Hole probabilities for Random Taylor series

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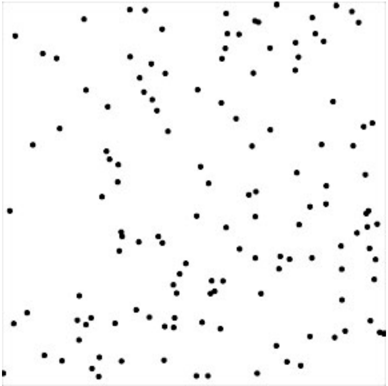
- Which is the probability that a disk of radius r contains no points of $\mathfrak{X}$, as $r \to \infty$.
- We will denote this probability by $p_H(r)$.
- The center is not important if the process is translation invariant.
- One can also consider other domains\geometries.

Poisson point process



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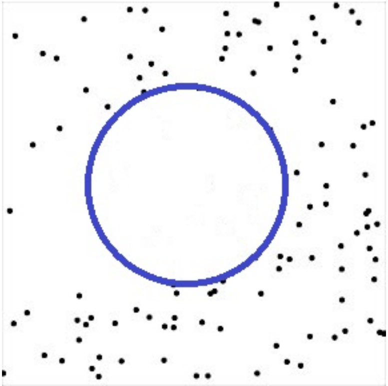

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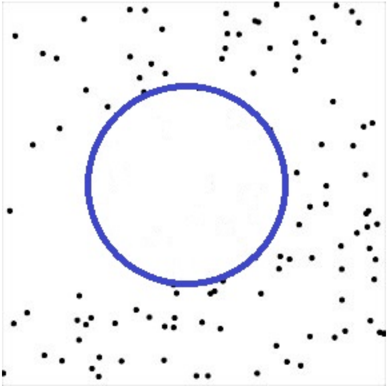
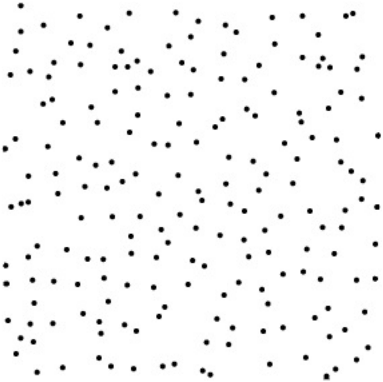


- Number of points in circle of radius r has Poisson distribution with mean $\lambda\pi r^2$.
- The hole probability is

$$e^{-\lambda\pi r^2}$$

Random Matrices - Ginibre Ensemble



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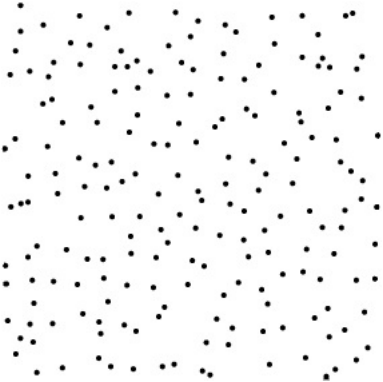
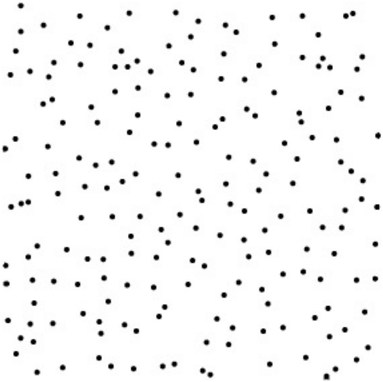


- Finite Ginibre - The eigenvalues of the $N \times N$ random matrix with i.i.d. standard complex Gaussian coefficients.
- Ginibre - The limit of this process as $N \to \infty$.
- Determinantal point process.

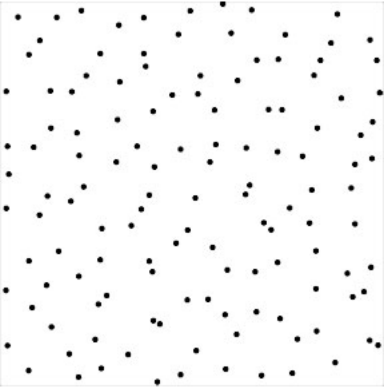
Random Matrices - Ginibre Ensemble



- Set of radii $\left\{|z_1|^2, |z_2|^2, \ldots\right\}$ is distributed like independent set $\{\Gamma(1,1), \Gamma(2,1), \ldots\}$.
- Hole probability:

$$p_H(r) = e^{-\frac{1}{4}r^2(1+o(1))}$$

GEF - Gaussian Entire function



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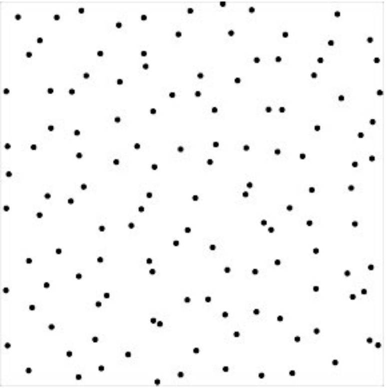
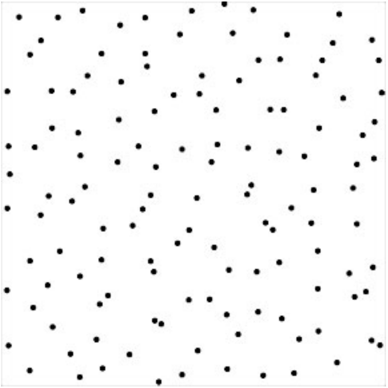


- Zeros of entire function given by the Gaussian Taylor series

$$F(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}$$

- ξ_n i.i.d. standard complex Gaussian.
- Zero set is translation invariant (unique for Gaussian analytic functions).

GEF - Gaussian Entire function



- Not a determinantal point process.
- Hole probability?

Random Taylor series

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- ξ_n i.i.d. random variables (mostly std. complex Gaussians).
  - $\mathbb{E}\xi_n = 0$, $\mathbb{E}|\xi_n|^2 = 1$
- a_n non-random sequence (can assume $a_n \geq 0$)
- f is (a.s.) an entire function when $\limsup a_n^{1/n} = 0$.
  - Analytic in the unit disk when $\limsup a_n^{1/n} = 1$.

Random Taylor series

$$f(z) = \sum_{n=0}^{\infty} \xi_n a_n z^n$$

- Properties are determined by covariance kernel:

$$K_f(z,w) = \mathbb{E}\left\{ f(z)\overline{f(w)} \right\} = \sum_{n=0}^{\infty} |a_n|^2 (z\overline{w})^n$$

- For the GEF:

$$K_F(z,w) = e^{z\overline{w}}$$

- One can consider similar functions in several variables.

Zero set

$$Z_f = \{z : f(z) = 0\}$$

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- Understood to include multiplicities (multiset).
- Random point set (or point process)
  - Simple in the Gaussian case (maybe except non-rand. zeros).

Zero set

$$Z_f = \{z : f(z) = 0\}$$

- Interested in linear statistics.

$$L_f(h) = \int_{\mathbb{C}} h \, \mathrm{d}n_f = \sum_{\alpha \in Z_f} h(\alpha)$$

  - $h : \mathbb{C} \to \mathbb{R}$ meas. function with compact support
- Formula for complex Gaussian (Edelman-Kostlan)

$$\mathbb{E}L_f(h) = \frac{1}{4\pi} \int_{\mathbb{C}} h \cdot \Delta \log K(z,z) \, \mathrm{d}m(z)$$

- For the GEF

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  - Also might be related to moderate deviations in other cases.
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- Problem related to large\very large deviations in some cases.
  - Also might be related to moderate deviations in other cases.
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- Physics - The zeros represent a gas of (interacting) particles in quantum systems.

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  - Ideas were used in many different settings.

Hole probabilities (cont.)

- Special case: $r\mathbb{D}$ - disk around zero with large radius r.
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  - Ideas were used in many different settings.
- GEF: N. (2010)

$$\log p_H(r) = -\frac{e^2}{4} \cdot r^4 + o\left(r^4\right).$$

  - In fact error term is $O\left(r^2 \log r\right)$.

How to guess the right constant

- Idea for the lower bound.
- Works surprisingly well for different cases.
  - Unit disk, higher dimensions, etc.

How to guess the right constant

- Generate an event where there are no zeros of F inside $r\mathbb{D}$:
  - ξ_0 is large
  - ξ_1, ..., ξ_N are small (N will depend on r)
  - $\xi_{N+1}, \ldots$ are 'typical' (not too large).
- Then for $|z| \leq r$:

$$|f(z)| \geq |\xi_0| - \sum_{n=1}^{N} |\xi_n| \frac{r^n}{\sqrt{n!}} - \text{'tail'} > 0.$$

- Use independence:

$$\mathbb{P}(|\xi_1| \leq t_1, \ldots, |\xi_N| \leq t_N) = \prod_{n=1}^{N} \mathbb{P}(|\xi_n| \leq t_n).$$

The 'general' case (entire functions)

- We need the following function

$$S(r) = 2 \cdot \sum_{n=0}^{\infty} \log^{+}(a_n r^n)$$

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- A measurable set $E \subset [1, \infty)$ has finite logarithmic length if

$$\int_E \frac{1}{t}\, dt < \infty.$$

- Remark: We have to assume that $a_0 \neq 0$

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Theorem

(N. 2012) For $\varepsilon \in \left(0, \frac{1}{2}\right)$, exists set $E \subset [1, \infty)$ of finite log. length, such that

$$\log p_H(r) = -S(r) + O\left(S(r)^{1/2+\varepsilon}\right), \quad r \to \infty, r \notin E.$$

Remark: Log. length of E depends on ε and on the sequence $\{a_n\}$.

Other random variables?

Difficult to say something for ‘general’ random variables. Make the following assumptions:

- Let $K \subset \mathbb{C}$ be a compact set and $0 \notin K$.
- Suppose that $\xi_k \in K$ for each n, and that $a_n = \frac{1}{\sqrt{n!}}$.

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Theorem

There exists $r_0 = r_0(K) < \infty$ so that $f(z)$ must vanish somewhere in the disk $\{|z| \le r_0\}$.

- This result depends on the (specific) nature of the deterministic coefficients a_n.

Gist of the proof (upper bound)

- ($a_0 = 1$) Assume $f \neq 0$ inside $r\mathbb{D}$, so $\log|f|$ harmonic there and for $\rho < r$

$$\frac{1}{2\pi}\int \log\left|f\left(\rho e^{\theta}\right)\right| \mathrm{d}\theta = \log|f(0)| = \log|\xi_0|$$

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$$\frac{1}{N}\sum_{j=1}^{N} \log|f(z_j)| \leq A'$$
- The vector $(f(z_1), \ldots, f(z_N))$ has multivariate complex Gaussian distribution.

Gist of the proof (cont.)

- Have to find an upper bound for the following expression:

$$I = \frac{1}{\pi^N \det \Sigma} \int_{\mathscr{A}} \exp\left(-\zeta^* \Sigma^{-1} \zeta\right) \mathrm{d}\zeta$$

  - $\mathscr{A} = \left\{\zeta \in \mathbb{C}^N : |\zeta_j| \leq B \text{ and } \prod |\zeta_j| \leq N \cdot A'\right\}$
  - $\Sigma_{jk} = \mathbb{E}\left\{f(z_j)\overline{f(z_k)}\right\}$
- Choice of $\{z_j\}$ is important for getting a good lower bound for $\det \Sigma$.
- Upper bound for the integral is $\mathrm{vol}_{\mathbb{C}^N}(\mathscr{A})$

Gist of the proof (cont.)

- Outline of estimate for the GEF:
  - We choose N (dep. on r), equidistributed points.
- Then write $\Sigma = VV^*$ with

$$V = \begin{pmatrix} 1 & \frac{z_1}{\sqrt{1!}} & \dots & \frac{z_1^N}{\sqrt{N!}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 1 & \frac{z_N}{\sqrt{1!}} & \dots & \frac{z_N^N}{\sqrt{N!}} & \dots \end{pmatrix}$$

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- Collecting common terms and projecting on the largest N columns we get:

$$\det \Sigma \geq \prod_{n=1}^{N} \frac{|z_n|^2}{n!} \cdot \prod_{1 \leq j < k \leq N} |z_j - z_k|^2$$

Further questions\Future

- Another approach (for the upper bound): Large deviation principle for empirical measure of zeros of random polynomials.
- [Joint w. Ghosh, following Zeitouni and Zelditch (2010)] Set $N = \lceil \alpha L^2 \rceil$, where $\alpha \geq e$ and consider the polynomial

$$P_N(z) = \sum_{n=1}^{N} \xi_n \frac{(Lz)^n}{\sqrt{n!}}$$

  - $\mathrm{d}\mu_\xi^N \in \mathscr{M}(\mathbb{C})$ - empirical measure of the polynomial P_N.
  - This is a scaling of the original problem to the unit disk (r is replaced by L).

Further questions\Future (cont.)

- Roughly: For Borel set $A \subset \mathscr{M}(\mathbb{C})$

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\mathrm{d}\mu_\xi^N \in A\right) = -\inf_{\sigma\in A} I(\sigma)$$

- Here the rate function is:

$$I(\sigma) = \sup_{z\in\mathbb{C}} \left\{ 2\cdot U_\sigma(z) - \frac{|w|^2}{\alpha} \right\} - \Sigma_\sigma + C_\alpha$$

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- $U_\sigma(z) = \int_{\mathbb{C}} \log|z-w| \, \mathrm{d}\sigma(w)$
- $\Sigma_\sigma = \iint_{\mathbb{C}^2} \log|z-w| \, \mathrm{d}\sigma(z) \mathrm{d}\sigma(w)$

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  - Minimizing measure might shed light on the conditional distribution of zeros.
- Let $n(r)$ be the number of zeros of P_N inside the disk of radius 1. Can consider $\mathbb{P}\left(\left|n(r) - L^2\right| > \delta L^2\right)$.

Thank you for listening

'Gravitational' allocation of the plane to the zeros of the GEF.

Proof for compact coefficients

- (Simple) proof is by contradiction.
  - There is a sequence of entire function $f_k(z)$ that does not vanish in $r_k\mathbb{D}$, with $r_k \to \infty$.
  - Using compactness, Hurwitz's theorem and the fact that $0 \notin K$, we find a limiting function that does not vanish in the complex plane, and have coefficients from the set K.
  - Since the function is of order 2 we know that it is of the form

$$g(z) = \exp\left(\alpha z^2 + \beta z + \gamma\right), \qquad \alpha, \beta, \gamma \in \mathbb{C}.$$

  - Calculate the asymptotics of the Taylor coefficients of $g(z)$.
  - Arrive at a contradiction, since $\xi_n \cdot \frac{1}{\sqrt{n!}}$ has a different asymptotics.