Overview

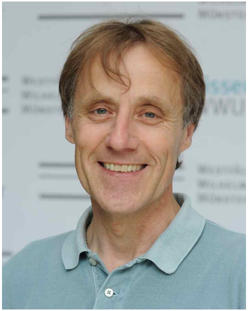
Ordinary random walk

Perturbed random walk

Moments of first passage times and related quantities for ordinary and perturbed random walks

Alexander Iksanov, Kiev, Ukraine

Workshop on Persistence probabilities and related fields, Darmstadt, Germany, July 15-18, 2014



Gerold Alsmeyer (Münster)

[1] Alsmeyer, G., Iksanov, A. and Meiners, M. (2014). Power and exponential moments of the number of visits and related quantities for perturbed random walks. *JTP*, to appear.

[2] Iksanov, A. and Meiners, M. (2010). Exponential moments of first passage times and related quantities for random walks. *ECP*. **15**, 365–375.

Matthias Meiners (Darmstadt)

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- ordinary random walk

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  - definition of the main objects
  - simple cases and previously known results
  - criteria for the finiteness of exponential moments

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  - illustrating example

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Ordinary random walk:
definition of the main objects

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Perturbed random walk

Let $(\xi_k)_{k\in\mathbb{N}}$ be independent copies of a real-valued random variable ξ.

A random sequence $(S_n)_{n\in\mathbb{N}_0}$ defined by

$$S_0 := 0, \quad S_n := \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}$$

is called *zero-delayed (ordinary) random walk*.

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definition of the main objects

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Main objects

- the *first passage time* into (x, ∞)

$$\tau(x) \ := \ \inf\{n \in \mathbb{N}_0 : S_n > x\}, \quad x \geq 0;$$

- the *number of visits* to the interval $(-\infty, x]$

$$N(x) \ : \ = \ \#\{n \in \mathbb{N} : S_n \leq x\} \ = \ \sum_{n\geq 1} \mathbb{1}_{\{S_n \leq x\}}, \quad x \geq 0;$$

- the *last exit time* from $(-\infty, x]$

$$\rho(x) \ = \ \sup\{n \in \mathbb{N}_0 : S_n \leq x\}, \quad x \geq 0.$$

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When $\mathbb{P}\{\xi \geq 0\} = 1$ and $\mathbb{P}\{\xi = 0\} < 1$,

$$\tau(x) - 1 \;=\; N(x) \;=\; \rho(x), \quad x \geq 0.$$

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Proposition (Beljaev & Maksimov (1963))
Assume that $\mathbb{P}\{\xi \geq 0\} = 1$ and let $\beta := \mathbb{P}\{\xi = 0\} \in [0, 1)$. Then for $a > 0$ the following conditions are equivalent:

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some (hence every) } x \geq 0;$$

$$a < -\log \beta$$

where $-\log \beta := \infty$ if $\beta = 0$. The same holds for $N(x)$ and $\rho(x)$.

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where $-\log \beta := \infty$ if $\beta = 0$. The same holds for $N(x)$ and $\rho(x)$.
Proof: If $a < -\log \beta$, pick $r > 0$ s.t. $e^a \mathbb{E}e^{-r\xi} < 1$

$$\begin{aligned}\sum_{k\geq 0} e^{ak}\mathbb{P}\{\tau(x) > k\} \;&=\; \sum_{k\geq 0} e^{ak}\mathbb{P}\{S_k \leq x\} \\ &\leq\; e^{rx}\sum_{k\geq 0}(e^a\mathbb{E}e^{-r\xi})^k < \infty\end{aligned}$$

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If $a \ge -\log \beta$,

$$\sum_{k\ge 0} e^{ak}\mathbb{P}\{S_k \le x\} \;\ge\; \sum_{k\ge 0} e^{ak}\mathbb{P}\{S_k = 0\} \;=\; \sum_{k\ge 0} e^{ak}\beta^k = \infty.$$

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General case: $\mathbb{P}\{\xi < 0\} \geq 0$.
FINITENESS:

- Borovkov (1962): if

$$a < R := -\log \inf_{t \geq 0} \mathbb{E}e^{-t\xi},$$

then $\mathbb{E}e^{a\tau(0)} < \infty$;
if $a > R$, then $\mathbb{E}e^{a\tau(0)} = \infty$.

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- Heyde (1964): assuming that $\mathbb{E}\xi \in (0, \infty)$,

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some } a > 0 \qquad \text{iff}$$

$$\mathbb{E}e^{b\xi^-} < \infty \text{ for some } b > 0.$$

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- Doney (1989); Bertoin & Doney (1996) – other relevant results.

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ASYMPTOTICS:

- Heyde (1966): whenever $\mathbb{E}\xi \in (0, \infty)$ and $\mathbb{E}e^{a\tau(0)} < \infty$,
$$\mathbb{E}e^{a\tau(x)} \sim \text{const}\, e^{(a/\mathbb{E}\xi)x}, \quad x \to \infty.$$

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Criteria for the finiteness of exponential moments

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Theorem (I. & Meiners (2010))
Let $a>0$ and $\mathbb{P}\{\xi<0\}>0$. Then the following conditions are equivalent:

$$\mathbb{E}e^{a\tau(x)}<\infty \text{ for some (hence every) } x\geq 0;$$

$$\mathbb{E}e^{aN(x)}<\infty \quad \text{for some (every)} \quad x\geq 0;$$

$$\sum_{n\geq 1}\frac{e^{an}}{n}\mathbb{P}\{S_n\leq x\} \;<\; \infty \quad \text{for some (every)} \quad x\geq 0;$$

$$a\leq R:=-\log\inf_{t\geq 0}\,\mathbb{E}e^{-t\xi}.$$

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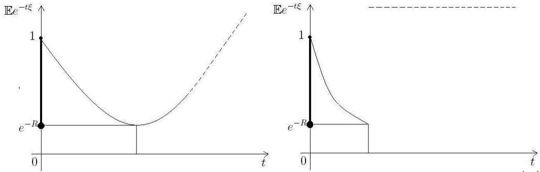

The bold segments indicate the sets of values of e^{-a} for which both $\mathbb{E}e^{a\tau(x)}$ and $\mathbb{E}e^{aN(x)}$ are finite.

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$$\text{either } a < R=-\log\inf_{t\geq 0}\mathbb{E}e^{-t\xi}, \quad \text{or}$$

$$a = R \text{ and } \mathbb{E}\xi e^{-\gamma_0\xi}>0,$$

where γ_0 is the unique positive number such that $\mathbb{E}e^{-\gamma_0\xi}=e^{-R}$.

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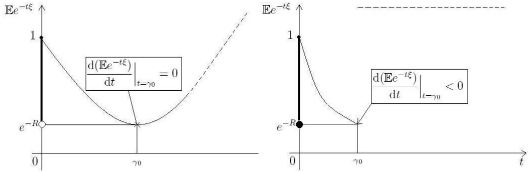

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Perturbed random walk

Example (simple random walk)
Let $p \in (1/2, 1)$ and

$$\mathbb{P}\{\xi = 1\} = p = 1 - \mathbb{P}\{\xi = -1\} =: 1 - q.$$

Then

$$\mathbb{E}e^{-t\xi} = pe^{-t} + qe^{t} \quad \text{and}$$

$$R = -\log \inf_{t \geq 0} \mathbb{E}e^{-t\xi} = -\log(2\sqrt{pq}).$$

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One can check that

$$\mathbb{P}\{\tau(0) = 2n-1\} = \frac{1}{2q} \frac{\binom{2n}{n}}{2^{2n}(2n-1)} (2\sqrt{pq})^{2n} = \frac{1}{2q} \frac{\binom{2n}{n}}{2^{2n}(2n-1)} e^{-2nR}, \quad n \in \mathbb{N};$$

$$\mathbb{P}\{\rho(0) = 2n\} = (p-q)\binom{2n}{n}(pq)^n = (p-q)\frac{\binom{2n}{n}}{2^{2n}} e^{-2nR}, \quad n \in \mathbb{N}_0.$$

Stirling's formula yields

$$\frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}, \quad n \to \infty,$$

which implies that

$$\mathbb{E}e^{R\tau(0)} < \infty \quad \text{and} \quad \mathbb{E}e^{R\rho(0)} = \infty.$$

Ordinary random walk:
Asymptotics of the exponential moments

NONNEGATIVE CASE

Theorem (I. & Meiners (2010))
Let $a > 0$,

$$\mathbb{P}\{\xi \geq 0\} = 1 \text{ and } \mathbb{P}\{\xi = 0\} < 1.$$

Assume that $\mathbb{E}e^{a\tau(x)} < \infty$ for some (hence every) $x \geq 0$. Then, as $x \to \infty$,

$$\mathbb{E}e^{a\tau(x)} \sim \text{const}\, e^{\gamma x}$$

where

$$\text{const} = \begin{cases} \frac{1-e^{-a}}{\gamma \mathbb{E}\xi e^{-\gamma\xi}}, & \text{if } \mathcal{L}(\xi) \text{ is non-lattice,} \\ \frac{\lambda(1-e^{-a})}{(1-e^{-\lambda\gamma})\mathbb{E}\xi e^{-\gamma\xi}}, & \text{if } \mathcal{L}(\xi) \text{ is } \lambda\text{-lattice} \end{cases}$$

γ is the unique positive number such that

$$\mathbb{E}e^{-\gamma\xi} = e^{-a},$$

and in the λ-lattice case the limit is taken over $x \in \lambda\mathbb{N}$.

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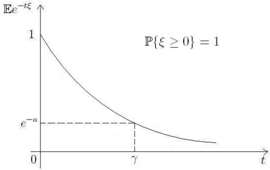

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$M := \inf_{n\geq 1} S_n$; γ is a minimal positive number such that

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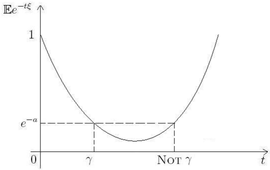

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Finiteness

(Kesten & Maller (1996)) For $p > 0$

$$\mathbb{E}(\tau(x))^{p+1} < \infty \quad \Leftrightarrow \quad \mathbb{E}(N(x))^p < \infty \quad \Leftrightarrow$$

$$\mathbb{E}(\rho(x))^p < \infty.$$

(!) It may happen that there exists $q > 0$

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(I. & Meiners (2010)) Provided that

$$R = -\log \inf_{t\geq 0} \mathbb{E}e^{-t\xi} > 0,$$

it holds that

$$\mathbb{E}e^{a\tau(x)} < \infty, \ \mathbb{E}e^{aN(x)} < \infty \text{ iff } a \leq R, \ \text{ and}$$
$$\mathbb{E}e^{a\rho(x)} < \infty \text{ if } a < R,$$

whereas $\mathbb{E}e^{R\rho(x)}$ is finite in some cases and infinite in others.

Ordinary random walk:
Exponential moments vs power moments

ASYMPTOTICS

Let $T(x)$ denote any of the rvs $\tau(x)$, $N(x)$ or $\rho(x)$.

(Kesten & Maller (1996)) If $\mathbb{E}(T(0))^p < \infty$ then

$$\mathbb{E}(T(x))^p \asymp \left(\frac{x}{\mathbb{E}\min(\xi^+, x)}\right)^p, \quad x \to \infty.$$

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- Simple cases and previously known results
- Criteria for the finiteness of exponential moments
- Illustrating example
- Asymptotics of the exponential moments
- Exponential moments vs power moments

Perturbed random walk

Asymptotics

Let $T(x)$ denote any of the rvs $\tau(x)$, $N(x)$ or $\rho(x)$.

(Kesten & Maller (1996)) If $\mathbb{E}(T(0))^p < \infty$ then

$$\mathbb{E}(T(x))^p \asymp \left(\frac{x}{\mathbb{E}\min(\xi^+, x)}\right)^p, \quad x \to \infty.$$

(I. & Meiners (2010)) If $\mathbb{E}e^{aT(0)} < \infty$ then

$$\mathbb{E}e^{aT(x)} \asymp e^{\gamma x}, \quad x \to \infty.$$

Perturbed random walk:
Definition and large-time asymptotics

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- Criteria for the finiteness of moments of the last exit time
- A curious observation
- Criteria for the finiteness of moments of the number of visits
- Random processes with immigration

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be independent copies of a random vector (ξ, η) with arbitrary dependence between ξ and η.

By a *perturbed random walk* is meant a random sequence $(T_n)_{n\in\mathbb{N}}$ defined by

$$T_n := S_{n-1} + \eta_n = \xi_1 + \ldots + \xi_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

Perturbed random walk:
Definition and large-time asymptotics

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The so defined PRW is related to

- perpetuities
- shot noise processes
- the Bernoulli sieve
- GI/G/∞ queues

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$$\sum_{n\geq 1} e^{T_n}$$

- shot noise processes
- the Bernoulli sieve
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$$T_n = S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

Proposition (Goldie & Maller (2000))
Either
$\lim_{n\to\infty} T_n = +\infty$ a.s. (*drift to* $+\infty$), or
$\lim_{n\to\infty} T_n = -\infty$ a.s. (*drift to* $-\infty$), or
$-\infty = \liminf_{n\to\infty} T_n < \limsup_{n\to\infty} T_n = +\infty$ a.s. (*oscillation*).

(!) It may happen that the perturbed random walk T_n oscillates when the corresponding ordinary random walk S_n drifts to $\pm\infty$.

Perturbed random walk:
Criteria for the finiteness of moments of the last exit time

$$T_n = S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

For $x \in \mathbb{R}$, the *last exit time* from $(-\infty, x]$ is

$$\rho^*(x) = \begin{cases} \sup\{n \in \mathbb{N} : T_n \le x\}, & \text{if } \inf_{k\ge 1} T_k \le x, \\ 0, & \text{if } \inf_{k\ge 1} T_k > x \end{cases}$$

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Criteria for the finiteness of moments of the last exit time

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Power moments.

Theorem (Alsmeyer, I. & Meiners (2014))
Let (T_n) be positively divergent and $p > 0$. The following assertions are equivalent:

$$\mathbb{E}(\rho^*(x))^p < \infty \text{ for some (hence every) } x \in \mathbb{R};$$

$$\mathbb{E}(\rho(y))^p < \infty \quad \text{for some (hence every)} \quad y \ge 0 \quad \text{and}$$
$$\mathbb{E}J^{p+1}(\eta^-) < \infty,$$

where $J(z) := \frac{z}{\mathbb{E}(\xi^+ \wedge z)}$, $z > 0$, and

$$\rho(y) \;=\; \sup\{n \in \mathbb{N}_0 : S_n \le y\}, \quad y \ge 0.$$

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EXPONENTIAL MOMENTS.

Theorem (Alsmeyer, I. & Meiners (2014))
Let (T_n) be positively divergent, $\mathbb{P}\{\xi < 0\} > 0$ and $a > 0$. The following assertions are equivalent:

$$\mathbb{E}\exp(a\rho^*(x)) < \infty \text{ for some (hence every) } x \in \mathbb{R};$$

$$\mathbb{E}\exp(a\rho(y)) < \infty \text{ for some (hence every) } y \ge 0 \quad \text{and}$$

$$\mathbb{E}e^{-\gamma\eta} < \infty,$$

where γ is the minimal positive number satisfying $\mathbb{E}e^{-\gamma\xi} = e^{-a}$, and

$$\rho(y) = \sup\{n \in \mathbb{N}_0 : S_n \le y\}, \quad y \ge 0.$$

Perturbed random walk:
A curious observation

Alsmeyer, I. & Meiners (2014): for $a > 0$

$$\begin{aligned} \mathbb{E}e^{a\rho^*(x)} &< \infty \quad \text{for some (hence every)} x \in \mathbb{R} \quad \Leftrightarrow \\ \sum_{n\geq 1} e^{an}\mathbb{P}\{T_n \leq x\} &< \infty \quad \text{for some/all } x \in \mathbb{R}. \end{aligned}$$

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Kesten & Maller (1996): for $p > 0$

$$\begin{aligned} \mathbb{E}\rho(x)^p &< \infty \quad \text{for some (hence every) } x \geq 0 \quad \Leftrightarrow \\ \sum_{n\geq 1} n^{p-1}\mathbb{P}\{S_n \leq x\} &< \infty \quad \text{for some/all } x \geq 0. \end{aligned}$$

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However

$$\mathbb{E}\rho^*(x)^p < \infty \quad \not\Leftrightarrow \quad \sum_{n\geq 1} n^{p-1}\mathbb{P}\{T_n \leq x\} < \infty!!!$$

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Indeed,

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\{T_n \leq x\} < \infty \quad \Leftrightarrow \quad \mathbb{E}\rho(x)^p < \infty \ \text{ and } \ \mathbb{E}J(\eta^-)^p < \infty,$$

whereas

$$\mathbb{E}\rho^*(x)^p < \infty \quad \Leftrightarrow \quad \mathbb{E}\rho(x)^p < \infty \ \text{ and } \ \mathbb{E}J(\eta^-)^{p+1} < \infty.$$

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$$T_n = S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

For $x \in \mathbb{R}$, the *number of visits* to $(-\infty, x]$ is

$$N^*(x) \;=\; \#\{n \in \mathbb{N} : T_n \leq x\}$$

Perturbed random walk:

Criteria for the finiteness of moments of the number of visits

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Here

$$N(y) \ = \ \#\{n \in \mathbb{N}_0 : S_n \le y\}, \quad y \ge 0.$$

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Exponential moments.

Theorem (Alsmeyer, I. & Meiners (2014))
Let (T_n) be positively divergent.

(a) If $\mathbb{P}\{\xi < 0\} > 0$, then the following assertions are equivalent:

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Perturbed random walk:
Criteria for the finiteness of moments of the number of visits

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Exponential moments.

Theorem (Alsmeyer, I. & Meiners (2014))
Let (T_n) be positively divergent.

(b) If $\xi \geq 0$ a.s., then the assertions

$$\mathbb{E}\exp(aN^*(x)) < \infty,$$
$$e^a\,\mathbb{P}\{\xi = 0, \eta \leq x\} + \mathbb{P}\{\xi = 0, \eta > x\} < 1$$

are equivalent for each $a > 0$ and $x \in \mathbb{R}$.

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DEFINITION:

- $X = (X(t))_{t\in\mathbb{R}}$ is a random process with paths in the Skorokhod space $D(\mathbb{R})$;

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- $(S_n)_{n \in \mathbb{N}_0}$ is the zero-delayed random walk with jumps ξ_k, i.e.,

$$S_0 := 0, \quad S_n := \xi_1 + \xi_2 + \ldots + \xi_n, \quad n \in \mathbb{N}.$$

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Random process with immigration

The process $Y := (Y(t))_{t\in\mathbb{R}}$ defined by

$$Y(t) := \sum_{k\geq 0} X_{k+1}(t - S_k), \quad t \in \mathbb{R}$$

will be called random process with immigration at the epochs of a renewal process.

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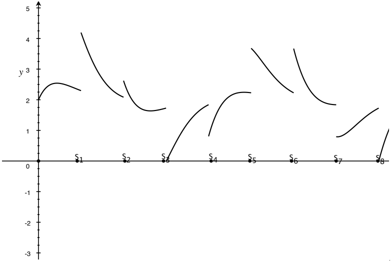
INTERPRETATION in the case when $\xi > 0$ a.s. and $X(t) = 0$, $t < 0$.



At time $S_0 = 0$ the immigrant 1 arrives at the system and runs a random process X_1.

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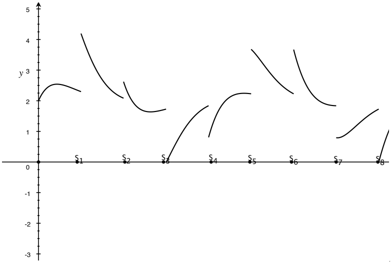


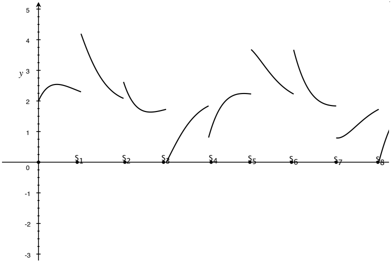
At time $S_0 = 0$ the immigrant 1 arrives at the system and runs a random process X_1.

. . . .

At time S_k the immigrant $k+1$ arrives at the system and runs a random process X_{k+1} which evolves along with already existing processes $X_1, \ldots, X_k$.

Perturbed random walk:
Random processes with immigration

INTERPRETATION: in the case when $\xi > 0$ a.s. and $X(t) = 0$, $t < 0$.



$Y(t)$ is then the sum of all processes run by the immigrants up to and including time t, i.e.,

$$Y(t) = \sum_{k \geq 0} X_{k+1}(t - S_k).$$

Perturbed random walk:
Random processes with immigration

$$\begin{aligned} N^*(t) &= \#\{k \in \mathbb{N} : T_k \le t\} \\ &= \sum_{k \ge 0} \mathbb{1}_{\{\eta_{k+1} \le t - S_k\}} \\ &= \sum_{k \ge 0} X_{k+1}(t - S_k) = Y(t), \quad t \in \mathbb{R}. \end{aligned}$$

Therefore $(N^*(t))_{t \in \mathbb{R}}$ is a random process with immigration with $X_k(t) = \mathbb{1}_{\{\eta_k \le t\}}$, $k \in \mathbb{N}$.

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THANK YOU FOR YOUR ATTENTION