Least Energy Functions Accompanying Wiener Process

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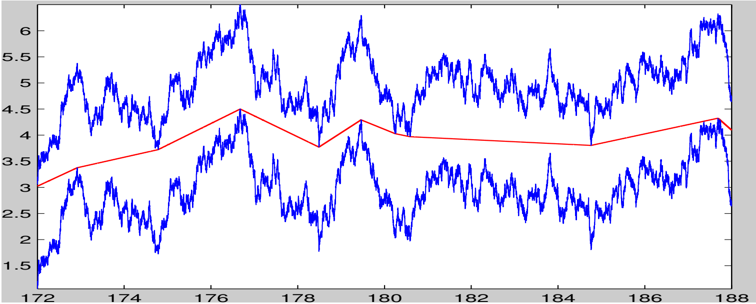
This is a joint work with E. Setterqvist from Linköping university, Sweden, and Z. Kabluchko, Ulm university.

Taut string

$$\begin{cases} \int_0^T f'(t)^2 dt \searrow \min \\ f(0) = w(0), \quad f(T) = w(T), \\ w(t) - r \le f(t) \le w(t) + r, \qquad 0 \le t \le T. \end{cases} \qquad \text{or(!!)} \int_0^T \varphi(f'(t))dt \searrow \min$$



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Let W be a Wiener process. We are mostly interested in its approximation characteristics

$$I_W(T,r) := \inf\{|h|_T; h \in AC[0,T], ||h - W||_T \le r, h(0) = 0\}$$

and

$$I_W^0(T,r) := \inf\{|h|_T; h \in AC[0,T], ||h - W||_T \le r, h(0) = 0, h(T) = W(T)\}.$$

Main results

Theorem

There exists $\mathcal{C} \in (0, \infty)$ such that for any $q > 0$ if $\frac{r}{\sqrt{T}} \to 0$, then

$$\frac{r}{T^{1/2}} I_W(T,r) \xrightarrow{L_q} \mathcal{C} \qquad \textit{and} \qquad \frac{r}{T^{1/2}} I_W^0(T,r) \xrightarrow{L_q} \mathcal{C}.$$

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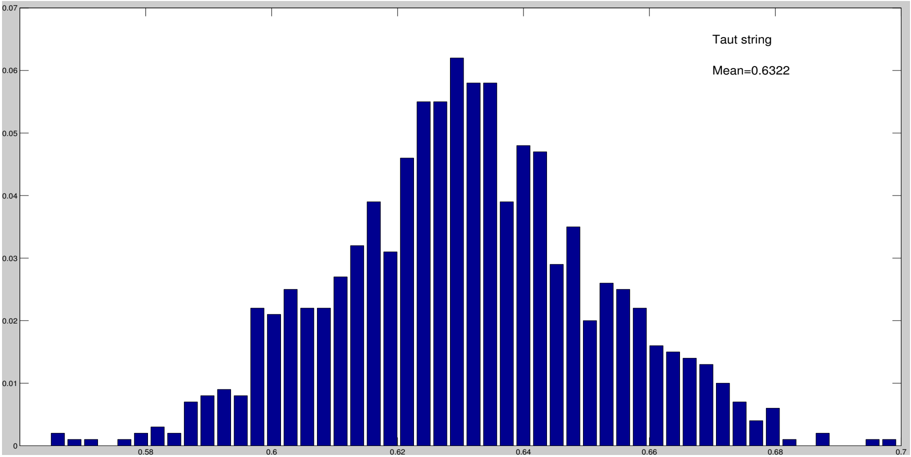
Empirical modelling of $\mathcal{C}$



$$\mathcal{C} \approx 0.63$$

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$$\mathbb{P}\left(|I_W - med(I_W)| > \rho\right) \le \mathbb{P}\left(|N| > \rho\right)$$

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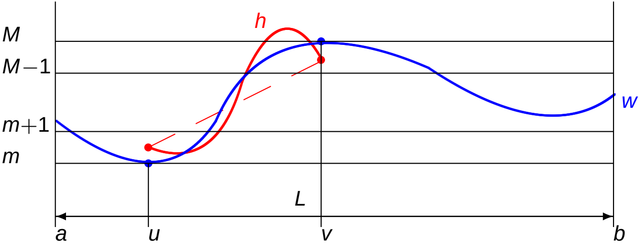
Subadditivity in time:

$$I^0_W(T_1 + T_2, r)^2 \le I^0_W(T_1, r)^2 + I^0_{\widetilde{W}}(T_2, r)^2.$$

with independent W and $\widetilde{W}$.

Lower bound (Talagrand idea)

Assume that $|w(t) - h(t)| \leq 1$ on an interval $[a, b]$ of length L.



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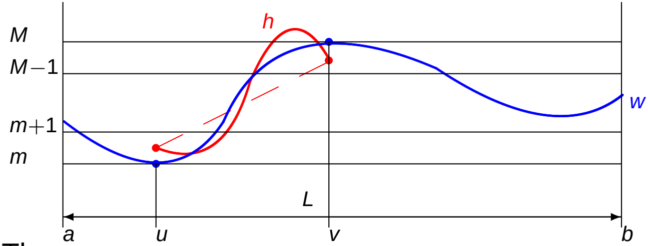

Then

$$\int_a^b h'(t)^2 dt \ge \int_u^v h'(t)^2 dt \ge \frac{|h(u) - h(v)|^2}{|u - v|} \ge \frac{(M - m - 2)_+^2}{L}.$$

For Wiener process, $\frac{(M-m-2)_+^2}{L}$ scales to $(R - 2L^{-1/2})_+^2$ where R is the range of W on the unit interval.

Lower bound (continued)

By taking $T = nL$ and splitting $[0, T]$ into n intervals of length L we obtain

$$\frac{|h|_T^2}{T} \geq \frac{1}{nL} \sum_{j=1}^{n} (R^{(j)} - 2L^{-1/2})_+^2 \to L^{-1}\, \mathbb{E}(R - 2L^{-1/2})_+^2.$$

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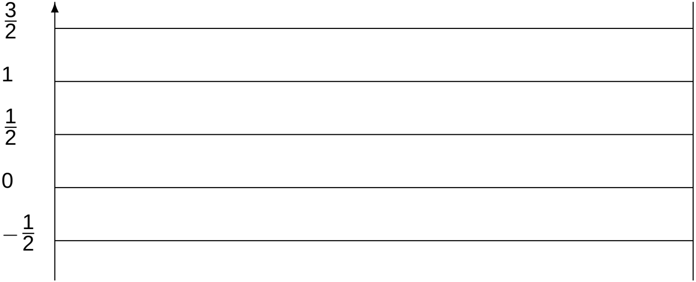
$$\mathcal{C} \geq 0.38$$

which is in agreement with empirical data.

Upper bound: free-knot approximation

Let $\tau_{n+1} := \inf\left\{t \geq \tau_n \middle| \, |W(t) - W(\tau_n)| \geq \frac{1}{2}\right\}$ Let $h(t)$ interpolate between the points $(\tau_n, W(\tau_n))$.



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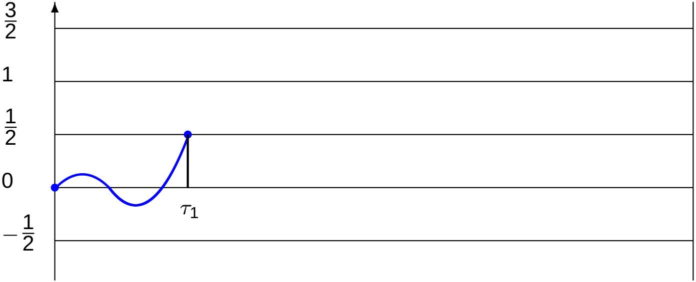

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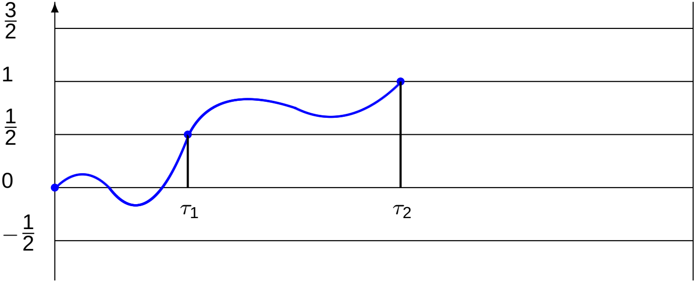

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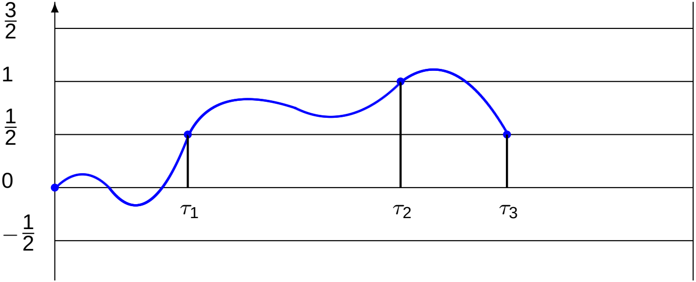

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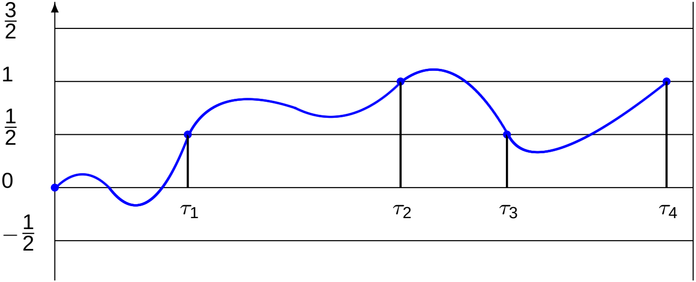

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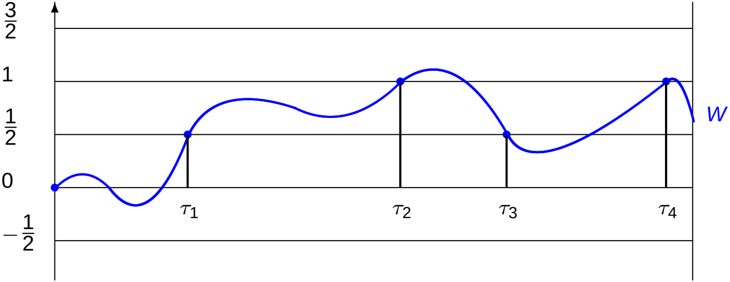

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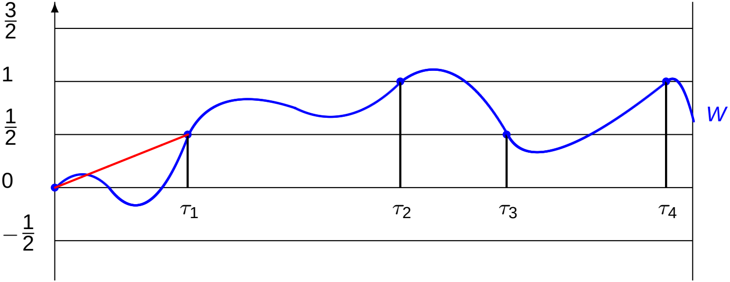

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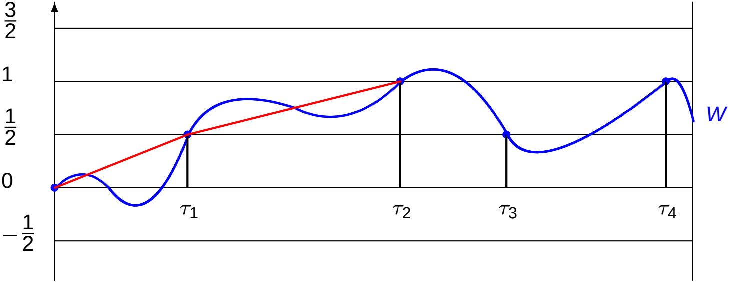

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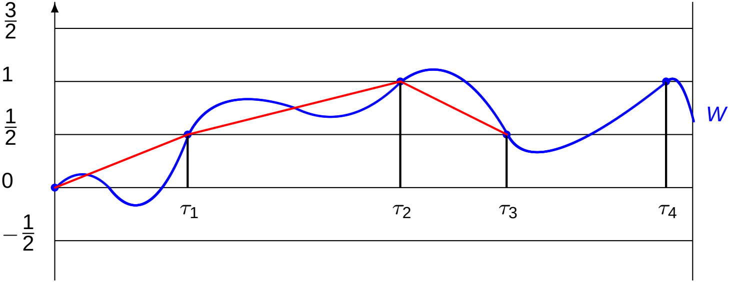

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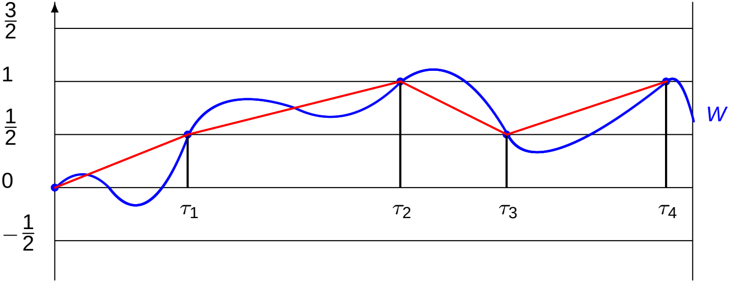

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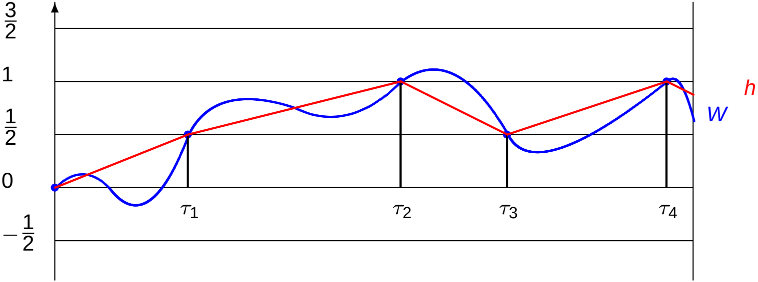

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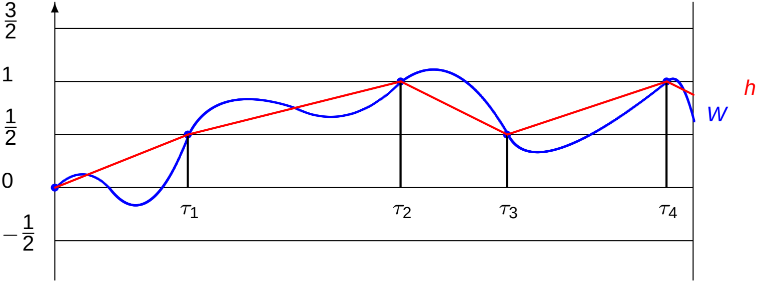

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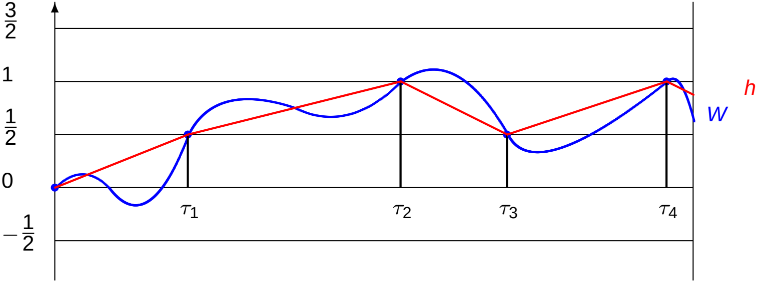

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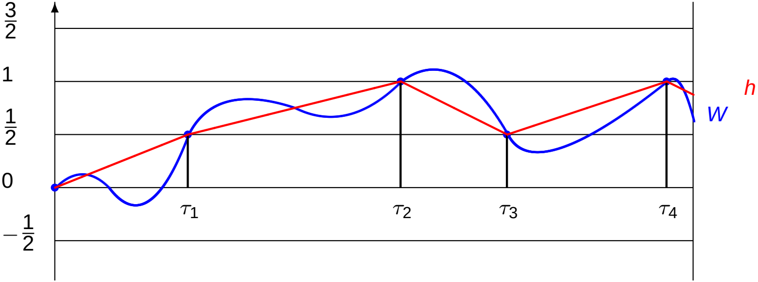

Then $\forall t$ we have $|h(t) - W(t)| \leq 1$ and

$$\int_{\tau_n}^{\tau_{n+1}} h'(t)^2 dt = \frac{(h(\tau_{n+1}) - h(\tau_n))^2}{\tau_{n+1} - \tau_n} = \frac{1}{4(\tau_{n+1} - \tau_n)}$$

are i.i.d. random variables.

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Thus $\mathcal{C} \le 2\sqrt{1.832} \approx 2.7$.

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$$\ln \mathbb{P}(W \in \varepsilon U) \sim -\frac{\pi^2}{8\varepsilon^2},$$

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where $U := \{x : ||x||_1 \leq 1\}$ and $K := \{h : |h|_1 \leq 1\}$.
By Gaussian isoperimetric inequality,

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Since $\Phi^{-1}(p) \sim -\sqrt{2|\ln p|}$, and by small ball asymptotics

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whenever $c > \frac{\pi}{2}$. It follows that $\mathcal{C} \leq \frac{\pi}{2}$. This is the best known upper bound but it is totally non-constructive.

Markov pursuit: problem setting

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How to keep the Brownian dog on a leash in the energy saving mode?

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You must follow it by moving with a finite speed and always stay not more than 1 away from the dog.

If $x(t)$ is your trajectory, then the goal is to follow the dog by expending minimal energy per unit of time

$$\frac{1}{T}\int_0^T x'(t)^2\,dt$$

in a long run, $T \to \infty$.

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The difference between the construction of pursuit and the taut string is huge: the former is built "online" based on the past and present trajectory of W while the latter requires the knowledge of entire trajectory of W.

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By ergodic theorem, in the stationary regime

$$\frac{1}{T}\int_0^T x'(t)^2\, dt \to \frac{1}{4}\int_{-1}^1 b(x)^2\, p(x)\, dx = \frac{1}{4}\int_{-1}^1 \frac{p'(x)^2}{p(x)^2}\, p(x)\, dx := \frac{1}{4} I(p).$$

We have to minimize Fisher information I(p) !

Solution: optimal strategy for Markov pursuit

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We get

$$0.63 \approx \mathcal{C} \le \frac{I(p)^{1/2}}{2} = \frac{\pi}{2} \approx 1.51.$$

This is a price to pay for not knowing the future.

Bounded time interval

Now we consider the optimal pursuit strategy on the finite time interval $[0, t]$.

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Introduce the minimal average pursuit energy

$$F(y,t) := \mathbb{E} \int_0^t x'(s)^2 ds = \mathbb{E} \int_0^t b(x(s) - W(s), t - s)^2 ds$$

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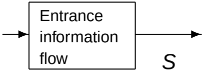
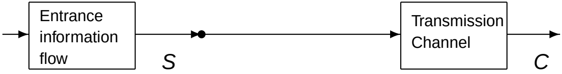
$$b(y,t) = \frac{\mathcal{P}'_y(y,t)}{\mathcal{P}(y,t)}\ .$$

Related discrete applied problem



Related discrete applied problem



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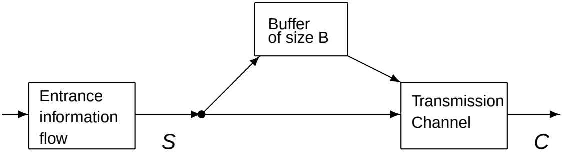

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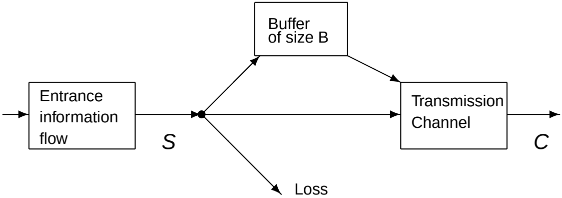

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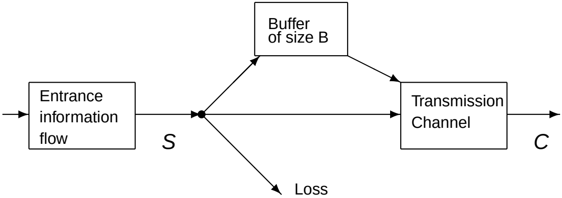

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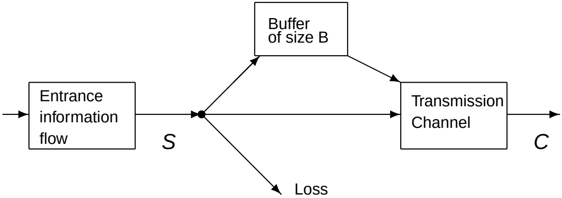

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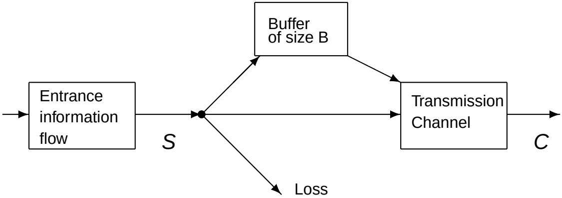

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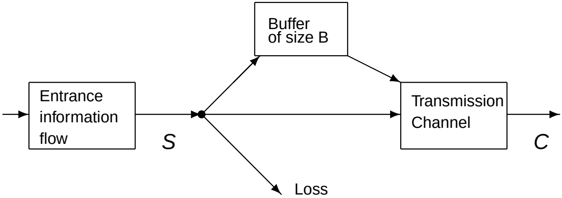

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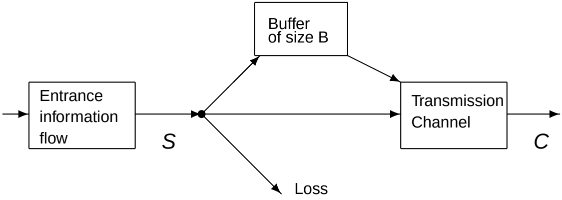

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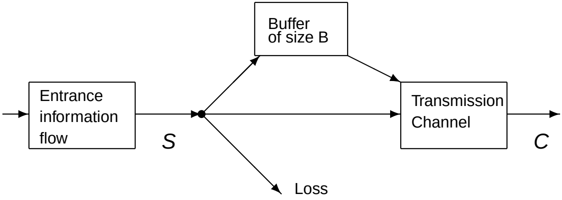

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$$F := \sum_{j=1}^{n} \varphi\left(\frac{L_j}{S_j}\right) S_j \searrow \min\ .$$

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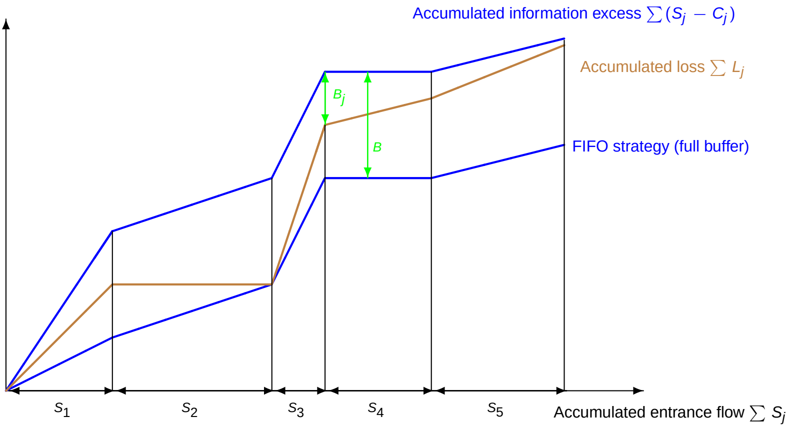
Now buffer bounds $0 \le B_k \le B$ mean that

$$\sum_{j=1}^{k} \left(S_j - C_j\right) - B \le \sum_{j=1}^{k} L_j \le \sum_{j=1}^{k} \left(S_j - C_j\right).$$

That is $\sum_{j=1}^{k} L_j$ must go within a (random) band of fixed width B.

Buffer balance: graph



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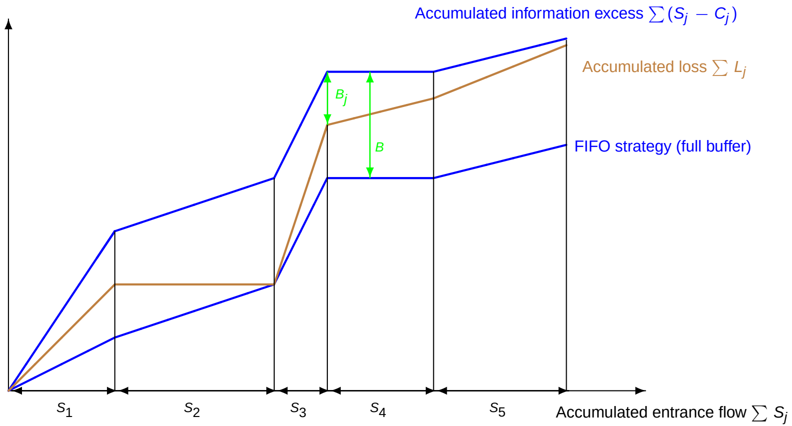

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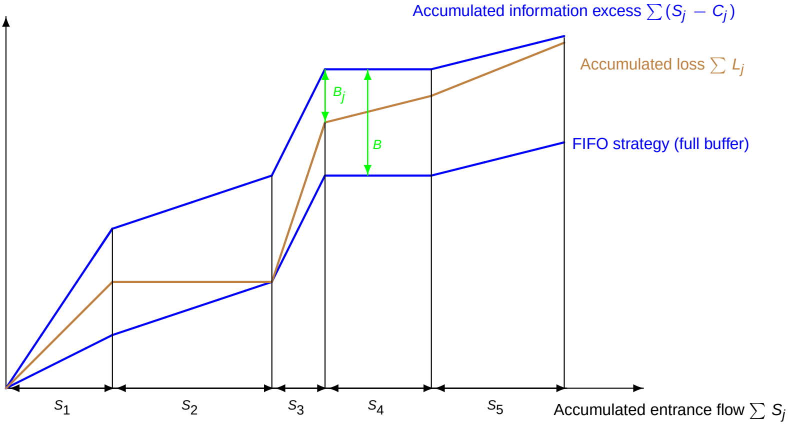

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In terms of the tout string energy $I_W(\cdot,\cdot)$ we have

$$\limsup_{T\to\infty} \frac{I_W(T, c_1(2T)^{1/2}(\ln\ln T)^{-1/6})}{(2\ln\ln T)^{1/2}} > 1,$$

etc.

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$$c_3 < \liminf_{T\to\infty} (\ln\ln T) \inf_{|h|_1\le 1} \left\| \frac{W(\cdot T)}{\sqrt{2T\ln\ln T}} - h \right\|_1 < c_4 \qquad \text{a.s.}$$

In terms of the tout string energy $I_W(\cdot,\cdot)$ we have

$$\limsup_{T\to\infty} \frac{I_W(T, c_1(2T)^{1/2}(\ln\ln T)^{-1/6})}{(2\ln\ln T)^{1/2}} > 1,$$

etc. Here the tube is much wider and the string energy is much lower than in our case.