Random walks and Brownian motion with drift in cones

Kilian Raschel

Joint with Rodolphe Garbit (Université d'Angers)

Workshop "Persistence probabilities and related fields"
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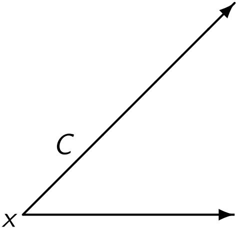
Random processes (RW and BM) in cones

First exit time from the cone C



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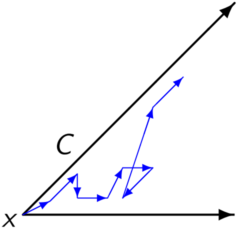

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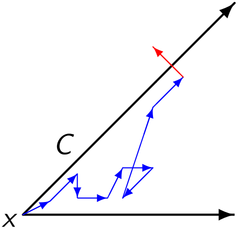

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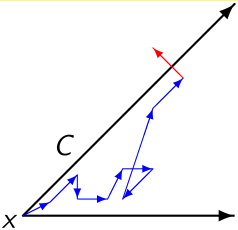

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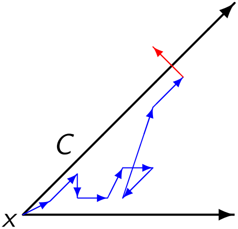

Exact and asymptotic expressions for the non-exit probab.

$\mathbf{P}_x[\tau_C > n]$ $(n \to \infty)$ & $\mathbf{P}_x[T_C > t]$ $(t \to \infty)$

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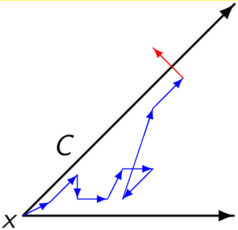

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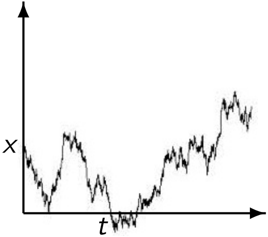
Motivations

- Constructing RW and BM *conditioned on staying in cones* (e.g., Weyl chambers);
- Constructing BM *conditioned on starting at the origin of cones* (Brownian meander);
- Many processes are naturally in cones (non-colliding processes, eigenvalues of certain random matrices, etc.).

Dimension 1: two characterizations of $\mathbf{P}_x[T_{(0,\infty)} > t]$

Exact computation of the non-exit probability



$$\mathbf{P}_x[T_{(0,\infty)} > t] = \mathbf{P}_0[\min_{u\in[0,t]} B(u) > -x]$$
$$= \mathbf{P}_0[|B(t)| < x]$$
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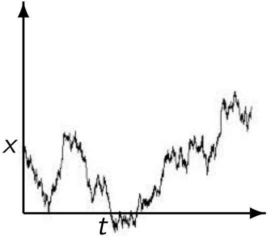

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Heat equation

The function $g(t;x) = \mathbf{P}_x[T_{(0,\infty)} > t]$ satisfies the heat equation:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) g(t;x) = 0, & \forall x \in (0,\infty),\ \forall t \in (0,\infty), \\ g(0;x) = 1, & \forall x \in (0,\infty), \\ g(t;0) = 0, & \forall t \in (0,\infty). \end{cases}$$

Dimension d: explicit expression for $\mathbf{P}_x[T_C > t]$

Heat equation [Doob '55]

For essentially *any domain* C in *any dimension* d, $\mathbf{P}_x[T_C > t]$ and $p^C(t; x, y)$ ($\mathbf{P}_x[T_C > t] = \int_C p^C(t; x, y) \mathrm{d}y$) satisfy *heat equations*.

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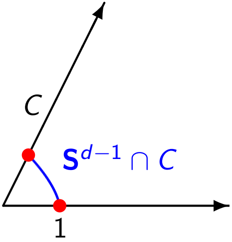
Dirichlet eigenvalues problem [e.g., Chavel '84]



$$\begin{cases} \Delta_{\mathbf{S}^{d-1}} m = -\lambda m & \text{in } \mathbf{S}^{d-1}\cap C, \\ m = 0 & \text{in } \partial(\mathbf{S}^{d-1}\cap C). \end{cases}$$

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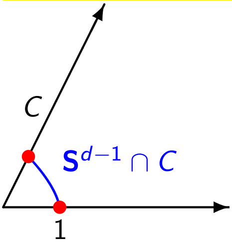

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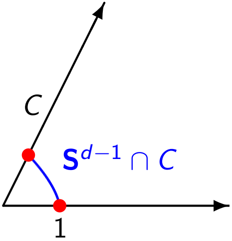

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Series expansion [DeBlassie '87, Bañuelos & Smits '97]

$$\mathbf{P}_x[T_C > t] = \int_C p^C(t; x, y) \mathrm{d}y = \sum_{j=1}^{\infty} B_j(|x|^2/t) m_j(x/|x|).$$

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- $\kappa > 0$.

Dimension 1: Brownian motion with drift $a \in \mathbf{R}$

Density of Brownian motion with drift

$$\mathbf{P}_x[B(t) \in \mathrm{d}y] = e^{a(y-x)-ta^2/2} \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} \mathrm{d}y.$$

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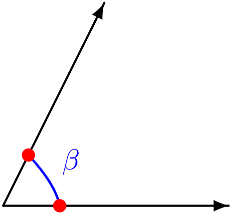
Asymptotic behavior of the non-exit probability

For all starting point $x > 0$, as $t \to \infty$,

$$\mathbf{P}_x[T_{(0,\infty)} > t] = (1 + o(1)) \begin{cases} \dfrac{x e^{-ax} e^{-ta^2/2}}{\sqrt{2\pi} a^2 t^{3/2}} & \text{if} \quad a < 0, \\ \dfrac{\sqrt{2} x}{\sqrt{\pi t}} & \text{if} \quad a = 0, \\ 1 - e^{-2ax} & \text{if} \quad a > 0. \end{cases}$$

Dimension 2: simple cases

Cones in dimension 2



In dimension 2, any cone is a rotation of $\{\rho e^{i\theta} : \rho > 0, 0 < \theta < \beta\}$, for some $\beta \in (0, 2\pi]$.

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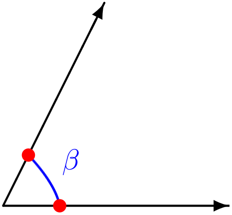

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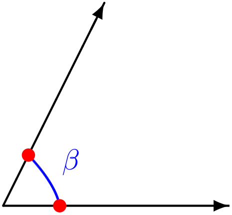

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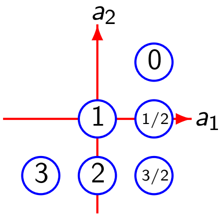
Quarter plane $\mathbf{R}_+^2$ ($\beta = \pi/2$) with drift $(a_1, a_2) \in \mathbf{R}^2$

$$\begin{aligned}
\mathbf{P}_x[T_{\mathbf{R}_+^2} > t] \\
&= \mathbf{P}_{x_1}[T_{(0,\infty)}(B^{(1)}) > t] \\
&\times \mathbf{P}_{x_2}[T_{(0,\infty)}(B^{(2)}) > t] \\
&= \kappa h(x) t^{-\alpha} e^{-\gamma t}(1 + o(1))
\end{aligned}$$



Dimension d: general case of drift $a \in \mathbf{R}^d$ (1/3)

Exact expression of the non-exit probability

Using

- the expression of the heat kernel $p^C(t; x, y)$ for the zero drift case,
- Girsanov theorem,

one obtains

$$\mathbf{P}_x[T_C > t] = e^{\langle -a,x\rangle - t|a|^2/2} \int_C e^{\langle a,y\rangle} p^C(t; x, y) \mathrm{d}y.$$

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Universal exponential decay [Garbit & R. '13]

Distance between the drift and the cone:

$$\gamma = \frac{1}{2} d(a, C)^2 = \frac{1}{2} \min_{y \in \overline{C}} |a - y|^2.$$

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Cones and polar cones

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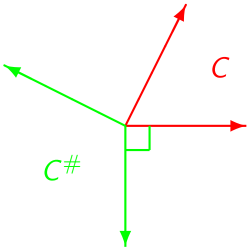

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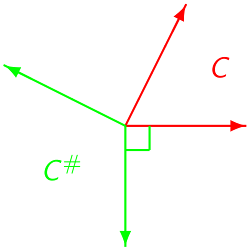

Six possible cases

- polar interior drift: $a \in (C^{\sharp})^o$;
- zero drift: $a = 0$;
- interior drift: $a \in C$;
- boundary drift: $a \in \partial C \setminus \{0\}$;
- non-polar exterior drift: $a \in \mathbf{R}^d \setminus (\overline{C} \cup C^{\sharp})$;
- polar boundary drift: $a \in \partial C^{\sharp} \setminus \{0\}$.

Dimension d: general case of drift $a \in \mathbf{R}^d$ (3/3)

Main remark [Garbit & R. '13]

- Universal exponential decay $e^{-\gamma t}$ with $\gamma = \frac{1}{2} d(a, C)^2$;

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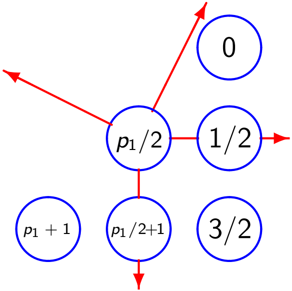
Results in dimension 2 [Garbit & R. '13]



Framework and different approaches

Random walk in C

- Let $(S(n))_{n\geq 0}$ be a *random walk*:

$$S(n) = x + X(1) + \cdots + X(n),$$

  where the $X(i)$ are i.i.d.
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Many different approaches

- *Analytic approach* [Fayolle *et al.* '99];
- *Representation theory* [Biane *et al.* '91];
- *Comparison with BM* [Shimura '84, Garbit '07, Denisov & Wachtel '11].

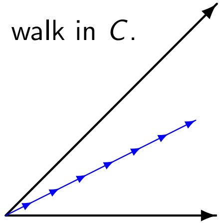
Non-exponential decay of the non-exit probability

Result [Garbit '07]

If $\mathbf{E}[X(i)] = 0$ then

$$\lim_{n\to\infty} \mathbf{P}_x[\tau_C > n]^{1/n} = 1.$$

Proof: Push the random walk in C.



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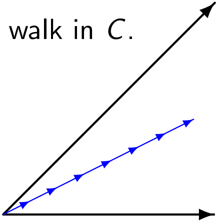

$$\mathbf{P}_0[\tau_C > n] \geq \mathbf{P}_0[X_1 = z, \ldots, X_{\sqrt{n}} = z, \tau_C > n]$$

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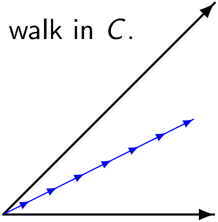

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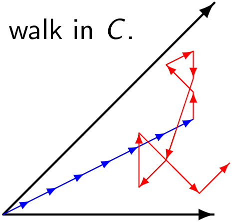

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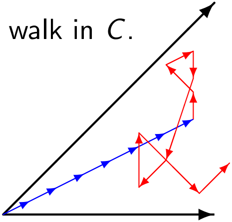

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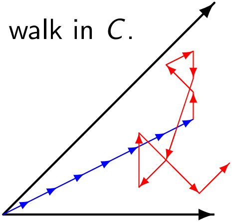

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More precise asymptotics

Upper and lower bounds [Varopoulos '99]

One has

$$c_1 n^{-p_1} \leq \mathbf{P}_x[\tau_C > n] \leq c_2 n^{-p_1},$$

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More [Denisov & Wachtel '11]

- *Local limit theorems*;
- Application in *enumerative combinatorics*;
- *Convergence* of conditioned RW to conditioned BM.

Exponential decay [Garbit & R. '13]

- $C \subset \mathbf{R}^d$ $(d \geq 2)$ is a convex cone, and $-C^\sharp$ is the dual cone;
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Walks in half-spaces

The exponential growth depends on the starting point.

Some ideas of proof

Exponential change of measure (Girsanov or Cramer)

$$\mathbf{P}[X(i) = s] = \mu(s) \qquad \rightarrow \qquad \mathbf{P}^z[X(i) = s] = \mu(s)\frac{e^{\langle s,z\rangle}}{L_\mu(z)}$$

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Lower bound

More complicated: locate the minimum of $L_\mu(z)$ on the dual cone.

Summing local limit theorems

Precise asymptotics in a specific case [Duraj '13]

There is the identity:

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Results [Duraj '13]

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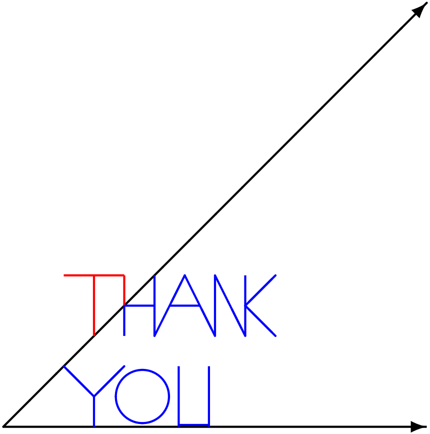

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