First-passage properties in diffusion with stochastic resetting

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  $\Rightarrow$ new Nonequilibrium Steady State

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- Summary and Conclusion

Search Problems

Search problems are ubiquitous in nature

- search for Holy Grail
- search for Higgs boson

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- data search (Google)
- animals searching for food
- protein searching for a binding site on a DNA
- Visual search: locating a face in the crowd

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A robust class of models: Intermittent target search strategies combine

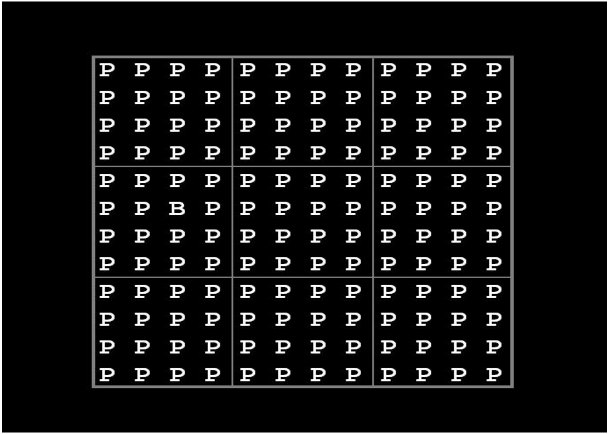
(i) phases of slow motion (target detection)

(ii) phases of fast motion (searcher relocates but not reactive)

[O. Bénichou et. al. Rev. Mod. Phys. 83, 81 (2011)]

Visual search: a face in a crowd

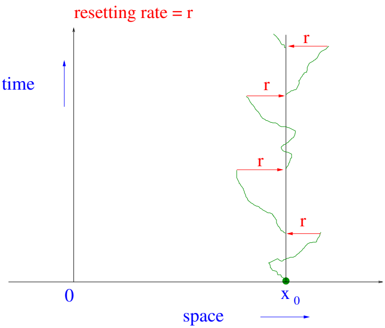
Visual search in psychology



Search via diffusion and resetting

I : The Model and its Steady State

Diffusion with stochastic resetting: The model



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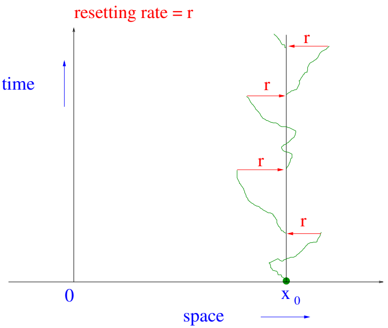

Dynamics: In a small time interval Δt

$$x(t+\Delta t) = x_0 \quad \text{with prob. } r\Delta t \quad \text{(resetting)}$$

$$= x(t) + \eta(t)\,\Delta t \quad \text{with prob. } 1 - r\Delta t \quad \text{(diffusion)}$$

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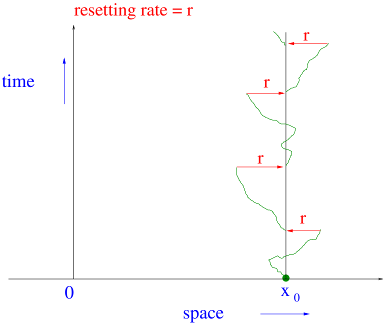

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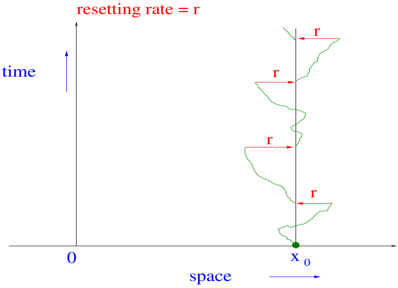
$\eta(t) \to$ Gaussian white noise: $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = 2\,D\,\delta(t-t')$

[M.R. Evans & S.M., PRL, 106, 160601 (2011)]

Prob. density $p(x,t)$ with resetting rate $r > 0$



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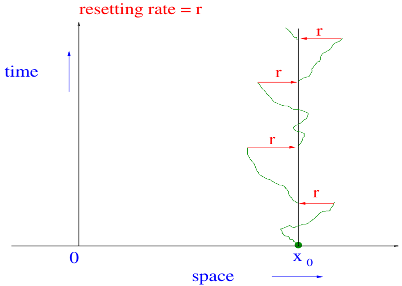

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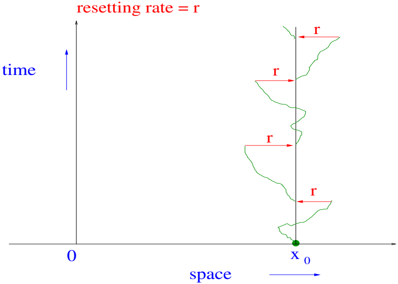

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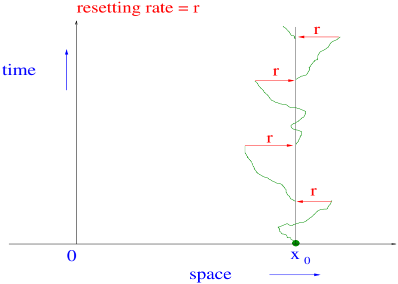

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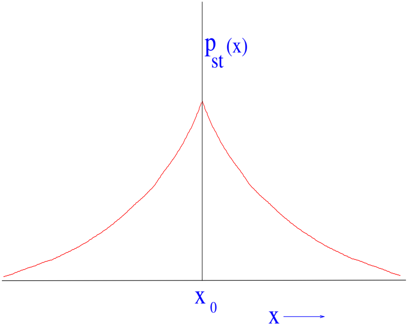


$\to$ nonequilibrium steady state

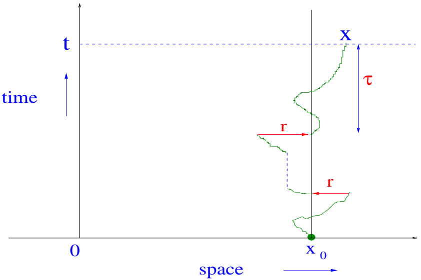
$\Rightarrow$ current carrying with detailed balance $\to$ violated

$p_{\rm st}(x) = \alpha_0\,\exp[-V_{\rm eff}(x)]$

effective potential:

$V_{\rm eff}(x) = \alpha_0|x-x_0|$

An intuitive derivation:



$\tau \rightarrow$ time since the last resetting during which free diffusion $\Rightarrow$

If $0 < \tau < t$
$p(x,t) = \frac{1}{\sqrt{4\pi\, D\, \tau}}\, \exp[-(x-x_0)^2/4D\tau]$

If $\tau > t$ (no resetting in $[0,t]$)
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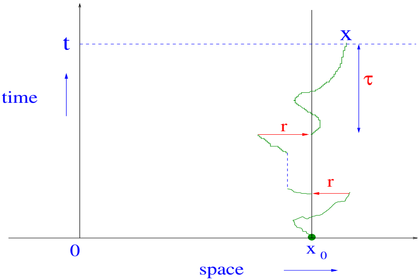

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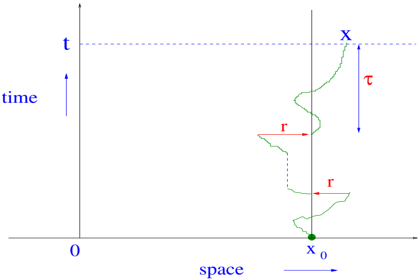

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$p(x,t) = \int_0^t d\tau\, (r\, e^{-r\,\tau})\, G(x,\tau) + e^{-r\,t}\, G(x,t)$
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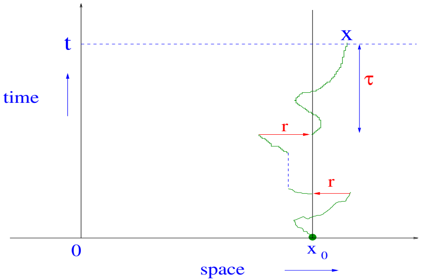

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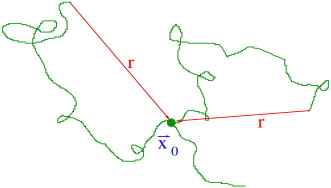
- as $t \rightarrow \infty$, $p_{\mathrm{st}}(x) = \frac{\alpha_0}{2} \exp[-\alpha_0 |x - x_0|]$ with $\alpha_0 = \sqrt{r/D}$

Generalization to higher dimensions



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and resets to $\vec{x}_0$ with rate r

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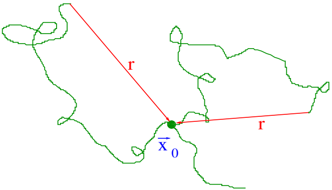

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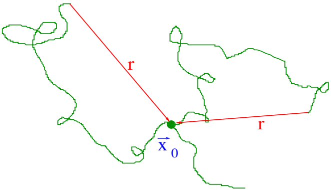

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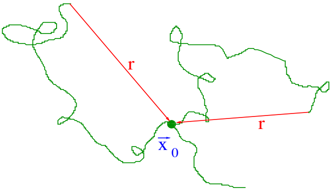

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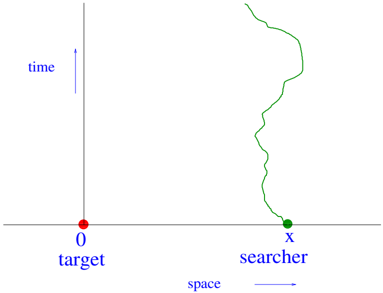
- For $d = 1$, it reduces to $\to p_{\rm st}(x) = \frac{\alpha_0}{2}\, \exp[-\alpha_0\, |x - x_0|]$

II : Persistence and First-passage properties

Search of a fixed target by a purely diffusive searcher in $d = 1$ without resetting



$Q(x, t) \to$ persistence/survival prob. of the target

backward Fokker-Planck equation

$\partial_t Q(x, t) = D\partial_x^2 Q(x, t)$ for $x \geq 0$

boundary cond. :
$Q(x = 0, t) = 0$ and $Q(x \to \infty, t) = 1$

intial cond. :
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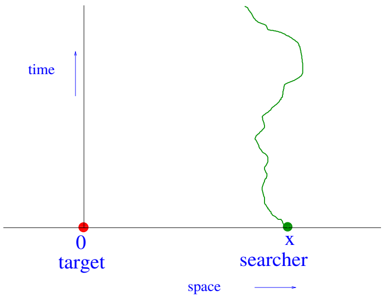

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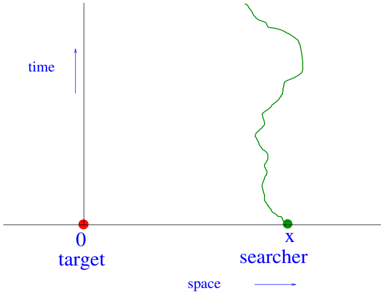

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- first-passage prob. :

$$F(x,t) = -\partial_t Q(x,t) = \frac{x}{\sqrt{4\pi D t^3}} \exp[-x^2/4Dt] \xrightarrow[t\to\infty]{} t^{-3/2}$$

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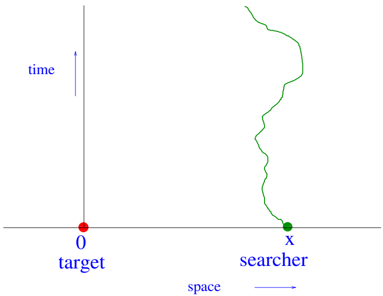

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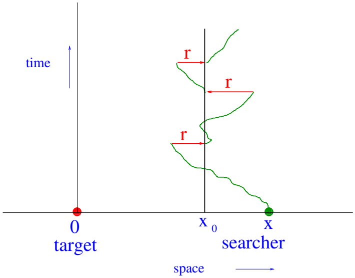
- mean capture time $\to$ $\bar{T} = \int_0^\infty t\, F(x,t)\, dt = \infty$

Target search via diffusion with resetting



- starting position $x \to$ 'variable', resetting to x_0 with rate r

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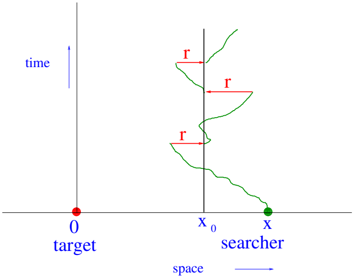

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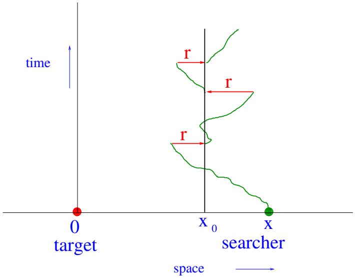

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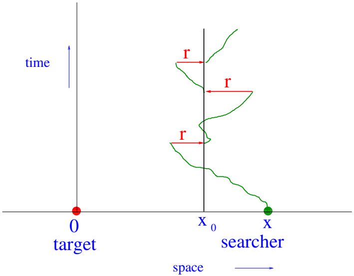

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Solution via Laplace transform

- Laplace transform: $\tilde{Q}(x,s) = \int_0^\infty Q(x,t)\, e^{-s\, t}\, dt$

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- Exact solution:

$$\tilde{Q}(x,s) = \frac{1 + r\,\tilde{Q}(x_0,s)}{r+s} \left[1 - \exp\left(-\sqrt{(r+s)/D}\, x\right)\right]$$

Solution via Laplace transform

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$\Rightarrow$ As $t \to \infty$, $Q(x_0,t) \approx \exp\left[-r\,t\,e^{-\sqrt{r/D}\,x_0}\right]$

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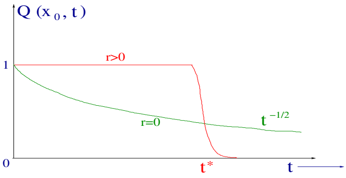
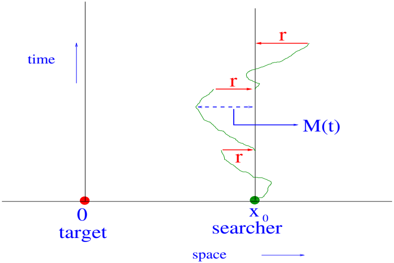


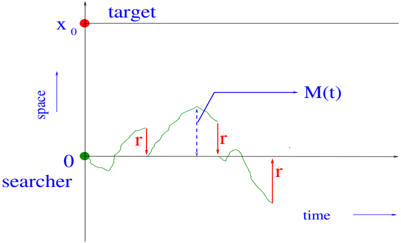
For $r > 0$, $Q(x_0,t) \approx \exp[-t/t^*]$

where $t^* \approx (1/r)\, e^{\sqrt{r/D}\,x_0}$

Survival Probability $\Longleftrightarrow$ Extreme Value Statistics

ROTATE & SHIFT

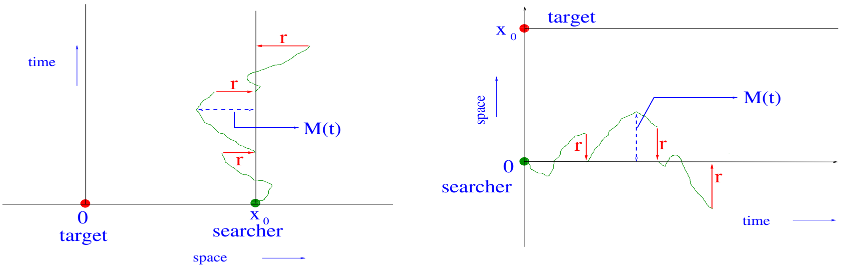




- $M(t) \to$ maximum of the process up to time t

Survival Probability ⟺ Extreme Value Statistics

ROTATE & SHIFT



- $M(t) \to$ maximum of the process up to time t
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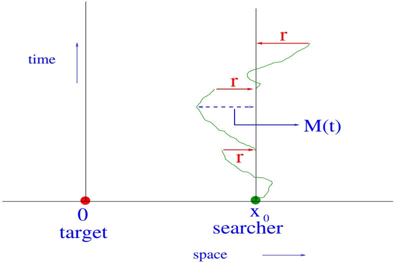
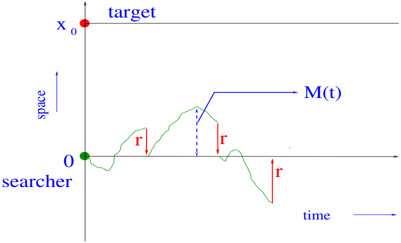
Survival Probability $\Longleftrightarrow$ Extreme Value Statistics

ROTATE & SHIFT





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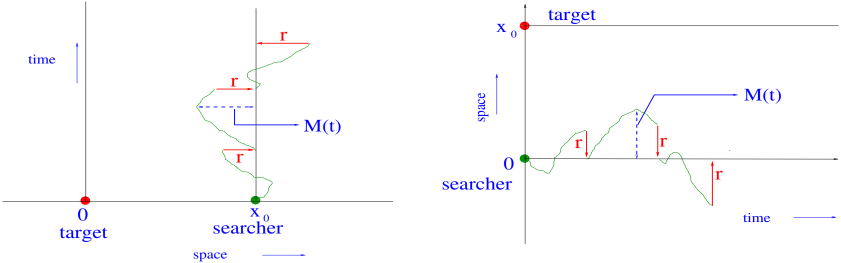
- Correlation time $\tau = 1/r$

ROTATE & SHIFT



- $M(t) \rightarrow$ maximum of the process up to time t
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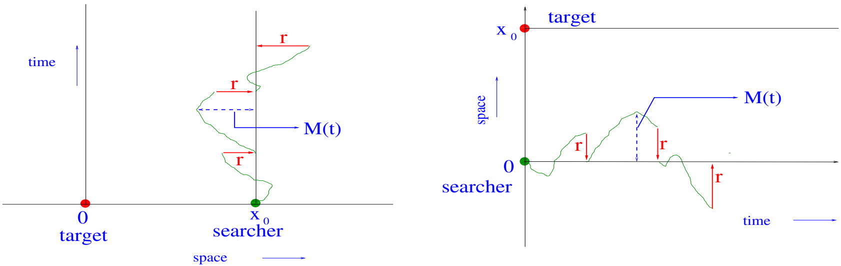
ROTATE & SHIFT



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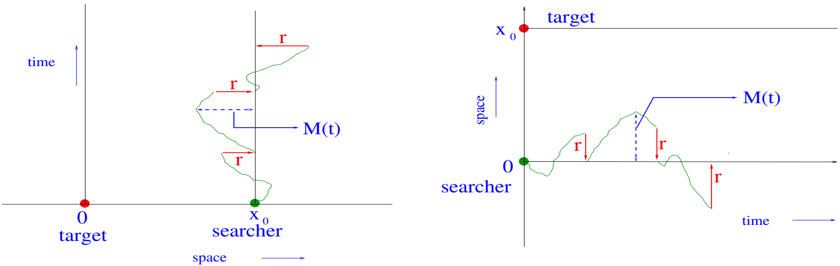
- $Q(x_0, t) \approx \exp\left[-rt\, e^{-\sqrt{r/D}\, x_0}\right]$

ROTATE & SHIFT



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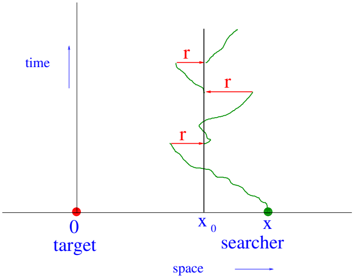
- Correlation time $\tau = 1/r \to N_{\text{eff}} = t/\tau = rt$ effectively independent blocks in the time interval $[0, t]$
- $Q(x_0, t) \approx \exp\left[-rt\, e^{-\sqrt{r/D}\, x_0}\right] \approx \exp\left[-N_{\text{eff}}\, e^{-\sqrt{r/D}\, x_0}\right]$

$\Rightarrow$ classical Gumbel distribution for the maximum of a set of N_{eff} exponentially distributed independent random variables

Mean capture/search time



mean capture time: $\bar{T} = \int_0^\infty t\,[-\partial_t Q(x_0, t)]\,dt = \tilde{Q}(x_0, s = 0)$

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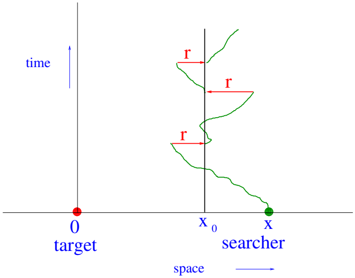

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$$\boxed{\bar{T}(r, x_0) = \frac{1}{r}\left[\exp\left(\sqrt{r/D}\,x_0\right) - 1\right]}$$

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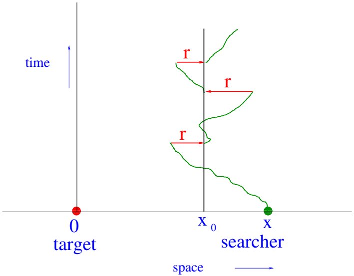

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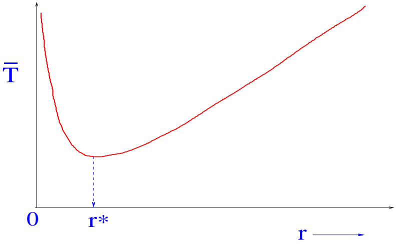
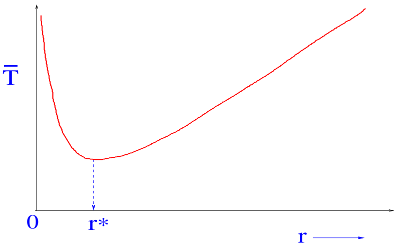
$\Rightarrow$ mean capture time is ∞ for $r = 0$, but finite when $r > 0$

Optimal resetting rate



$$\bar{T}(r, x_0) = \frac{1}{r}\left[\exp\left(\sqrt{r/D}\, x_0\right) - 1\right]$$



$$\bar{T}(r, x_0) = \frac{1}{r}\left[\exp\left(\sqrt{r/D}\, x_0\right) - 1\right]$$

- For fixed x_0 and D, the mean capture time $\bar{T}(r, x_0)$ diverges as $r \to 0$ and also as $r \to \infty$

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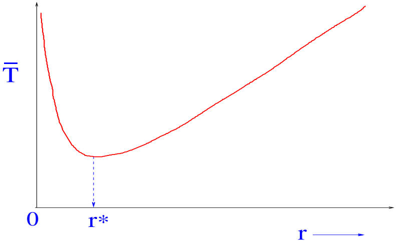

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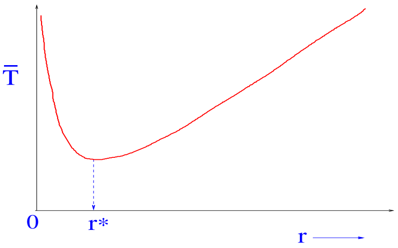

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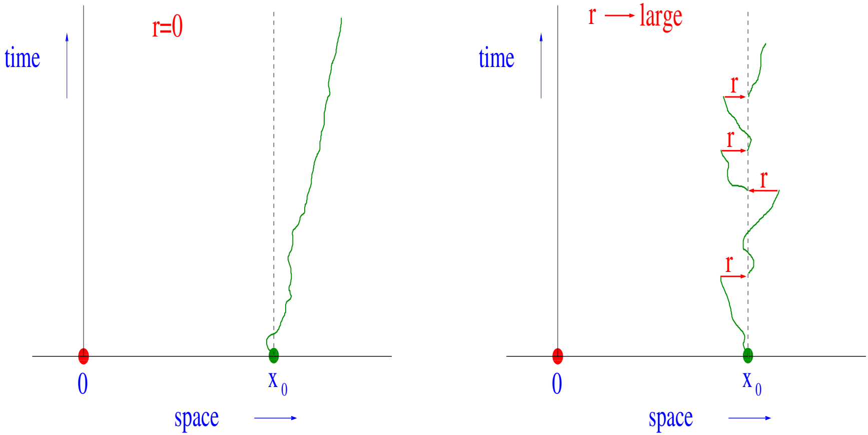
optimal resetting rate r^* is given by:

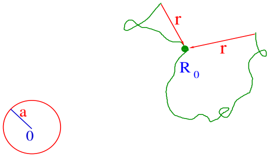
$$\boxed{r^* = \gamma^2 \frac{D}{x_0^2}} \text{ where } \boxed{\gamma - 2\left(1 - e^{-\gamma}\right) = 0} \Rightarrow \gamma = 1.59362\ldots$$

(M.R. Evans and S.M., Phys. Rev. Lett. 106, 160601 (2011))

Typical trajectories for $r \to 0$ and $r \to \infty$



Target search via diffusion with resetting in $d > 1$



stationary target of radius a at 0 in $d > 2$

searcher starts at $R_0 > a$, diffuses, and resets with rate r

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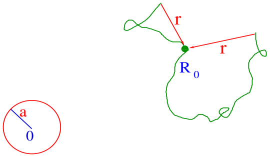

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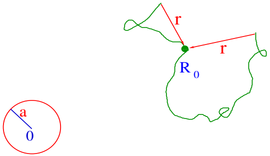

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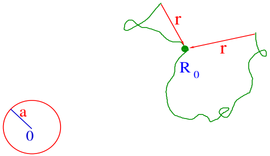

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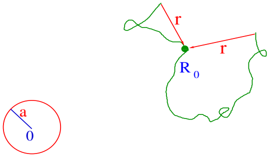

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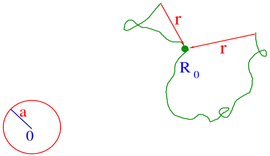
Target search via diffusion with resetting in $d > 1$



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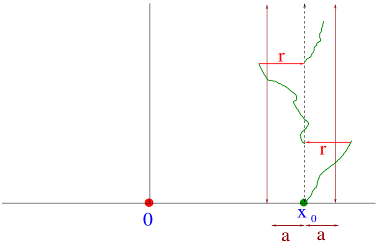
where $\nu = 1 - d/2$

- Once again, there is an optimal r^* that minimizes $\bar{T}(r, R_0)$ in all d

[M.R. Evans and S.M., J. Phys. A: Math. Theo. 47, 285001 (2014)]

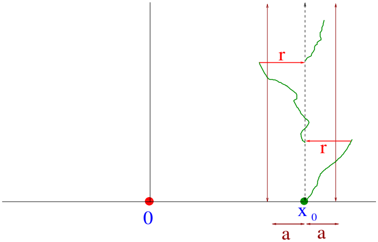
Various Generalisations



stationary target at 0, searcher diffuses and resets to x_0 with rate r only if it goes outside the box $[x_0 - a, x_0 + a]$ $\longrightarrow$ otherwise no resetting

- space-dependent resetting rate $r(x)$: What is the optimization strategy?

Various Generalisations



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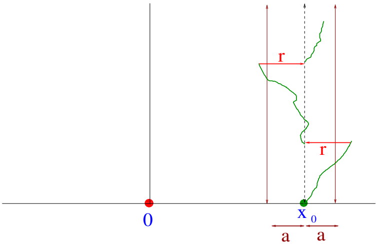
Various Generalisations



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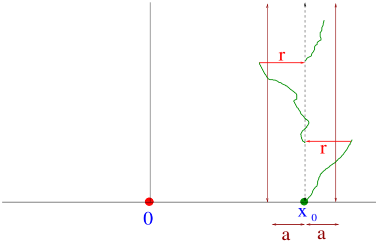
[M.R. Evans & S.M., J. Phys. A: Math. Theo. 44, 435001 (2011)]

Various Generalisations



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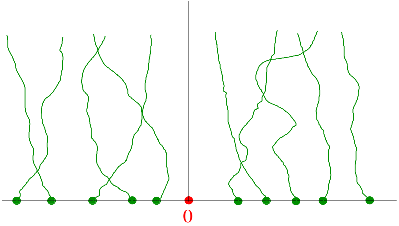
  [M.R. Evans & S.M., J. Phys. A: Math. Theo. 44, 435001 (2011)]
- Nonequilibrium reset dynamics vs. equilibrium dynamics

  " Nonequilibrium beats equilibrium "

  [M.R. Evans, S.M. & K. Mallick, J. Phys. A: Math. Theor. 46, 185001 (2013)]

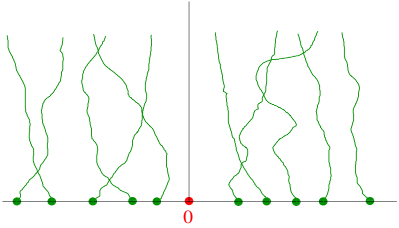
III : Multiparticle Problem

Target search by multiple searchers $d = 1$



stationary target at 0 surrounded by a sea of independent searchers (traps), initially distributed with uniform density ρ

Target search by multiple searchers $d = 1$

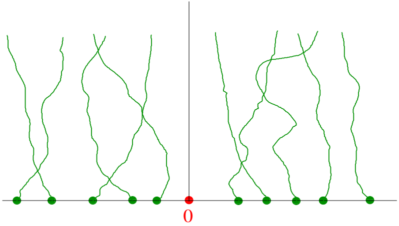


stationary target at 0 surrounded by a sea of independent searchers (traps), initially distributed with uniform density ρ

- target survival probability: $P_s(t) = \prod_{i=1}^{N} Q(x_i, t)$

  $Q(x_i, t) \rightarrow$ prob. that the i-th searcher starting initially at x_i does not hit the origin up to time t

Target search by multiple searchers $d = 1$



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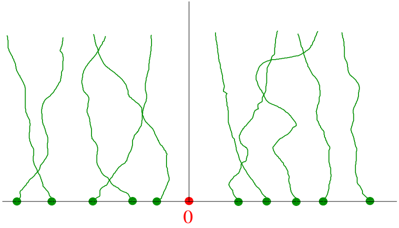
  $Q(x_i, t) \to$ prob. that the i-th searcher starting initially at x_i does not hit the origin up to time t

- Average surv. prob. of the target: $\langle P_s(t) \rangle = \left\langle \prod_{i=1}^{N} Q(x_i, t) \right\rangle$

  $\langle \rangle \to$ average over x_i's each drawn independently and uniformly from a box $[-L/2, L/2]$

Target search by multiple searchers $d = 1$



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  $\langle \rangle \to$ average over x_i's each drawn independently and uniformly from a box $[-L/2, L/2]$

- Eventually $N \to \infty$ and $L \to \infty$ with their ratio $N/L = \rho$ fixed

Average target survival probability

- $\langle P_s(t) \rangle = \left\langle \prod_{i=1}^{N} Q(x_i, t) \right\rangle$

Average target survival probability

- $\langle P_s(t)\rangle = \left\langle \prod_{i=1}^{N} Q(x_i,t) \right\rangle = \prod_{i=1}^{N} \left[1 - \langle (1 - Q(x_i,t)) \rangle\right]$

Average target survival probability

- $\langle P_s(t) \rangle = \left\langle \prod_{i=1}^{N} Q(x_i, t) \right\rangle = \prod_{i=1}^{N} \left[1 - \langle (1 - Q(x_i, t)) \rangle\right]$

$$= \left[1 - \frac{1}{L} \int_{-L/2}^{L/2} (1 - Q(x, t))\, dx\right]^N$$

Average target survival probability

- $\langle P_s(t)\rangle = \left\langle \prod_{i=1}^{N} Q(x_i,t)\right\rangle = \prod_{i=1}^{N}\left[1-\langle(1-Q(x_i,t))\rangle\right]$

$$= \left[1-\frac{1}{L}\int_{-L/2}^{L/2}(1-Q(x,t))\,dx\right]^N$$

$$\Rightarrow \boxed{\langle P_s(t)\rangle = \exp\left[-2\,\rho\int_0^\infty (1-Q(x,t))\,dx\right]}$$

Average target survival probability

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(Zumofen, Klafter, Blumen '83, Tachiya '83, Burlatsky & Ovchinnikov '87)

Link to Extreme Value Statistics

- $\langle P_s(t) \rangle = \exp\left[-2\,\rho\,\int_0^\infty (1 - Q(x,t))\,dx\right] = \exp\left[-2\,\rho\,E[M(t)]\right]$

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$\Rightarrow$ general result valid for any trap process

(J. Franke and S.M., JSTAT, P05024 (2012))

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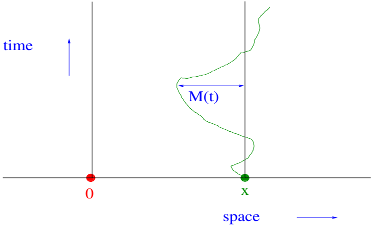
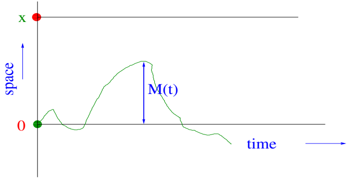
ROTATE & SHIFT





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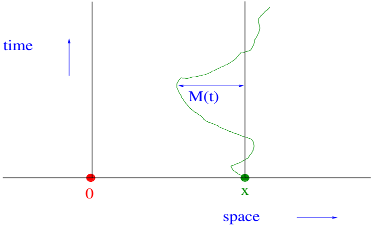

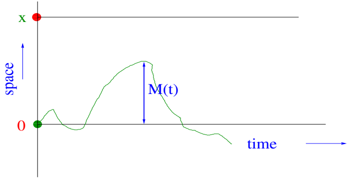

- Several exact results for $E[M(t)]$ for subdiffusive and superdiffusive (Lévy flights) processes (J. Franke and S.M., 2012)

Average vs. Typical

- average of a random variable may be different from typical

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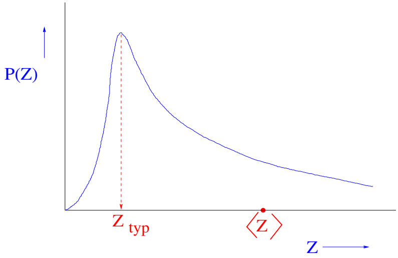

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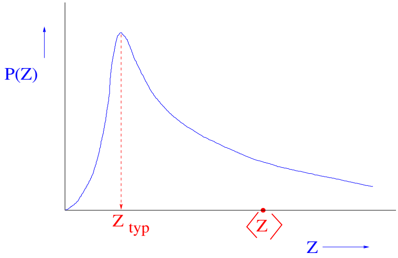

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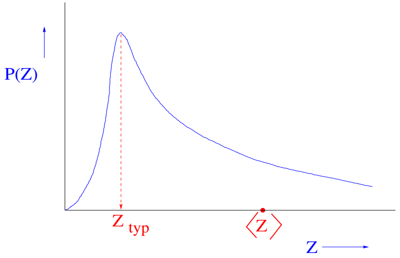

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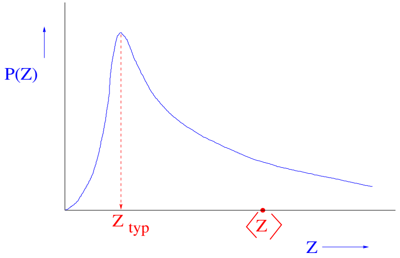

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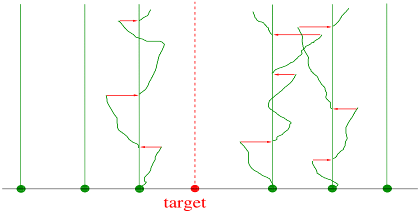
$\Rightarrow \langle P_s(t)\rangle \sim P_s^{\rm typ}(t)$ and both decay stretched-exponentially

Target search by multiple searchers with resetting



stationary target at 0 surrounded by a sea of independent diffusive searchers or traps (each with reset rate r), initially distributed with uniform density ρ

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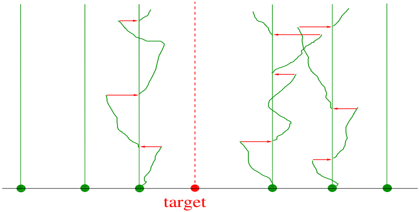

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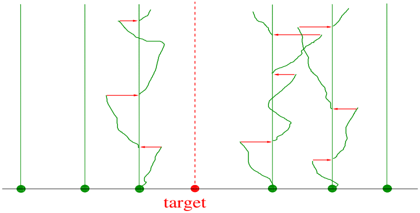

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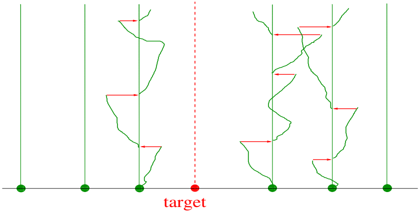

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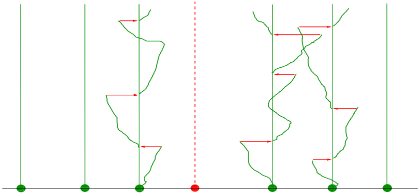

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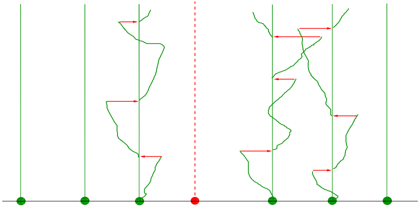

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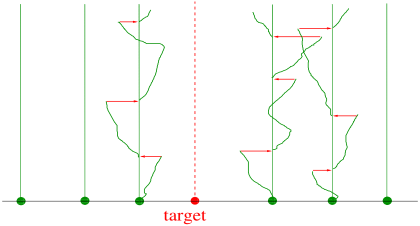

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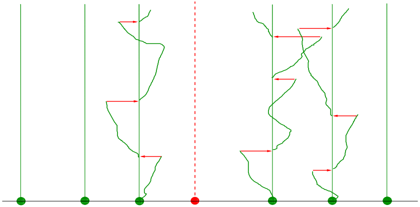

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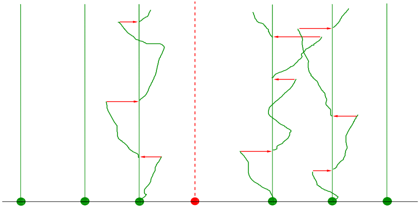

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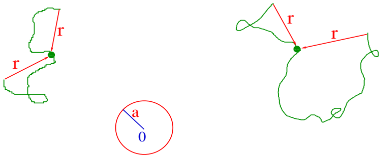
- In presence of resetting ($r > 0$): $P_s^{\rm typ}(t) << \langle P_s(t)\rangle$

Rare trajectories dominate the average

Multiple searchers with reset in $d > 1$



stationary target of radius a at 0 in $d > 2$

searchers diffuse and reset with rate r independently

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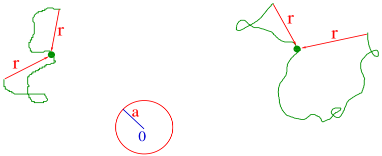

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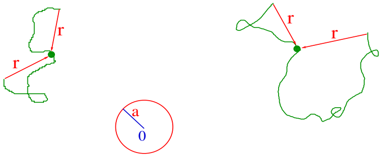

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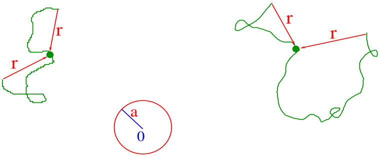

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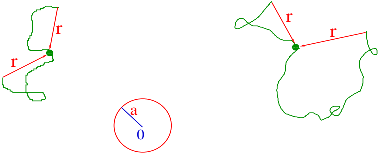

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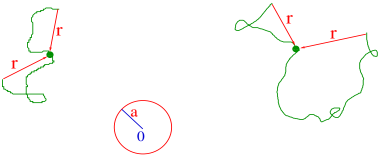

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[M.R. Evans and S.M., J. Phys. A: Math. Theo. 47, 285001 (2014)]

Summary and Conclusion

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For example, in $d = 1$, for large time t,

$$\langle P_s(t) \rangle \sim t^{-2\,\rho\,\sqrt{D/r}}$$

$$P_s^{\mathrm{typ}}(t) \sim \exp\left[-8\,(1 - \ln 2)\,\sqrt{r\,D}\,\rho\,t\right]$$

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Various generalisations:

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Resetting $\rightarrow$ rich and interesting static and dynamic phenomena

Collaborators and References

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References:

- M.R. Evans and S.N. Majumdar, Phys. Rev. Lett. 106, 160601 (2011).
- M.R. Evans and S.N. Majumdar, J. Phys. A: Math. Theor. 44, 435001 (2011).
- M. R. Evans, S.N. Majumdar, K. Mallick, J. Phys. A: Math. Theor. 46, 185001 (2013).
- J. Whitehouse, M. R. Evans, and S. N. Majumdar, Phys. Rev. E 87, 022118 (2013).
- S. Gupta, S.N. Majumdar, G. Schehr, Phys. Rev. Lett. 112, 220601 (2014).
- M.R. Evans and S. N. Majumdar J. Phys. A: Math. Theor. 47, 285001 (2014).

References

For a recent extensive survey on persistence from physics perspectives, see:

"Persistence and First-Passage Properties in Non-equilibrium Systems"

A.J. Bray, S.N. Majumdar and G. Schehr,

Adv. in Phys. **62**, 225-361 (2013)

also available at arXiv: 1304.1195