Brownian motion with limited local time

M.Savov (joint work with M.Kolb)

In this talk we discuss one dimensional Brownain motion with *zero* drift $(B_s)_{s\geq 0}$ and local time at zero defined as

$$L_t = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t 1_{\{|B_s|\leq\epsilon\}}ds.$$

We know that L_t is a non-decreasing stochastic process which increases only on $\{s \geq 0 : B_s = 0\}$.

We are interested in models in which a deterministic, continuous and increasing function $f : [0, \infty) \mapsto (0, \infty)$ restricts the rate of return to zero L_t, i.e. we shall investigate the conditional measures

$$\mathbb{P}_t(.) = \mathbb{P}\left(.\middle| L_s \leq f(s),\, 0 \leq s \leq t\right).$$

We show for a class of functions f that the Brownian motion with restricted local time exists and describe its properties, i.e. show that

$$\lim_{t \to \infty} \mathbb{P}_t(.) = \mathbb{Q}(.)$$

and discuss $\mathbb{Q}$.

Example: When $f(t) \equiv 0$ then we have that $\mathbb{Q}$ is the law of the three dimensional Bessel process started from 0. This corresponds to the extreme case when the Brownian motion is not allowed to return to zero.

Previous work: This problem is introduced and studied by I. Benjamini and N. Berestycki. Their contributions are the following:

1. Always $(\mathbb{P}_t)_{t\geq 0}$ is a tight sequence of measures
2. If
$$I(f) := \int_0^\infty f(s)\frac{ds}{s^{\frac{3}{2}}} < \infty$$
then every **possible** limit $\mathbb{Q}$ of $\mathbb{P}_t$ is the law of a transient process namely
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They proposed the following open problems:

1. If $I(f) < \infty$ then is it true that $\mathbb{P}_t \to \mathbb{Q}$ and can we explicitly describe $\mathbb{Q}$?
2. If $I(f) = \infty$ then is it true that $\mathbb{P}_t \to \mathbb{Q}$? Is the limiting process recurrent to zero ?
3. If $I(f) = \infty$ can we describe the ***descreasing to zero*** functions $w(t)$ such that

$$\lim_{t\to\infty} \mathbb{Q}\left(L_t \leq w(t)f(t)\right) = 1.$$

$\mathcal{D} = \{w : \lim_{t\to\infty} \mathbb{Q}\left(L_t \leq w(t)f(t)\right) = 1\}$ is called the **entropic repulsion** envelop. *Entropic repulsion* is the phenomenon whence the easiest way for a process to satisfy an imposed condition is to satisfy even more stringent condition.

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The starting point of our study is the following immediate observation

$$\mathcal{O}_t = \{\tau_s \geq g(s), s \leq t\} = \{L_s \leq f(s), s \leq g(t)\},$$

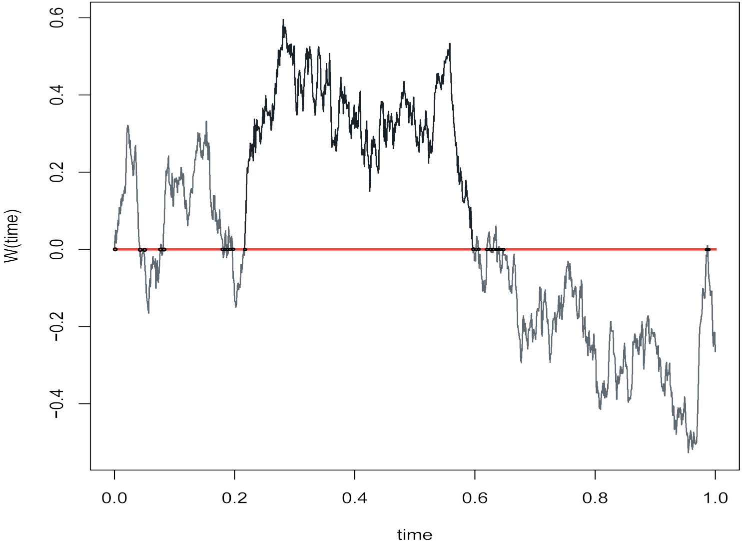
where $\tau = L^{-1}$ is the right inverse of L and $g = f^{-1}$.

We know that $(\tau_t)_{t \geq 0}$ is a stable subordinator (increasing Lévy process) with index $1/2$, namely

$$\mathbb{E}\left[e^{-\lambda \tau_t}\right] = e^{-t\lambda^{\frac{1}{2}}},\ \lambda > 0.$$

Also $\tau_t = \sum_{s \leq t} \Delta\tau_s$. The jumps $\Delta\tau$ are the length of the excursions away from zero.

A Typical Path of Brownian Motion



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$$\lim_{t\to\infty} \mathbb{P}\left(\tau \in .\middle|\mathcal{O}_t\right) = \mathbb{Q}\left(\tau \in .\right)$$

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Some ideas for the proof:

1. Using the Markov property we have that

$$\mathbb{P}\left(\tau_a \in dx \middle| \mathcal{O}_t\right) = \frac{\mathbb{P}\left(\tau_a \in dx, \mathcal{O}_t\right)}{\mathbb{P}\left(\mathcal{O}_t\right)} =$$

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$$\frac{\mathbb{P}\left(\mathcal{O}^{x,a}_{t-a}\right)}{\mathbb{P}\left(\mathcal{O}_t\right)}\mathbb{P}\left(\tau_a \in dx, \mathcal{O}_a\right).$$

2. Above $g^{x,a}(s) = g(s+a) - x$ and

$$\mathcal{O}^{x,a}_t = \{\tau_s > g^{x,a}(s),\ s \le t\}.$$

3. However, to prove recurrence, we need ***uniform*** asymptotic of $\mathbb{P}\left(\mathcal{O}^{x,a}_{t-a}\right)$, so as to apply DCT in

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- Since τ_s is non-decreasing a jump Δ exceeding $g(t)$ at time $T < t$ will ensure that $\mathcal{O}_t = \{\tau_v > g(v), v \leq t\} = \mathcal{O}'_T \cap \{T \leq t\}$.
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$$\Phi(t) = \Phi(t_0)e^{\int_{t_0}^{t} \frac{2K}{\sqrt{g(s)}}ds + \int_{t_0}^{t} R_g(s)ds}$$

- When $\int_{t_0}^{\infty} R_g(s)ds < \infty$ the asymptotic is entirely determined by $\int_{t_0}^{t} \frac{2K}{\sqrt{g(s)}}ds$. When τ is stable with index $\alpha = 1/2$ this is the case when $\liminf_{t\to\infty} g(t)/(t^2 \ln^{\frac{8}{5}+\epsilon}(t)) = \infty$.
- When $\int_{t_0}^{\infty} \frac{2K}{\sqrt{g(s)}}ds < \infty$ we have our criterion $I(f) < \infty$ and $\Phi(\infty) = \int_0^{\infty} \mathbb{P}(\mathcal{O}_s)\, ds < \infty$.
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If $\int_{t_0}^{\infty} R_{g_i}(s) ds < \infty, i = 1, 2$. Then the asymptotic of $\mathbb{P}(\tau_s > g_i(s), s \leq t)$ are directly comparable and tightly related if furthermore

$$\int_{t_0}^{\infty} \left| \frac{1}{\sqrt{g_1(s)}} - \frac{1}{\sqrt{g_2(s)}} \right| ds < \infty.$$

This is how we prove ***uniformity*** in the asymptotics of $\mathbb{P}(\tau_s > g(s+a) - x,\ s \leq t) = \mathbb{P}(\tau_s > g^{x,a}(s),\ s \leq t)$

Under

$$I(f) = \int_1^\infty \frac{f(s)}{s^{\frac{3}{2}}} ds < \infty$$

we have that the limiting measure corresponds to the process:

1. Sample from the random variable X with density $\Phi^{-1}(\infty)\mathbb{P}\left(\mathcal{O}_s\right) ds$
2. Conditional on $X = s$ run a Brownian motion with the restriction $\{L(v) \leq f(v), v \leq s\}$.
3. At the moment s choose with equal probability the value $Y = \pm 1$.
4. We choose independent Bessel process $B^{(3)}$ and from time s we attach $YB^{(3)}$ (*the attachment of this process plays the role of an excursion away of zero of infinite length at time X*).

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When $I(f) = \infty$ and with $g = f^{-1}$. Then when $\liminf_{t\to\infty} g(t)/t^2 \ln^{\frac{8}{5}+\epsilon}(t) = \infty$ we have that:

1. $\mathbb{P}_t \to \mathbb{Q}$
2. Under $\mathbb{Q}$ the process is recurrent at zero. *In this case $\Phi(\infty) = \infty$ and infinite excursion away from zero is not attached*
3. For a function $w(t) \downarrow 0$ we have that

$$\lim_{t\to\infty} \mathbb{Q}\left(L(t) \le f(t)w(t)\right) = 0 \iff \lim_{t\to\infty} \int_t^{f\left(\frac{g(t)}{w(t)}\right)} \frac{1}{\sqrt{g(s)}} ds = 0.$$

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1. $\mathbb{P}_t\to\mathbb{Q}$
2. Under $\mathbb{Q}$ the process is recurrent at zero. *In this case* $\Phi(\infty)=\infty$ *and infinite excursion away from zero is not attached*
3. For a function $w(t)\downarrow 0$ we have that

$$\lim_{t\to\infty}\mathbb{Q}\left(L(t)\leq f(t)w(t)\right)=0 \iff \lim_{t\to\infty}\int_t^{f\left(\frac{g(t)}{w(t)}\right)}\frac{1}{\sqrt{g(s)}}ds=0.$$

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Thank you!