $\mathbb{R}^d$...
... perturbed by small noise

$$\mathbf{dX}^\varepsilon = \mathbf{b}(\mathbf{X}^\varepsilon)\mathbf{dt} + \varepsilon \mathbf{dW}, \qquad \mathbf{X}(\mathbf{0}) = \mathbf{x} \in \mathbb{R}^\mathbf{d}, \qquad \varepsilon > \mathbf{0}$$

Huge literature on related topics:

- **First exit problems, Freidlin-Wentsell theory, Persistence probabilities**
- **Invariant measures, Hitting probabilities, Metastability etc.**

But in general:

$$\mathcal{L}(\mathbf{X}^\varepsilon) \xrightarrow{\mathbf{w}} \delta_{\mathbf{X}^\mathbf{0}}, \qquad \varepsilon \to \mathbf{0}+ .$$

Ill-posed ordinary differential equations in $\mathbb{R}^{\mathbf{d}}$...

$$\dot{\mathbf{X}} = \mathbf{b}(\mathbf{X}), \qquad \mathbf{X}(\mathbf{0}) = \mathbf{x} \in \mathbb{R}^{\mathbf{d}}$$

- **e.g. Characteristics of the transport equation**
- **Mild ill-posedness: Non-uniqueness of the solution**
- **Dimension 1 as a test case**

Ill-posed ordinary differential equations in $\mathbb{R}^d$...

$$\dot{X} = b(X), \qquad X(0) = x_0 \in \mathbb{R}^d$$

Peano phenomenon:[1] **Let** $b : \mathbb{R} \to \mathbb{R}$ **continuous.**

1. $b(x_0) \neq 0$, **there is a unique local solution around** x_0
2. $b(x_0) = 0$, x_0 **isolated zero of** b.
   Then there exists a non-constant local solution around x_0 **if and only if one of the two cases is satisfied**

   (a) $b(y) > 0$ **for** $y > x_0$ **and**

   $$\int_{x_0}^{x_0+r} \frac{1}{b(y)} dy < \infty$$

   (b) $b(y) < 0$ **for** $y < x_0$ **and**

   $$\int_{x_0}^{x_0-r} \frac{1}{b(y)} dy < \infty.$$

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[1] Petrov: Ordinary differential equations, Prentice-Hall, 1966

Paradigmatic Example:

$$\mathbf{b}(\mathbf{x}) = \mathbf{B}^{+}|\mathbf{x}|^{\beta^{+}}\mathbf{1}\{\mathbf{x} \geqslant \mathbf{0}\} - \mathbf{B}^{-}|\mathbf{x}|^{\beta^{-}}\mathbf{1}\{\mathbf{x} \geqslant \mathbf{0}\}$$

for $\mathrm{B}^{+}, \mathrm{B}^{-} > \mathbf{0}$ **and** $\beta^{+}, \beta^{-} \in (\mathbf{0}, \mathbf{1})$**.**

The set of solutions of $\dot{\mathrm{u}} = \mathrm{b}(\mathrm{u})$ **is known to be** $\mathrm{t} \geqslant \mathrm{t}' \geqslant 0$

$$\mathbf{u}(\mathbf{t};\mathbf{t}',\mathbf{x}) = \begin{cases} \left(\mathbf{B}^{+}(\mathbf{1}-\beta^{+})(\mathbf{t}-\mathbf{t}') + \mathbf{x}^{\mathbf{1}-\beta^{+}}\right)^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{+}}} & \mathbf{x} \geqslant \mathbf{0} \\ (\mathbf{B}^{+}(\mathbf{1}-\beta^{+}))^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{+}}}(\mathbf{t}-\mathbf{t}'-\mathbf{s})_{+}^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{+}}} & \mathbf{x} = \mathbf{0}, \mathbf{s} \geqslant \mathbf{0} \\ (\mathbf{B}^{-}(\mathbf{1}-\beta^{-}))^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{+}}}(\mathbf{t}-\mathbf{t}'-\mathbf{s})_{+}^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{-}}} & \mathbf{x} = \mathbf{0}, \mathbf{s} \geqslant \mathbf{0} \\ \left(-\mathbf{B}^{-}(\mathbf{1}-\beta^{-})(\mathbf{t}-\mathbf{t}') + \mathbf{x}^{\mathbf{1}-\beta^{-}}\right)^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{-}}} & \mathbf{x} \leqslant \mathbf{0}' \end{cases}$$

Ill-posed ordinary differential equations in $\mathbb{R}^{\mathbf{d}}$...
... perturbed by noise

$$\mathbf{dX}^{\varepsilon} = \mathbf{b}(\mathbf{X}^{\varepsilon})\mathbf{dt} + \varepsilon\mathbf{dW}, \qquad \mathbf{X}(\mathbf{0}) = \mathbf{x} \in \mathbb{R}^{\mathbf{d}}$$

- **There exists a unique strong solution for any Wiener Process $\mathbf{W}$**
- **It has the strong Markov property**
- **Remains true for $\mathbf{W}$ replaced by an α-stable process $\mathbf{L}$ with absolutely continuous laws**

Bafico and Baldi: A solution selection problem [2],

Theorem:

Denote by $\mathbf{P}^{\mathbf{0}}_{\varepsilon}$ the law of $\mathbf{X}^{\varepsilon}$

$$\mathbf{dX}^{\varepsilon} = \mathbf{b}(\mathbf{X}^{\varepsilon})\mathbf{dt} + \varepsilon\mathbf{dW}, \qquad \mathbf{X}(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^{\mathbf{d}}, \varepsilon > \mathbf{0}$$

and $\tau^{\varepsilon}_{\mathbf{r}} = \inf\{\mathbf{t} > \mathbf{0} \mid \mathbf{X}^{\varepsilon}_{\mathbf{t}} \notin [-\mathbf{r}, \mathbf{r}]\}$ for $\mathbf{r} > \mathbf{0}$.

Then

$$\mathbf{P}^{\mathbf{0}}_{\varepsilon} \xrightarrow{\mathbf{w}} \mathbf{p}^{+}\delta_{\mathbf{x}^{+}} + \mathbf{p}^{-}\delta_{\mathbf{x}^{-}}, \qquad \text{as } \varepsilon \to \mathbf{0},$$

for the **extremal solutions**

$$\mathbf{x}^{\pm}(\mathbf{t}) = \pm\mathbf{C}^{\pm}\mathbf{t}^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{\pm}}}, \qquad \mathbf{t} \geqslant \mathbf{0}$$

$$\mathbf{p}^{+} = \begin{cases} \mathbf{1} & \text{if} \quad \beta^{+} < \beta^{-} \\ \dfrac{(\mathbf{B}^{-})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}}}{(\mathbf{B}^{+})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}} + (\mathbf{B}^{-})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}}} & \text{if} \quad \beta^{+} = \beta^{-} =: \beta \\ \mathbf{0} & \text{if} \quad \beta^{+} > \beta^{-}. \end{cases}$$

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[2] Small Random perturbations of Peano Phenomena, Stochastics, (6), 279–292

Proof of Bafico and Baldi [3],

- $\phi_\varepsilon(\mathbf{x}) := \mathbb{P}(\omega \in \mathcal{C}([\mathbf{0},\infty),\mathbb{R})$ with $\tau(\omega) < \infty$ and $\omega(\tau) = \mathbf{r})$ solves

$$\frac{\varepsilon^{\mathbf{2}}}{\mathbf{2}}\phi_\varepsilon''(\mathbf{x}) + \mathbf{b}(\mathbf{x})\phi_\varepsilon'(\mathbf{x}) = -\mathbf{1}, \qquad \phi_\varepsilon(-\mathbf{r}) = \mathbf{0}, \quad \phi_\varepsilon(\mathbf{r}) = \mathbf{1}.$$

- Explicit solution and explicit calculation

$$\phi_\varepsilon(\mathbf{x}) = \frac{-\mathbf{A}_\varepsilon(-\mathbf{r})}{\mathbf{A}_\varepsilon(\mathbf{r}) - \mathbf{A}_\varepsilon(-\mathbf{r})}$$

$$\mathbf{A}_\varepsilon(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \exp\big(-\frac{\mathbf{2}}{\varepsilon^{\mathbf{2}}}\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{b}(\mathbf{s})\mathbf{ds}\big)\mathbf{dt}.$$

[3] Small Random perturbations of Peano Phenomena, Stochastics, (6), 279–292

Flandoli, Delarue: Reproof based on self-similarity [4]

Proposition 1:
Let $\mathbf{B} = \mathbf{B}^{+} = \mathbf{B}^{-}$ and $\beta = \beta^{+} = \beta^{-}$.
There are functions $(\mathbf{t}_{\varepsilon}, \boldsymbol{\Theta}_{\varepsilon}) \searrow \mathbf{0}+$ such that

$$\lim_{\varepsilon \to \mathbf{0}} \mathbf{P}_{\varepsilon}^{\mathbf{0}}(\tau_{\boldsymbol{\Theta}_{\varepsilon}} > \mathbf{t}_{\varepsilon}) = \mathbb{P}(|\mathbf{W_1}| \leqslant \mathbf{2}) < \mathbf{1}.$$

and as a consequence for any $\tilde{\mathbf{t}}_{\varepsilon}/\mathbf{t}_{\varepsilon} \to \infty$ as $\varepsilon \to \mathbf{0}$

$$\lim_{\varepsilon \to \mathbf{0}} \mathbf{P}_{\varepsilon}^{\mathbf{0}}(\tau_{\boldsymbol{\Theta}_{\varepsilon}} > \tilde{\mathbf{t}}_{\varepsilon}) = \mathbf{0}.$$

[4] The transition point in the zero noise limit for a 1 D Peano example, *Discrete Contin. Dyn. Syst.*, **34** (2014), n.10, 4071-4083.

Flandoli and Delarue: A proof based on self-similarity [5]

Proof: Assume there are $(\mathbf{t}_\varepsilon, \mathbf{\Theta}_\varepsilon)$ as above.

- $\tau_{\mathbf{\Theta}_\varepsilon} > \mathbf{t}_\varepsilon$ implies $|\mathbf{X}^\varepsilon_\mathbf{t}| \leqslant \mathbf{\Theta}_\varepsilon$ for all $\mathbf{t} \leqslant \mathbf{t}_\varepsilon$

$$\varepsilon|\mathbf{W_t}| \leqslant |\mathbf{X}^\varepsilon_\mathbf{t}| + \int_0^\mathbf{t} |\mathbf{b}(\mathbf{X}^\varepsilon_\mathbf{x})|\mathbf{dt} \leqslant \mathbf{\Theta}_\varepsilon + \mathbf{B}\mathbf{t}_\varepsilon\mathbf{\Theta}^\beta_\varepsilon$$

- Assume $\mathbf{B}\mathbf{t}_\varepsilon\mathbf{\Theta}^\beta_\varepsilon = \mathbf{\Theta}_\varepsilon$. Then

$$\varepsilon|\mathbf{W_t}| \leqslant \mathbf{2\Theta}_\varepsilon \qquad \forall\, \mathbf{t} \leqslant \mathbf{t}_\varepsilon,$$

  in particular $\varepsilon|\mathbf{W}_{\mathbf{t}_\varepsilon}| \leqslant \mathbf{2\Theta}_\varepsilon$.

- Hence

$$\mathbf{P}^\mathbf{0}_\varepsilon(\tau_{\mathbf{\Theta}_\varepsilon} > \mathbf{t}_\varepsilon) \leqslant \mathbf{P}^\mathbf{0}_\varepsilon(\varepsilon|\mathbf{W}_{\mathbf{t}_\varepsilon}| \leqslant \mathbf{2\Theta}_\varepsilon) = \mathbf{P}^\mathbf{0}_\varepsilon(|\mathbf{t}_\varepsilon^{-\frac{1}{2}}\mathbf{W}_{\mathbf{t}_\varepsilon}| \leqslant \mathbf{2}\frac{\mathbf{\Theta}_\varepsilon}{\varepsilon\mathbf{t}_\varepsilon^{\frac{1}{2}}})$$

- Assume further $\mathbf{\Theta}_\varepsilon = \varepsilon\mathbf{t}_\varepsilon^{\frac{1}{2}}$ then

$$\mathbf{P}^\mathbf{0}_\varepsilon(\tau_{\mathbf{\Theta}_\varepsilon} > \mathbf{t}_\varepsilon) = \mathbb{P}(|\mathbf{W_1}| \leqslant \mathbf{2}) < \mathbf{1}.$$

[5] The transition point in the zero noise limit for a 1 D Peano example, *Discrete Contin. Dyn. Syst.*, **34** (2014), n.10, 4071-4083.

- Solving

$$\mathbf{B}\mathbf{t}_\varepsilon\boldsymbol{\Theta}_\varepsilon^\beta = \boldsymbol{\Theta}_\varepsilon$$
$$\boldsymbol{\Theta}_\varepsilon = \varepsilon\mathbf{t}_\varepsilon^{\frac{1}{2}}$$

yields

$$\mathbf{t}_\varepsilon = \frac{\varepsilon^{\mathbf{2}\frac{\mathbf{1}-\beta}{\mathbf{1}+\beta}}}{\mathbf{B}^{\mathbf{1}+\beta}} = \frac{\varepsilon^{\mathbf{2}\frac{\mathbf{1}-\beta}{\mathbf{2}+\beta-\mathbf{1}}}}{\mathbf{B}^{\mathbf{1}+\beta}}, \qquad \boldsymbol{\Theta}_\varepsilon = \frac{\varepsilon^{\frac{\mathbf{2}}{\mathbf{1}+\beta}}}{\mathbf{B}^{\mathbf{1}+\beta}} = \frac{\varepsilon^{\frac{\mathbf{2}}{\mathbf{2}+\beta-\mathbf{1}}}}{\mathbf{B}^{\frac{\mathbf{1}+\beta}{\mathbf{2}}}} = \frac{\varepsilon^{\frac{\mathbf{2}(\mathbf{1}-\beta)}{\mathbf{2}+\beta-\mathbf{1}+\beta(\mathbf{2}+\beta-\mathbf{1})}}}{\mathbf{B}^{\frac{\mathbf{1}+\beta}{\mathbf{2}}}}.$$

Note that

$$\boldsymbol{\Theta}_\varepsilon = \mathbf{x}^+_{\mathbf{t}_\varepsilon}.$$

  – $(\mathbf{t}_\varepsilon, \boldsymbol{\Theta}_\varepsilon)$ defines a scale of space-time points, close to which the strenght of the noise matches the drift.

Flandoli, Delarue: larger values [6]

Proposition 2:
For $\mathbf{B} = \mathbf{B}^+ = \mathbf{B}^-$ and $\beta = \beta^+ = \beta^-$, $\gamma \in (\mathbf{0}, \mathbf{1})$
and any $\tilde{\mathbf{\Theta}}_\varepsilon / \mathbf{\Theta}_\varepsilon \to \infty$ we have

$$\lim_{\varepsilon \to \mathbf{0}} \inf_{\mathbf{x} \geqslant \tilde{\mathbf{\Theta}}_\varepsilon} \mathbb{P}(\mathbf{X}_\mathbf{t}^{\mathbf{x},\varepsilon} \geqslant (\mathbf{1} - \gamma)\mathbf{x}_\mathbf{t}^+ \quad \forall\, \mathbf{t} \geqslant \mathbf{0}) = \mathbf{1}.$$

- Indirect argument using the **continuity of the paths of X^ε**.
- Proposition 1, the strong Markov property and Proposition 2 together prove the desired result, in this case

$$\mathbf{P}_\varepsilon^\mathbf{0} \to \frac{\mathbf{1}}{\mathbf{2}}\delta_{\mathbf{x}^+} + \frac{\mathbf{1}}{\mathbf{2}}\delta_{\mathbf{x}^-}, \qquad \text{for } \varepsilon \to \mathbf{0}$$

[6] The transition point in the zero noise limit for a 1 D Peano example, *Discrete Contin. Dyn. Syst.*, **34** (2014), n.10, 4071-4083.

Extension to α-stable perturbations

- Let $\mathbf{L}$ be a strictly α-stable process, for $\alpha \in (\mathbf{0}, \mathbf{2})$.
- Selfsimiliarity: Then there exists $\gamma_{\mathbf{0}} \in \mathbb{R}$ such that for any $\mathbf{a} > \mathbf{0}$.

$$(\mathbf{X_{at}})_{\mathbf{t} \geqslant \mathbf{0}} \stackrel{\mathbf{d}}{=} (\mathbf{a}^{\frac{\mathbf{1}}{\alpha}} \mathbf{X_t})_{\mathbf{t} \geqslant \mathbf{0}}.$$

- Consider

$$\mathbf{dX}^{\varepsilon} = \mathbf{b}(\mathbf{X}^{\varepsilon})\mathbf{dt} + \varepsilon \mathbf{dL}, \qquad \mathbf{X}^{\varepsilon}(\mathbf{0}) = \mathbf{0}.$$

with $\mathbf{b}(\mathbf{x}) = \mathbf{B}^{+}|\mathbf{x}|^{\beta^{+}} \mathbf{1}\{\mathbf{x} \geqslant \mathbf{0}\} - \mathbf{B}^{-}|\mathbf{x}|^{\beta^{-}} \mathbf{1}\{\mathbf{x} \geqslant \mathbf{0}\}$

for $\mathbf{B}^{+}, \mathbf{B}^{-} > \mathbf{0}, \beta^{+}, \beta^{-} \in (\mathbf{0}, \mathbf{1})$.

Extension to α-stable perturbations

Question:

$$\mathbf{P}_{\varepsilon}^{\mathbf{0}} \to \mathbf{p}^{+}\delta_{\mathbf{x}^{+}} + \mathbf{p}^{-}\delta_{\mathbf{x}^{-}}$$

Difficulties:

- Integral/ inegro-differential equation to solve
- $\mathbf{L}$, has arbitrarily large jumps, for any $\varepsilon > \mathbf{0}$, $|\mathbf{X}^{\varepsilon} - \mathbf{x}^{+}|_{\infty}$ will be very large.
- Asymmetries of $\mathbf{B}^{+}, \mathbf{B}^{-} > \mathbf{0}, \beta^{+}, \beta^{-} \in (\mathbf{0}, \mathbf{1}) \in \mathbb{R}$

A proof based on self-similarity [7]

Proposition 1: Under the previous assumptions and

$$\alpha > \mathbf{1} - (\beta^+ \wedge \beta^-)$$

there are functions $(\mathbf{t}_\varepsilon, \mathbf{\Theta}_\varepsilon^+, \mathbf{\Theta}_\varepsilon^-) \searrow \mathbf{0}+$ such that

$$\lim_{\varepsilon \to \mathbf{0}} \mathbb{P}\left(\tau_{-\mathbf{\Theta}_\varepsilon^-, \mathbf{\Theta}_\varepsilon^+} > \mathbf{t}_\varepsilon\right) < \mathbf{1}$$

and as a consequence for any $\tilde{\mathbf{t}}_\varepsilon / \mathbf{t}_\varepsilon \to \infty$ as $\varepsilon \to \mathbf{0}$ we have

$$\lim_{\varepsilon \to \mathbf{0}} \mathbf{P}_\varepsilon^{\mathbf{0}}(\tau_{-\mathbf{\Theta}_\varepsilon^-, \mathbf{\Theta}_\varepsilon^+} > \tilde{\mathbf{t}}_\varepsilon) = \mathbf{0}.$$

[7] Flandoli, Högele: A solution selection problem for small stable perturbations, http://arxiv.org/abs/1407.3469

A proof based on self-similarity [8]

Proof: Assume there are $(\mathbf{t}_\varepsilon, \mathbf{\Theta}_\varepsilon^+, \mathbf{\Theta}_\varepsilon^-)$ as above.

- $\tau_{-\mathbf{\Theta}_\varepsilon^-, \mathbf{\Theta}_\varepsilon^+} > \mathbf{t}_\varepsilon$ implies $-\mathbf{\Theta}_\varepsilon^- \leqslant \mathbf{X}_\mathbf{t}^\varepsilon \leqslant \mathbf{\Theta}_\varepsilon^+$ for all $\mathbf{t} \leqslant \mathbf{t}_\varepsilon$

$$\varepsilon \mathbf{L}_\mathbf{t} \leqslant \mathbf{X}_\mathbf{t}^\varepsilon + \int_\mathbf{0}^\mathbf{t} \mathrm{B}^-(\mathbf{X}_\mathbf{s}^\varepsilon)^{\beta^-} \mathbf{ds} \leqslant \mathbf{\Theta}_\varepsilon^+ + \mathbf{B}^- \mathbf{t}_\varepsilon (\mathbf{\Theta}_\varepsilon^-)^{\beta^-}$$

- Assume $\mathbf{B}^- \mathbf{t}_\varepsilon (\mathbf{\Theta}_\varepsilon^-)^{\beta^-} = \mathbf{\Theta}_\varepsilon^+$. Then

$$\varepsilon \mathbf{L}_\mathbf{t} \leqslant \mathbf{2\Theta}_\varepsilon \qquad \forall\, \mathbf{t} \leqslant \mathbf{t}_\varepsilon.$$

- Assume $\mathbf{B}^+ \mathbf{t}_\varepsilon (\mathbf{\Theta}_\varepsilon^+)^{\beta^+} = \mathbf{\Theta}_\varepsilon^+$. Then by symmetry

$$\varepsilon \mathbf{L}_\mathbf{t} \geqslant -\mathbf{2\Theta}_\varepsilon^- \qquad \forall\, \mathbf{t} \leqslant \mathbf{t}_\varepsilon.$$

- In particular

$$-\mathbf{2\Theta}_\varepsilon^- \leqslant \varepsilon \mathbf{L}_{\mathbf{t}_\varepsilon} \leqslant \mathbf{2\Theta}_\varepsilon^+.$$

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[8] A solution selection problem for small stable perturbations,

- In particular

$$-2\Theta_\varepsilon^- \leqslant \varepsilon L_{t_\varepsilon} \leqslant 2\Theta_\varepsilon^+.$$

- Hence

$$P_\varepsilon^0(\tau_{\Theta_\varepsilon} > t_\varepsilon) \leqslant \mathbb{P}(-2\Theta_\varepsilon^- \leqslant \varepsilon L_{t_\varepsilon} \leqslant 2\Theta_\varepsilon^+) = \mathbb{P}(-2\frac{\Theta_\varepsilon^-}{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}} \leqslant t_\varepsilon^{-\frac{1}{\alpha}} L_{t_\varepsilon} \leqslant 2\frac{\Theta_\varepsilon^+}{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}})$$

- Assume in addition $\Theta_\varepsilon^+ \wedge \Theta_\varepsilon^- = \varepsilon t_\varepsilon^{\frac{1}{\alpha}}$, w.l.o.g. $\Theta_\varepsilon^+ = \Theta_\varepsilon^+ \wedge \Theta_\varepsilon^-$. Then

$$\lim_{\varepsilon\to 0} P_\varepsilon^0(\tau_{\Theta_\varepsilon} > t_\varepsilon) \leqslant \mathbb{P}(L_1 \leqslant 2) < 1.$$

- Solving

$$\mathbf{B}^-\mathbf{t}_\varepsilon(\mathbf{\Theta}_\varepsilon^-)^{\beta^-} = \mathbf{\Theta}_\varepsilon^+$$
$$\mathbf{B}^+\mathbf{t}_\varepsilon(\mathbf{\Theta}_\varepsilon^+)^{\beta^+} = \mathbf{\Theta}_\varepsilon^+$$
$$\mathbf{\Theta}_\varepsilon^+ \wedge \mathbf{\Theta}_\varepsilon^- = \varepsilon\mathbf{t}_\varepsilon^{\frac{\mathbf{1}}{\alpha}}$$

yields

$$\mathbf{t}_\varepsilon = \frac{\varepsilon^{\frac{\alpha(\mathbf{1}-\beta^*\beta^\circ)}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}}{\left(\mathbf{B}^\circ\right)^{\frac{\alpha\beta^*}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}\left(\mathbf{B}^*\right)^{\frac{\alpha}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}},$$

$$\mathbf{\Theta}_\varepsilon^+ = \frac{\left(\mathbf{B}^-\right)^{\frac{\mathbf{1}}{\mathbf{1}-\beta^\circ\beta^*}}\left(\mathbf{B}^+\right)^{\frac{\beta^-}{\mathbf{1}-\beta^\circ\beta^*}}}{\left(\mathbf{B}^\circ\right)^{\frac{\alpha\beta^*}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}\left(\mathbf{B}^*\right)^{\frac{\alpha}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}}\,\varepsilon^{\frac{\alpha(\mathbf{1}+\beta^-)}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}$$

$$\mathbf{\Theta}_\varepsilon^- = \frac{\left(\mathbf{B}^+\right)^{\frac{\mathbf{1}}{\mathbf{1}-\beta^\circ\beta^*}}\left(\mathbf{B}^-\right)^{\frac{\beta^+}{\mathbf{1}-\beta^\circ\beta^*}}}{\left(\mathbf{B}^\circ\right)^{\frac{\alpha\beta^*}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}\left(\mathbf{B}^*\right)^{\frac{\alpha}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}}\,\varepsilon^{\frac{\alpha(\mathbf{1}+\beta^+)}{\alpha+\beta^*-\mathbf{1}+\beta^*(\alpha+\beta^\circ-\mathbf{1})}}$$

where $\beta^* = \beta^+ \vee \beta^-$ and $\beta^\circ = \beta^+ \wedge \beta^-$

- We see $\alpha > \mathbf{1} - \beta^\circ$ is necessary for $\mathbf{t}_\varepsilon \to \mathbf{0}$!

Exit probabilities from very small neighborhoods of 0 [9]

Proposition 2: Under the previous assumptions and

$$\alpha > \mathbf{1} - (\beta^+ \wedge \beta^-)$$

and functions $(\mathbf{t}_\varepsilon, \boldsymbol{\Theta}_\varepsilon^+, \boldsymbol{\Theta}_\varepsilon^-) \searrow \mathbf{0}+$ as in Proposition 1 and $\chi = \tau_{-\boldsymbol{\Theta}_\varepsilon^-, \boldsymbol{\Theta}_\varepsilon^+}$ we have

$$\lim_{\varepsilon\to\mathbf{0}} \mathbb{P}(\mathbf{X}_\chi^\varepsilon \geqslant \boldsymbol{\Theta}_\varepsilon^+) = \lim_{\varepsilon\to\mathbf{0}} \mathbb{P}(\varepsilon \mathbf{L}_{\chi^{\mathbf{L}}}^\varepsilon \geqslant \boldsymbol{\Theta}_\varepsilon^+)$$

$$= \lim_{\varepsilon\to\mathbf{0}} \frac{\boldsymbol{\Theta}_\varepsilon^-}{\boldsymbol{\Theta}_\varepsilon^+ + \boldsymbol{\Theta}_\varepsilon^-} = \begin{cases} \mathbf{1} & \text{if } \beta^+ < \beta^- \\ \left(\mathbf{1} + \left(\frac{\mathbf{B}^+}{\mathbf{B}^-}\right)^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}}\right)^{-\mathbf{1}} & \text{if } \beta = \beta^+ = \beta^- \\ \mathbf{0} & \text{if } \beta^+ < \beta^-. \end{cases}$$

- Standard arguments for hitting probabilities for $\varepsilon\mathbf{L}$ analogous to Brownian motion.

[9] A solution selection problem for small stable perturbations

Small perturbations [10]

Proposition 3:
Under the previous assumptions there is $\theta^* > \mathbf{0}$ and $\delta_\varepsilon \searrow 0$ such that for any $\mathbf{\Delta}_\varepsilon/\delta_\varepsilon \to \infty$ with $\mathbf{\Delta}_\varepsilon \searrow \mathbf{0}$ that

$$\lim_{\varepsilon\to\mathbf{0}} \sup_{\mathbf{3}\delta_\varepsilon<\mathbf{x}<\mathbf{\Delta}_\varepsilon} \mathbb{P}(\sup_{\mathbf{t}\in[\mathbf{0},\varepsilon^{-\theta^*}]} |\mathbf{X}_{\mathbf{t}}^{\varepsilon,\mathbf{x}} - \mathbf{x}_{\mathbf{t}}^{+}| > \mathbf{\Delta}_\varepsilon^{\beta^*}) = \mathbf{0}.$$

Sketch of proof:

- For $\rho \in (\mathbf{0}, \mathbf{1})$ let

$$\mathbf{T_1} := \inf\{\mathbf{t} > \mathbf{0} \mid |\mathbf{\Delta L_t}| > \varepsilon^{-\rho}\}$$
$$\mathbf{T_1} \sim \mathbf{EXP}(\lambda_\varepsilon), \quad \lambda_\varepsilon = \nu(\mathbb{R} \setminus [-\varepsilon^{-\rho}, \varepsilon^{-\rho}]) \sim \varepsilon^{\rho\alpha}$$

- Let η^ε be the CPP of all jumps $|\mathbf{\Delta L_t}| > \varepsilon^{-\rho}$, and $\xi^\varepsilon = \mathbf{L} - \eta^\varepsilon$

$$\mathbf{dY}^\varepsilon = \mathbf{b}(\mathbf{Y}^\varepsilon)\mathbf{dt} + \varepsilon\xi^\varepsilon, \qquad \mathbf{Y}^\varepsilon = \mathbf{x}.$$

[10] Flandoli, H.: A solution selection problem for small stable perturbations, http://arxiv.org/abs/1407.3469

- Elementary comparison argument: On $\{\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T_1}]}|\varepsilon\xi^\varepsilon_\mathbf{t}|\leqslant\delta_\varepsilon\}$

$$\mathbf{Y}^{\mathbf{x},\varepsilon}_\mathbf{t}\geqslant\mathbf{u}(\mathbf{t};\mathbf{x}-\delta_\varepsilon)-\delta_\varepsilon$$

- Hence

$$\begin{aligned}
&\sup_{\mathbf{x}\geqslant\mathbf{3}\delta_\varepsilon}\mathbb{P}(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T_1}]}(\mathbf{Y}^{\mathbf{x},\varepsilon}_\mathbf{t}-\mathbf{x}^+_\mathbf{t})<\mathbf{0})\\
&\leqslant\sup_{\mathbf{x}\geqslant\mathbf{3}\delta_\varepsilon}\mathbb{P}(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T_1}]}(\mathbf{Y}^{\mathbf{x},\varepsilon}_\mathbf{t}-(\mathbf{u}(\mathbf{t};\mathbf{x}-\delta_\varepsilon)-\delta_\varepsilon))<\mathbf{0})\\
&\leqslant\mathbb{P}(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T_1}]}|\varepsilon\xi^\varepsilon|>\delta_\varepsilon)\\
&\leqslant\mathbb{P}(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{R}^\varepsilon]}|\varepsilon\xi^\varepsilon_\mathbf{t}|>\delta_\varepsilon)+\mathbb{P}(\mathbf{T_1}>\mathbf{R}^\varepsilon)\\
&\leqslant\exp(-\frac{\delta_\varepsilon}{\varepsilon^{\mathbf{1}-\rho}\mathbf{R}^\varepsilon})+\exp(-\varepsilon^{\alpha\rho}\mathbf{R}^\varepsilon).
\end{aligned}$$

- Need

$$\varepsilon^{\alpha\rho}\mathbf{R}^\varepsilon\to\infty\quad\text{and}\quad\frac{\delta_\varepsilon}{\varepsilon^{\mathbf{1}-\rho}\mathbf{R}^\varepsilon}\to\infty.$$

- Need

$$\varepsilon^{\alpha\rho}\mathbf{R}^{\varepsilon} \to \infty \quad \text{and} \quad \frac{\delta_{\varepsilon}}{\varepsilon^{\mathbf{1}-\rho}\mathbf{R}^{\varepsilon}} \to \infty.$$

$$\mathbf{R}^{\varepsilon} := \frac{|\ln(\varepsilon)|}{\varepsilon^{\alpha\rho}}$$
$$\delta_{\varepsilon} := \varepsilon^{\mathbf{1}-\rho(\mathbf{1}+\alpha)}|\ln(\varepsilon)|^{\mathbf{2}}$$
$$\rho < \frac{\mathbf{1}}{\alpha+\mathbf{1}}.$$

- Using the subadditivity we get for $\mathbf{r}^{\varepsilon} = \delta_{\varepsilon}^{-\frac{\beta}{\mathbf{2}}}$ and any $\mathbf{\Delta}_{\varepsilon}/\delta_{\varepsilon} \to \infty$ and $\mathbf{\Delta}_{\varepsilon} \to \mathbf{0}$ that

$$\sup_{\mathbf{3}\delta_{\varepsilon}<\mathbf{x}<\mathbf{\Delta}_{\varepsilon}} \mathbb{P}\Big(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{r}^{\varepsilon}]} \mathbf{X}^{\varepsilon}_{\mathbf{t}} - \mathbf{x}^{+}_{\mathbf{t}} > \mathbf{\Delta}_{\varepsilon}^{\mathbf{1}-\beta}\Big) \leqslant \mathbb{P}(\mathbf{T}_{\mathbf{1}} > \mathbf{1}) + \mathbb{P}\Big(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{R}^{\varepsilon}]} |\varepsilon\xi^{\varepsilon}_{\mathbf{t}}| > \delta_{\varepsilon}\Big) \to \mathbf{0}.$$

Bridging the initial values $\Theta^+ < \mathbf{x} < 4\delta_\varepsilon^+$

Proposition 4:
For $\varepsilon > 0$ and $x \in [\Theta_\varepsilon^+, 4\delta_\varepsilon]$ denote

$$\upsilon^{\mathbf{x},\varepsilon} := \inf\{\mathbf{t} > \mathbf{0} \mid \mathbf{X}_\mathbf{t}^{\varepsilon,\mathbf{x}} \geqslant \mathbf{4}\delta_\varepsilon\}.$$

There is an increasing, continuous function $\mathbf{s}_\cdot : (\mathbf{0},\mathbf{1}) \to (\mathbf{0},\mathbf{1})$ with $\mathbf{s}_\varepsilon \to \mathbf{0}$ as $\varepsilon \to \mathbf{0}$, such that

$$\lim_{\varepsilon\to\mathbf{0}} \sup_{\mathbf{\Theta}_\varepsilon^+ \leqslant \mathbf{x} \leqslant \mathbf{4}\delta_\varepsilon} \mathbb{P}(\upsilon^{\mathbf{x},\varepsilon} > \mathbf{s}_\varepsilon) = \mathbf{0}.$$

Bridging the initial values $\Theta^{+} < \mathbf{x} < 4\delta_{\varepsilon}^{+}$

- For an appropriate choice of a parameter $\pi \in \mathbb{R}$ we denote the time

$$\tilde{\mathbf{T}}_{\pi} = \tilde{\mathbf{T}}_{\pi}(\varepsilon) := \inf\{\mathbf{t} > \mathbf{0} \mid |\boldsymbol{\Delta}_{\mathbf{t}}\mathbf{L}| > \varepsilon^{-\pi}\}.$$

- Then on the events $\{\tilde{\mathbf{T}}_{\pi} > \mathbf{s}_{\varepsilon}\}$ and $\{\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{s}_{\varepsilon}]} |\varepsilon\mathbf{L}_{\mathbf{t}}| \leqslant \frac{\mathbf{B}}{\mathbf{2}}\boldsymbol{\Theta}_{\varepsilon}^{\beta}\mathbf{s}_{\varepsilon}\}$ we have for $\mathbf{t} \in [\mathbf{0}, \mathbf{s}_{\varepsilon}]$

$$\begin{aligned} \mathbf{X}_{\mathbf{t}}^{\varepsilon,\mathbf{x}} &= \mathbf{x} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{b}(\mathbf{X}_{\mathbf{s}}^{\varepsilon,\mathbf{x}})\mathbf{ds} + \varepsilon\mathbf{L}_{\mathbf{t}} \\ &\geqslant \boldsymbol{\Theta}_{\varepsilon} + \mathbf{B}\int_{\mathbf{0}}^{\mathbf{t}} \Big[\boldsymbol{\Theta}_{\varepsilon}^{\beta} + (\mathbf{X}_{\mathbf{s}}^{\varepsilon,\mathbf{x}} - \boldsymbol{\Theta}_{\varepsilon})\frac{(\mathbf{4}\delta_{\varepsilon})^{\beta} - \boldsymbol{\Theta}_{\varepsilon}^{\beta}}{\mathbf{4}\delta_{\varepsilon} - \boldsymbol{\Theta}_{\varepsilon}}\Big]\mathbf{ds} + \varepsilon\mathbf{L}_{\mathbf{t}}. \end{aligned}$$

Hence setting $\mathbf{s}_\varepsilon = \frac{\mathbf{4}}{\mathbf{B}}\delta_\varepsilon^{\frac{\mathbf{1}-\beta}{\mathbf{2}}}\ln(\frac{\mathbf{4}}{\mathbf{B}}\boldsymbol{\Theta}_\varepsilon^{-\beta}\delta_\varepsilon^{-\mathbf{2}(\mathbf{1}-\beta)})$ yields

$$\begin{aligned}\mathbf{X}_{\mathbf{s}_\varepsilon}^{\varepsilon,\mathbf{x}} \geqslant\ & \boldsymbol{\Theta}_\varepsilon + \frac{\mathbf{B}}{\mathbf{2}}\boldsymbol{\Theta}_\varepsilon^\beta \mathbf{s}_\varepsilon + \frac{\mathbf{8}}{\mathbf{B}}\boldsymbol{\Theta}_\varepsilon^\beta \delta_\varepsilon^{\mathbf{2}(\mathbf{1}-\beta)} \\ & \exp\Big(\frac{\mathbf{B}}{\mathbf{4}}\frac{\mathbf{s}_\varepsilon}{\delta_\varepsilon^{\mathbf{1}-\beta}}\Big)\underbrace{\Big(\mathbf{1} - (\mathbf{1} + (\frac{\mathbf{B}}{\mathbf{4}}\frac{\mathbf{s}_\varepsilon}{\delta_\varepsilon^{\mathbf{1}-\beta}}))\exp\big(-\frac{\mathbf{B}}{\mathbf{4}}\frac{\mathbf{1}}{\delta_\varepsilon^{\mathbf{1}-\beta}}\mathbf{s}_\varepsilon\big)\Big)}_{\nearrow \mathbf{1}} \\ & \gtrsim_\varepsilon \boldsymbol{\Theta}_\varepsilon + \frac{\mathbf{B}}{\mathbf{2}}\boldsymbol{\Theta}_\varepsilon^\beta \mathbf{s}_\varepsilon + \frac{\mathbf{4}}{\mathbf{B}}\boldsymbol{\Theta}_\varepsilon^\beta \delta_\varepsilon^{\mathbf{2}(\mathbf{1}-\beta)}\exp\Big(\frac{\mathbf{B}}{\mathbf{4}}\frac{\mathbf{s}_\varepsilon}{\delta_\varepsilon^{\mathbf{1}-\beta}}\Big) \\ & \geqslant \frac{\mathbf{1}}{\mathbf{2}}\exp\Big((\frac{\mathbf{1}}{\delta_\varepsilon})^{\frac{\mathbf{1}-\beta}{\mathbf{2}}}\Big) \gtrsim_\varepsilon \mathbf{4}\delta_\varepsilon \text{ as } \varepsilon \to \mathbf{0}.\end{aligned}$$

Flandoli and H.: A solution selection problem for small stable perturbations [11],

Theorem:

Let $\mathbf{L}$ be strictly α-stable. Denote by $\mathbf{P}^{\mathbf{0}}_{\varepsilon}(\theta)$ the law of $\mathbf{X}^{\varepsilon}|_{\mathbf{t}\in[\mathbf{0},\varepsilon^{-\theta}]}$

$$\mathbf{dX}^{\varepsilon} = \mathbf{b}(\mathbf{X}^{\varepsilon})\mathbf{dt} + \varepsilon\mathbf{dL}, \qquad \mathbf{X}(\mathbf{0}) = \mathbf{0} \in \mathbb{R}, \varepsilon > \mathbf{0}$$

If $\alpha > 1 - \beta^{+} \wedge \beta^{-}$ then there exists $\theta^{*} > \mathbf{0}$ such that

$$\mathbf{P}^{\mathbf{0}}_{\varepsilon}(\theta^{*}) \xrightarrow{\mathbf{w}} \mathbf{p}^{+}\delta_{\mathbf{x}^{+}} + \mathbf{p}^{-}\delta_{\mathbf{x}^{-}}, \qquad \text{as } \varepsilon \to \mathbf{0},$$

for the **extremal solutions**

$$\mathbf{x}^{\pm}(\mathbf{t}) = \pm\mathbf{C}^{\pm}\mathbf{t}^{\frac{\mathbf{1}}{\mathbf{1}-\beta^{\pm}}}, \qquad \mathbf{t} \geqslant \mathbf{0}$$

$$\mathbf{p}^{+} = \begin{cases} \mathbf{1} & \text{if} \quad \beta^{+} < \beta^{-} \\ \dfrac{(\mathbf{B}^{-})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}}}{(\mathbf{B}^{+})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}} + (\mathbf{B}^{-})^{-\frac{\mathbf{1}}{\mathbf{1}+\beta}}} & \text{if} \quad \beta^{+} = \beta^{-} =: \beta \\ \mathbf{0} & \text{if} \quad \beta^{+} > \beta^{-}. \end{cases}$$

[11] Flandoli, H.: A solution selection problem for small stable perturbations, http://arxiv.org/abs/1407.3469

Thank you very much!