Persistence of integrated stable processes

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Workshop "Persistence probabilities and related fields"

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Outline

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   - Study of L_{T_0}
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Introduction

Let $X = \{X_t,\ t \geq 0\}$ be a real-valued process starting at $x < 0$ and define

$$T_0 = \inf\{t > 0,\ X_t > 0\}.$$

In many interesting cases the survival function of T_0 has a polynomial decay :

$$\mathbb{P}_x(T_0 > t) \ = \ t^{-\theta + o(1)}, \qquad t \to +\infty,$$

where θ is a positive constant which is called the persistence exponent and usually does not depend on x.

Examples

1. If $X = B$ is a Brownian motion, then :

$$\mathbb{P}_x(T_0 > t) \underset{t\to+\infty}{\sim} |x|\sqrt{\frac{2}{\pi t}} \qquad (\theta = 1/2).$$

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2. If $X_t = x + \int_0^t B_u \, du$ with $B_0 = 0$, then :

$$\mathbb{P}_x(T_0 > t) \underset{t \to +\infty}{\sim} c \, \frac{|x|^{1/6}}{t^{1/4}} \qquad (\theta = 1/4).$$

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Theorem [Aurzada - Dereich]

The exponent remains 1/4 if one replaces B by any centered Lévy process having exponential moments.

Open problems

This result leads to some natural generalizations. What is the persistence exponent of :

1. twice integrated Brownian motion ?
   (r_2 in Prof. Dembo's talk yesterday, and $r_3, \dots$)
2. the integrated fractional Brownian motion ?
   (Gaussian but no longer Markov)
3. an integrated $\alpha-$stable Lévy process ?
   (Markov but no longer Gaussian, and with infinite variance)

Notations

Let L be a strictly $\alpha-$stable Lévy process starting from zero, with characteristic exponent :

$$\mathbb{E}\left[e^{i\lambda L_1}\right] = \exp\left(-(i\lambda)^\alpha e^{i\pi\alpha\rho \text{sgn}(\lambda)}\right) \qquad (\lambda \in \mathbb{R})$$

where

$$\begin{cases} \alpha \in (0,2] \text{ denote the self-similarity parameter, and} \\ \rho = \mathbb{P}[L_1 \geq 0] \text{ the positivity parameter.} \end{cases}$$

We set

$$X_t = x + \int_0^t L_u \, du$$

and denote by $\mathbb{P}_{(x,y)}$ the law of (X, L) when started from $X_0 = x$ and $L_0 = y$.

Remarks

1. If $\alpha < 1$, then $\rho \in [0,1]$. We shall exclude the cases $\rho = 0$ and $\rho = 1$ for which $|L|$ is a subordinator.

2. If $\alpha = 1$, then $\rho \in (0,1)$ and L is a Cauchy process with a linear drift.

3. If $1 < \alpha < 2$, then $\rho \in [\underbrace{1 - 1/\alpha}_{\text{no negative jumps}}, \underbrace{1/\alpha}_{\text{no positive jumps}}]$

When started from $(0,0)$, for $k > 0$:

$$\{L_{kt},\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{k^{1/\alpha} L_t,\, t \geq 0\} \quad \text{and} \quad \{X_{kt},\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{k^{1+1/\alpha} X_t,\, t \geq 0\}$$

Graphs

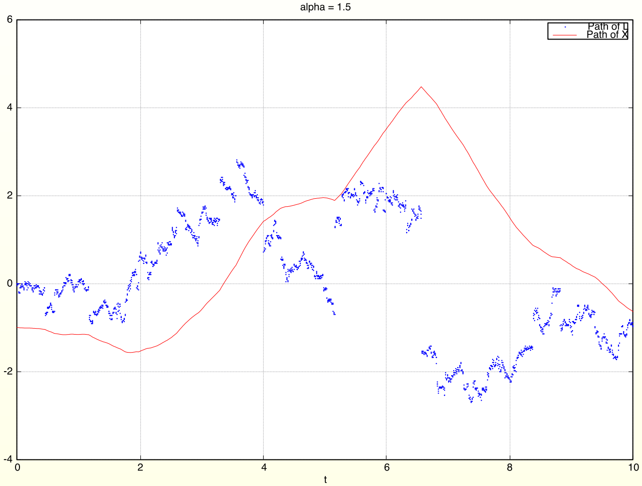

FIGURE: One path of L and X for $\alpha = 1.5$ and $\rho = 1/2$

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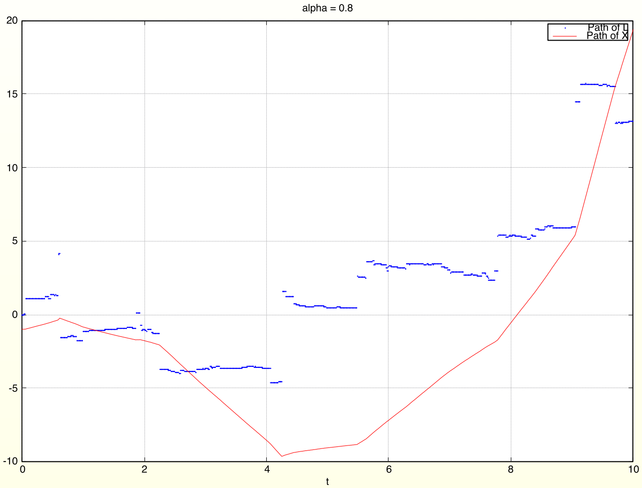

FIGURE: One path of L and X for $\alpha = 0.8$ and $\rho = 1/2$

Outline

Main result

Theorem

Assume that $x < 0$ or that $x = 0$ and $y < 0$. There exist two constants $0 < \kappa_1 \leq \kappa_2 < +\infty$ such that :

$$\frac{\kappa_1}{t^\theta} \leq \mathbb{P}_{(x,y)}\left(T_0 > t\right) \leq \frac{\kappa_2}{t^\theta} \qquad (t \to +\infty)$$

with

$$\theta = \frac{\rho}{1 + \alpha(1 - \rho)}.$$

In particular, if L is symmetric, the exponent reads : $\theta = \dfrac{1}{2 + \alpha}$.

Heuristic

Instead of studying directly T_0, we shall first focus on L_{T_0}.

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Indeed, by scaling, if T_0 and L were independent, then :

$$L_{T_0} \overset{\mathcal{L}}{=} T_0^{1/\alpha} L_1,$$

so we may hope that :

$$\mathbb{P}_{(x,y)}(L_{T_0} > z) = z^{-\alpha\theta + o(1)} \qquad (z \to +\infty)$$

Study of L_{T_0}

Theorem

For $x < 0$ or $x = 0$ and $y < 0$, the Mellin transform of L_{T_0} is given, for $s \in (0,1)$, by :

$$\mathbb{E}_{(x,y)}\left[L_{T_0}^{s-1}\right] = \frac{\pi \int_0^{+\infty} \mathbb{E}_{(x,y)}\left[X_t^{\frac{s}{1+\alpha}-1} 1_{\{X_t>0\}}\right] dt}{(1+\alpha)^{\frac{s}{1+\alpha}} \left(\Gamma(\frac{s}{1+\alpha})\right)^2 \Gamma(1-s)\sin(\pi(1-\gamma)s)}$$

where

$$\gamma = \frac{\rho\alpha}{1+\alpha}.$$

Sketch of the proof

By right-continuity, $L_{T_0} \geq 0$ a.s.

We first apply the Markov property, for $a \geq 0$:

$$\mathbb{P}_{(x,y)}(X_t \in da) = \int_0^\infty \int_0^t \mathbb{P}_{(0,z)}(X_{t-s} \in da)\, \mathbb{P}_{(x,y)}(T_0 \in ds, L_{T_0} \in dz).$$

We then integrate in time (to make T_0 disappear) :

$$\int_0^\infty \mathbb{P}_{(x,y)}(X_t \in da)\, dt = \int_0^\infty \left(\int_0^\infty \mathbb{P}_{(0,z)}(X_t \in da)\, dt \right) \mathbb{P}_{(x,y)}(L_{T_0} \in dz)$$

Sketch of the proof

We finally integrate in space with respect to $a^{-\nu}$:

$$\int_0^{\infty} \mathbb{E}_{(x,y)}[X_t^{-\nu} 1_{\{X_t>0\}}]dt = \int_0^{\infty} \mathbb{P}_{(x,y)}(L_{T_0} \in dz) \left(\int_0^{\infty} \mathbb{E}_{(0,z)}[X_t^{-\nu} 1_{\{X_t>0\}}]\, dt \right)$$

and the expressions $\mathbb{E}_{(x,y)}[X_t^{-\nu} 1_{\{X_t>0\}}]$ may be (partially) computed

thanks to the formula :

$$\int_0^{\infty} \lambda^{\nu-1}\, \mathbb{E}\left[\sin\left(\lambda X + \frac{\nu\pi}{2}\right)\right] d\lambda \;=\; \Gamma(\nu)\sin(\pi\nu)\, \mathbb{E}[X^{-\nu} 1_{\{X>0\}}]$$

$\square$

Corollary

When $x = 0$ (and $y < 0$) :

$$\mathbb{E}_{(0,y)}[L_{T_0}^{s-1}] \; = \; |y|^{s-1} \left(\frac{\sin(\pi\gamma s)}{\sin(\pi(1-\gamma)s)} \right).$$

Therefore under $\mathbb{P}_{(0,y)}$,

$$L_{T_0} \; \overset{\mathcal{L}}{=} \; \left(\mathbf{C}_{\alpha\theta}^{1-\gamma} \right)^{(1)}$$

where $\mathbf{C}_\mu$ denote a $\mu-$Cauchy random variable and $X^{(1)}$ the size bias of order 1 of X.

Corollary

When $y = 0$ (and $x < 0$) :

$$\mathbb{E}_{(x,0)}\left[L_{T_0}^{s-1}\right] = \frac{(1+\alpha)^{\frac{1-s}{1+\alpha}}\Gamma(\frac{\alpha+2}{\alpha+1})\Gamma(\frac{1-s}{\alpha+1})\sin(\pi\gamma)}{\Gamma(\frac{s}{\alpha+1})\Gamma(1-s)\sin(\pi(1-\gamma)s)}|x|^{\frac{s-1}{\alpha+1}}.$$

In particular,

1. when $\alpha = 1$, we deduce that under $\mathbb{P}_{(x,0)}$:

$$L_{T_0} \overset{\mathcal{L}}{=} \sqrt{2|x|}\left(\mathbf{C}^{1-\gamma}_{(1+\alpha\theta)/2}\right)^{(1)}$$

Corollary

When $y = 0$ (and $x < 0$) :

$$\mathbb{E}_{(x,0)}\left[L_{T_0}^{s-1}\right] = \frac{(1+\alpha)^{\frac{1-s}{1+\alpha}}\Gamma(\frac{\alpha+2}{\alpha+1})\Gamma(\frac{1-s}{\alpha+1})\sin(\pi\gamma)}{\Gamma(\frac{s}{\alpha+1})\Gamma(1-s)\sin(\pi(1-\gamma)s)}|x|^{\frac{s-1}{\alpha+1}}.$$

In particular,

2. when $\alpha = 2$, we deduce that under $\mathbb{P}_{(x,0)}$:

$$L_{T_0} \overset{\mathcal{L}}{=} |9x|^{1/3}\left(\frac{\mathbf{\Gamma}_{5/6}}{\mathbf{B}_{1/6,1/6}}\right)^{1/3}.$$

where $\mathbf{\Gamma}_c$ and $\mathbf{B}_{a,b}$ denote standard Gamma and Beta r.v.'s.

Reduction of the problem

From a converse mapping theorem for Mellin transforms, we deduce that when $\{x < 0$ and $y = 0\}$ or $\{x = 0$ and $y < 0\}$:

$$\mathbb{P}_{(x,y)}(L_{T_0} > z) \underset{z \to +\infty}{\sim} c\, z^{-\alpha\theta}$$

We shall in the following restrict our attention to these cases.

Proposition

Assume that $x < 0$. For every $y \in \mathbb{R}$, there exist $0 < \kappa_1 \leq \kappa_2 < +\infty$ such that :

$$\kappa_1\, \mathbb{P}_{(x,0)}(T_0 > t) \;\leq\; \mathbb{P}_{(x,y)}(T_0 > t) \;\leq\; \kappa_2\, \mathbb{P}_{(x,0)}(T_0 > t).$$

Outline

From L_{T_0} to T_0

From the Markov property :

$$\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{(x,y)}\left[(X_t^+)^{-\nu}\right]dt \;=\; \mathbb{E}_{(x,y)}\left[e^{-\lambda T_0}\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]$$

hence, integrating by parts :

$$\begin{aligned}&\mathbb{E}_{(x,y)}\left[(1-e^{-\lambda T_0})\int_0^\infty e^{-\lambda t}\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]\\ &=\mathbb{E}_{(x,y)}\left[\int_0^\infty e^{-\lambda t}\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]-\int_0^\infty e^{-\lambda t}\mathbb{E}_{(x,y)}\left[(X_t^+)^{-\nu}\right]dt\end{aligned}$$

From L_{T_0} to T_0

From the Markov property :

$$\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{(x,y)}\left[(X_t^+)^{-\nu}\right]dt \;=\; \mathbb{E}_{(x,y)}\left[e^{-\lambda T_0}\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]$$

hence, integrating by parts :

$$\begin{aligned}
&\mathbb{E}_{(x,y)}\left[\left(1-e^{-\lambda T_0}\right)\int_0^\infty e^{-\lambda t}\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]\\
&=\mathbb{E}_{(x,y)}\left[\int_0^\infty e^{-\lambda t}\mathbb{E}_{(0,L_{T_0})}\left[(X_t^+)^{-\nu}\right]dt\right]-\int_0^\infty e^{-\lambda t}\mathbb{E}_{(x,y)}\left[(X_t^+)^{-\nu}\right]dt\\
&=\lambda\int_0^\infty e^{-\lambda t}\int_t^\infty\left(\mathbb{E}_{(x,y)}\left[(X_u^+)^{-\nu}\right]-\mathbb{E}_{(x,y)}\left[\mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right]\right]\right)du\,dt
\end{aligned}$$

From L_{T_0} to T_0

This last relation may be inverted to obtain :

$$\mathbb{E}_{(x,y)}\left[\int_0^t 1_{\{T_0>t-u\}}\mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right]du\right] = H_{(x,y)}(t)$$

with

$$H_{(x,y)}(t) \;=\; \int_t^{+\infty}\left(\mathbb{E}_{(x,y)}\left[(X_u^+)^{-\nu}\right] - \mathbb{E}_{(x,y)}\left[\mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right]\right]\right)du$$

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Lemma

For all $\nu \in (\alpha(1-\theta)/(\alpha+1), 1)$ there exists $\kappa > 0$ such that

$$\mathbb{E}_{(x,y)}\left[\int_0^t 1_{\{T_0>t-u\}}\,\mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right]du\right] \underset{t\to+\infty}{\sim} \kappa\, t^{1-(1+1/\alpha)\nu-\theta}.$$

Upper bound

Fix $A > 0$ and $\nu \in (\alpha/(\alpha+1), 1)$. By continuity and positivity there exists $\varepsilon > 0$ such that for all $z \in [0, A]$,

$$\int_0^1 \mathbb{E}_{(0,z)}\left[(X_u^+)^{-\nu}\right] du \;\geq\; \varepsilon.$$

For all $t > 0$,

$$t^{(1+1/\alpha)\nu-1} H_{(x,y)}(t) \geq \; t^{(1+1/\alpha)\nu-1}\, \mathbb{E}_{(x,y)}\left[1_{\{T_0>t\}} \int_0^t \mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right] du\right]$$

$$= \mathbb{E}_{(x,y)}\left[1_{\{T_0>t\}} \int_0^1 \mathbb{E}_{(0,\frac{1}{t^{1/\alpha}}L_{T_0})}\left[(X_u^+)^{-\nu}\right] du\right]$$

Upper bound

Fix $A > 0$ and $\nu \in (\alpha/(\alpha+1), 1)$. By continuity and positivity there exists $\varepsilon > 0$ such that for all $z \in [0, A]$,

$$\int_0^1 \mathbb{E}_{(0,z)}\left[(X_u^+)^{-\nu}\right] du \;\geq\; \varepsilon.$$

For all $t > 0$,

$$\begin{aligned}
t^{(1+1/\alpha)\nu-1} H_{(x,y)}(t) &\geq t^{(1+1/\alpha)\nu-1}\, \mathbb{E}_{(x,y)}\left[1_{\{T_0>t\}} \int_0^t \mathbb{E}_{(0,L_{T_0})}\left[(X_u^+)^{-\nu}\right] du\right] \\
&= \mathbb{E}_{(x,y)}\left[1_{\{T_0>t\}} \int_0^1 \mathbb{E}_{(0,\frac{1}{t^{1/\alpha}}L_{T_0})}\left[(X_u^+)^{-\nu}\right] du\right] \\
&\geq \varepsilon\, \mathbb{P}_{(x,y)}(T_0 > t, L_{T_0} \leq A t^{1/\alpha}) \\
&\geq \varepsilon\left(\mathbb{P}_{(x,y)}(T_0 > t) - \mathbb{P}_{(x,y)}(T_0 > t, L_{T_0} \geq A t^{1/\alpha})\right)
\end{aligned}$$

Upper bound

$$t^{(1+1/\alpha)\nu-1}H_{(x,y)}(t)+\varepsilon\mathbb{P}_{(x,y)}(T_0>t,L_{T_0}\geq At^{1/\alpha})\ \geq\ \varepsilon\,\mathbb{P}_{(x,y)}(T_0>t)$$

Upper bound

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Recall that :

$$t^{(1+1/\alpha)\nu-1}H_{(x,y)}(t)\underset{t\to+\infty}{\sim}\frac{\kappa}{t^{\theta}}$$

and

$$\mathbb{P}_{(x,y)}(T_0>t,L_{T_0}\geq At^{1/\alpha})\leq\mathbb{P}_{(x,y)}(L_{T_0}\geq At^{1/\alpha})\underset{t\to+\infty}{\sim}\frac{c}{t^{\theta}}.$$

Therefore, there exists $\kappa_2>0$ such that :

$$\mathbb{P}_{(x,y)}(T_0>t)\ \leq\ \frac{\kappa_2}{t^{\theta}}\qquad \text{as } t\to+\infty.$$

Lower bound

Fix $\nu \in (\alpha(1-\theta)/(1+\alpha), \alpha/(1+\alpha))$ and observe that :

$$\mathbb{E}_{(0,y)}\left[(X_u^+)^{-\nu}\right] \;\leq\; K\, u^{-\nu(1+1/\alpha)}.$$

Set $\eta = \nu(1+1/\alpha) \in (0,1)$ and fix $\varepsilon \in (0,1)$. We decompose

$$t^{\eta-1} H_{(x,y)}(t) \leq K\, t^{\eta-1} \left(\int_0^{t(1-\varepsilon)} \frac{\mathbb{P}_{(x,y)}(T_0 > u)}{(t-u)^\eta}\, du \;+\; \int_{t(1-\varepsilon)}^{t} \frac{\mathbb{P}_{(x,y)}(T_0 > u)}{(t-u)^\eta}\, du \right)$$

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$$\begin{aligned}
t^{\eta-1}H_{(x,y)}(t) &\leq K\,t^{\eta-1}\left(\int_0^{t(1-\varepsilon)} \frac{\mathbb{P}_{(x,y)}(T_0 > u)}{(t-u)^\eta}\,du \;+\; \int_{t(1-\varepsilon)}^{t} \frac{\mathbb{P}_{(x,y)}(T_0 > u)}{(t-u)^\eta}\,du\right)\\
&\leq K\,\frac{\varepsilon^{-\eta}}{t}\int_0^t \mathbb{P}_{(x,y)}(T_0 > u)\,du \;+\; \frac{K\varepsilon^{1-\eta}}{1-\eta}\,\mathbb{P}_{(x,y)}(T_0 > t(1-\varepsilon))\\
&\leq K\,\frac{\varepsilon^{-\eta}}{t}\int_0^t \mathbb{P}_{(x,y)}(T_0 > u)\,du \;+\; \frac{K\varepsilon^{1-\eta}}{(1-\eta)(1-\varepsilon)^\theta}\,\frac{\kappa_2}{t^\theta}.
\end{aligned}$$

Lower bound

$$t^{\eta-1}H_{(x,y)}(t) - \frac{K\varepsilon^{1-\eta}}{(1-\eta)(1-\varepsilon)^{\theta}}\frac{\kappa_2}{t^{\theta}} \leq K\frac{\varepsilon^{-\eta}}{t}\int_0^t \mathbb{P}_{(x,y)}(T_0 > u)\,du$$

Taking ε small enough, we deduce that there exists $\kappa_1 > 0$ such that :

$$\frac{1}{t}\int_0^t \mathbb{P}_{(x,y)}(T_0 > u)\,du \geq \frac{\kappa_1}{t^{\theta}} \qquad \text{as } t \to +\infty,$$

and the result follows from the mean value theorem.

□

Thank you for your attention.