A conditional limit theorem for products of random matrices

I) Notations

G is the linear group of $d \times d$ invertible matrices.

The Euclidean norm in $V = \mathbb{R}^d$ is denoted by $\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}$

Denote by $\|g\| = \sup_{v \in V - \{0\}} \frac{\|gv\|}{\|v\|}$

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given an i.i.d sequence $(g_n)_{n \geq 1}$ of random matrices with values in G and of law μ.

We consider the product $G_n = g_n g_{n-1} \cdots g_1$ and the Markov chain $G_n v$ $n \geq 1$ in $V - \{0\}$ starting at point $v \in V - \{0\}$

Let B the closed unit ball in V

For any $v \in B^c$

define the exit time of the Markov chain $G_n v$ from B^c

$$\tau_v = \inf\{ n \geq 1 \quad G_n v \in B \}$$

The goal of this talk is to establish the asymptotic of the probability $\mathbb{P}_r(\tau_v > n)$ and to find the limit law of $\frac{1}{\sqrt{n}} \ln \| G_n v \|$ conditioned by $G_k v \in B^c$ for $1 \leq k \leq n$

II) Hypotheses and results

In the sequel we suppose that the following conditions P_1, P_2, P_3, P_4, P_5 are satisfied.

Let $N(g) = \max(\|g\| ; \|g^{-1}\|)$, supp μ the support of μ and $P(V)$ the projective space of V.

$P_1)$ there exists $\delta_0 > 0$ such that

$$\int (N(g))^{\delta_0} \mu(dg) < +\infty$$

$P_2)$ (<u>strong irreducibility</u>) The support supp μ of μ acts strongly irreducibly on V i.e no proper union of finite vector subspaces of V is invariant with respect to all elements of the group generated by the support of μ.

We say that that the sequence $(h_n)_{n \geq 1}$ in G is contracting for the projective space $P(V)$ if $\lim_n \frac{a_1(n)}{a_2(n)} = \infty$ where $a_1(n) \geq a_2(n) \dots \geq a_d(n)$ are the eigenvalues of the symmetric matrice ${}^t h_n h_n$.

$P_3)$ (<u>Proximality</u>) The closed semi-group generated by supp μ contains a contracting sequence for the projective space $P(V)$.

P_4) $\gamma = \lim_n \frac{1}{n} \ln \| G_n \| = 0$ a.s

P_5) There exists $\delta > 0$ such that

$$\inf_{s \in S_{d-1}} \mu \{ g \; \ln \| g s \| > \delta \} > 0$$

Remark

Under P_1, P_2, P_3, P_4 there exists a constant $\sigma > 0$ such that for any $v \in V - \{0\}$ and any $t \in \mathbb{R}$

$$\lim_n P_n \left(\frac{\ln \| G_n v \|}{\sigma \sqrt{n}} \leq t \right) = \phi(t)$$

where Φ is the standard normal distribution.

Under P_1, P_2, P_3, P_4, P_5 we have the following theorem.

Theorem.

1°) For any $v \in B^c$ the limit

$$V(v) = \lim_n E_{P_n} (\ln \| G_n v \| ; \tau_v > n)$$

exists and satisfies $V(v) > 0$. In addition for any $s \in S_{d-1}$ the function $t \to V(ts)$ is increasing on $]1, \infty[$, and there exists constants $a>0, c>0$ such that for $s \in S_{d-1}$ $t>1$

$$\sup(0, \ln t - a) \leq V(ts) \leq c(1 + \ln t)$$

and $$\lim_{t \to +\infty} \frac{V(ts)}{\ln t} = 1$$

2°) For any $v \in B^c$

$$P_v(\tau_v > n) \sim \frac{2V(v)}{\sigma\sqrt{2\pi n}} \quad \text{as } n \to +\infty$$

3°) For any $v \in B^c$ and $t \geq 0$

$$\lim_n P_v\left(\frac{\ln \|G_n v\|}{\sigma\sqrt{n}} \leq t \,\middle|\, \tau_v > n\right) = \Phi_+(t)$$

Where $\Phi_+(t)$ is the Raleigh distribution $\Phi_+(t) = 1 - \exp - \frac{t^2}{2}$

III Proof

a)

For the proof we introduce the Markov chain $(X_n)_{n \geq 0}$ with value in $X = G \times P(V)$ defined by

$X_0 = (g, \bar{v})$ $X_1 = (g_1, g \cdot \bar{v})$ $X_{n+1} = (g_{n+1}, G_n \cdot \bar{v})$ $n \geq 1$

The probability transition of $(X_n)_{n \geq 0}$ is given by

$$Pf(g, \bar{v}) = \int f(g_1, g \cdot \bar{v}) \, \mu(dg_1)$$

Under $P_1 - P_3$ this Markov chain has a unique invariant probability measure given by $\lambda(dg, d\bar{v}) = \mu(dg) \, \nu(d\bar{v})$ where ν is the unique probability measure on $P(V)$ ~~satisfying~~ such that $\mu * \nu = \nu$

We consider also the cocycle $\rho : G \times P(V) \to \mathbb{R}$ defined by

$$\rho(g, \bar{v}) = \ln \left(\frac{\| g v \|}{\| v \|} \right)$$

For $v \in V - \{0\}$ when $X_0 = (g, \bar{v})$ we have the formula

$$\ln \| G_n g v \| = \ln \| v \| + S_n \qquad n \geq 1$$

where $S_n = \sum_{k=1}^{n} \rho(X_k)$

So the initial problem can be solved by considering the Markov chain $(Z_n)_{n \geq 0}$ with values in $X \times \mathbb{R}$ defined by $Z_0 = (g, \bar{v}, y)$ $\quad Z_n = (X_n, y + S_n)$ $n \geq 1$ and studying the law of the stopping time $\tau_y = \inf \{ n \geq 1 \quad y + S_n \leq 0 \}$

Note that we have here

$$\int \rho(x) \, \lambda(dx) = 0$$

b) <u>Spectral properties of the Markov chain $(X_n)_{n \geq 0}$</u>

On the space $P(V)$ define the distance

$$d(\bar{u}, \bar{v}) = | \sin \theta(\bar{u}, \bar{v}) |$$

Denote by $\mathcal{C}_b$ the vector space of bounded, continuous functions from $X \to \mathbb{R}$ and put $\|f\|_\infty = \sup_{x \in X} |f(x)| \quad f \in \mathcal{C}_b$

For $\varepsilon < \delta_0$ and $f \in \mathcal{C}_b$ let

$$k_\varepsilon(f) = \sup_{\substack{\bar{u} \neq \bar{v} \\ g \in G}} \frac{|f(g,\bar{u}) - f(g,\bar{v})|}{d^\varepsilon(\bar{u},\bar{v}) \, (N(g))^{4\varepsilon}}$$

$$+ \sup_{g \neq h \;\; \bar{u} \in P(V)} \frac{|f(g,\bar{u}) - f(h,\bar{u})|}{\|g-h\|^\varepsilon \, (N(g)\,N(h))^{3\varepsilon}}$$

Define the vector space $\mathcal{B}$

$$\mathcal{B} = \{ f \in \mathcal{C}_b \quad k_\varepsilon(f) < +\infty \}$$

endowed with the norm

$$\|f\|_{\mathcal{B}} = \|f\|_\infty + k_\varepsilon(f).$$

We have the following result

<u>Proposition</u> (spectral gap) For $\varepsilon > 0$ small enough

a) The map $f \to Pf$ is a bounded operator on $\mathcal{B}$

b) $$P = \Pi + R$$

where Π is a one dimensional projector

$\Pi(\mathcal{B}) = \{ f ;\ f \text{ is constant} \}$

R is an operator on $\mathcal{B}$ satisfying

$\Pi R = R \Pi = 0$ and of spectral radius less than 1.

Consider also the perturbed operators

$$P(t) f = P\left(e^{it\varphi} f \right)$$

we have the following proposition.

<u>Proposition</u> There exists $\eta_0 > 0$ such that

a) for $|t| \leq \eta_0$ $P(t)$ is a bounded operator on $\mathcal{B}$

b) $$\sup_{|t| \leq \eta_0,\ n \geq 1} \| P^n(t) \|_{\mathcal{B} \to \mathcal{B}} \leq C(P) < +\infty$$

c) <u>Martingale approximation</u>

Following Gordin and using spectral properties of P we can construct approximation of $(S_n)_{n \geq 0}$

More precisely we have the following proposition.

Proposition There exist a constant a on a martingale $(M_n)_{n\geq 0}$ such that for any $x \in X$

$$\mathbb{P}_x \left(\sup_{n\geq 0} |M_n - S_n| \leq a \right) = 1$$

d) Coupling argument

Proposition

For any $n \geq 1$ and $x \in X$ there exist a sequence of independant standard normal random variables $(W_k)_{k\geq 1}$ and a version of $(S_k)_{k\geq 1}$ defined on an extension of the probability space $(X^{\mathbb{N}}, \mathcal{B}_X^{\otimes \mathbb{N}}, \mathbb{P}_x)$ such that for any $\varepsilon \in]0, \frac{1}{5}[$

$$\sup_{x \in X} \mathbb{P}_x \left(n^{-1/2} \sup_{1 \leq k \leq n} |S_k - \sigma B_k| > c n^{-\varepsilon} \right) \leq c(\varepsilon)\, n^{-1}$$

where $B_k = \sum_{i=1}^{k} W_i$, $c(\varepsilon)$ is a constant depending on ε

e) Following the ideas of Denisov and Wachtel in the independent case and these approximations we can prove the following theorem.

<u>Theorem</u>: Assume hypotheses $P_1 - P_5$

1) For any $x \in X$ and $y > 0$ the limit

$$V(x, y) = \lim_n E_x(y + S_n ; \tau_y > n)$$

exists and satisfies $V(x, y) > 0$

Moreover for any $x \in X$ the function $V(x, \cdot)$ is increasing in $\mathbb{R}^+_*$, satisfies $\sup(0, y - a) \leq V(x, y) \leq c(1+y)$

For any $y > 0$, $x \in X$ and some $a > 0$

and $\lim_{y \to +\infty} \frac{V(x,y)}{y} = 1$

2) For any $x \in X$ and $y > 0$

$$\mathbb{P}_x(\tau_y > n) \sim \frac{2V(x,y)}{\sigma\sqrt{2\pi n}} \quad \text{as } n \to +\infty$$

3) let $t \geq 0$, For any $x \in X$ and $y > 0$

$$\lim_{n \to \infty} \mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \,\middle|\, \tau_y > n\right) = \phi^+(t) = 1 - \exp -\frac{t^2}{2}$$