Real roots of random polynomials and zero crossing properties of diffusion equation

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G. S., S. N. Majumdar, Phys. Rev. Lett. **99**, 060603 (2007), arXiv:0705.2648,

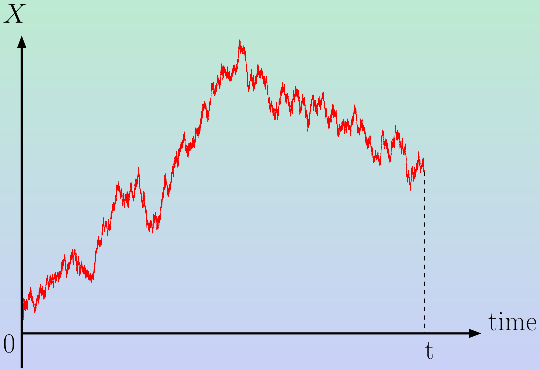
J. Stat. Phys. **132**, 235-273 (2008), arXiv:0803.4396

Introduction

Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time t, $\langle X(t) \rangle = 0$
- Persistence probability

$$p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t$$



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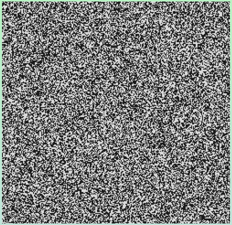
Persistence in spatially extended systems

- phase ordering kinetics

Introduction: Phase ordering kinetics

- Glauber dynamics of $2d$ Ising model at $T = 0$, $H_{\rm Ising} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$
  $\sigma_i = \pm 1$



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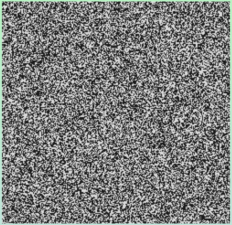

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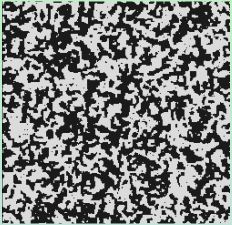

$t_2 = 10^2$

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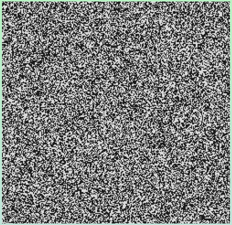

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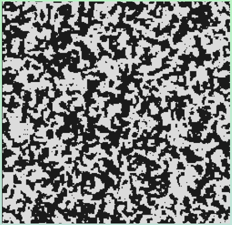

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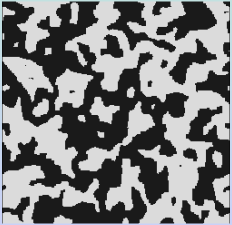

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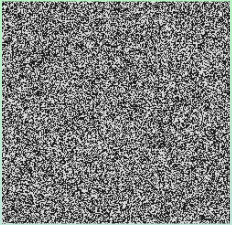

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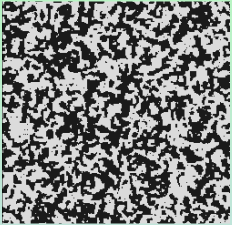

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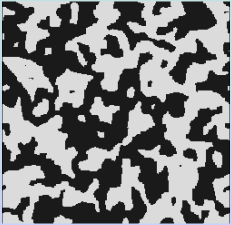

$t_3 = 10^4$

$t_4 = 10^6$

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Persistence in spatially extended systems

- phase ordering kinetics ('94-)

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- height of a fluctuating interface ('97-)
- ...

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$$p_0(t) \propto t^{-\theta_p}$$

A. J. Bray, S. N. Majumdar, G. S., Adv. Phys. **62**, pp 225-361 (2013), arXiv:1304.1195

"Persistence and First-Passage Properties in Non-equilibrium Systems"

Motivations : persistence for the diffusion equation

Diffusion equation with random initial conditions

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x-x')$$

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- Diffusion equation (or heat equation) is universal and ubiquitous in nature
- Ordering dynamics for $O(N)$-symmetric spin models in the limit $N \to \infty$
- see A. Dembo, S. Mukherjee 12

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, PRL 96

B. Derrida, V. Hakim and R. Zeitak, PRL 96

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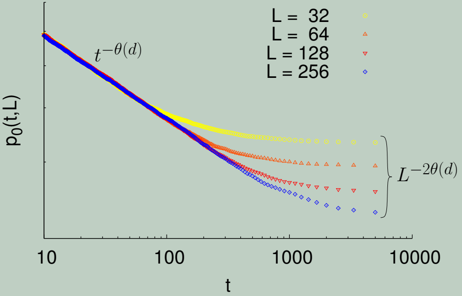

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$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

$$\theta(1) = 0.1207$$
$$\theta(2) = 0.1875 \quad , \quad \text{Numerics}$$

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STATISTICAL MECHANICS

Persistence Pays Off in Defining History of Diffusion

A. Watson, Science 96

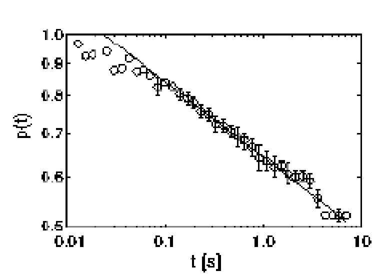
Persistence in 1d diffusion : NMR experiments on Xe

VOLUME 86, NUMBER 18 PHYSICAL REVIEW LETTERS 30 APRIL 2001

Measurement of Persistence in 1D Diffusion

Glenn P. Wong, Ross W. Mair, and Ronald L. Walsworth
Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, Massachusetts 02138

David G. Cory
Department of Nuclear Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 1 September 2000)



$$p_0(t,L) \propto t^{-\theta_{\exp}(1)}$$
$$\theta_{\exp}(1) \simeq 0.12$$

Motivations : Real random polynomials

Real Kac's polynomials

$$K_n(x) = \sum_{i=0}^{n-1} a_i x^i$$

$a_i \equiv$ Gaussian random variables,
$\langle a_i \rangle = 0$, $\langle a_i a_j \rangle = \delta_{ij}$

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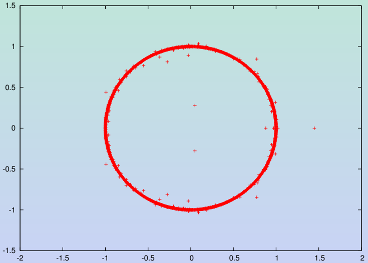
Complex roots



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Real roots

$\mathcal{N}_n \equiv$ mean number of roots on the real axis M.Kac '43

$$\mathcal{N}_n \sim \frac{2}{\pi} \log n$$

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RANDOM POLYNOMIALS HAVING FEW OR NO REAL ZEROS

AMIR DEMBO, BJORN POONEN, QI-MAN SHAO, AND OFER ZEITOUNI

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) \propto n^{-\gamma}$$

$$\text{with} \quad \gamma = 0.19(1) \quad \text{(Numerics)}$$

Purpose : a link between random polynomials & diffusion equation

Generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$a_i \equiv$ Gaussian random variables, $\langle a_i \rangle = 0$, $\langle a_i a_j \rangle = \delta_{ij}$

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

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G. S., S. N. Majumdar 07

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A. Dembo, S. Mukherjee 12

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   - The case of diffusion equation
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Persistence of diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x-x')$$

$$\phi(x,t) = \int_{|y|<L} d^d y G(x-y,t)\phi(y,0)$$

$$G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

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Mapping of $\phi(x,t)$ to a Gaussian stationary process

1. Normalized process $X(t) = \frac{\phi(x,t)}{\langle \phi(x,t)^2\rangle^{1/2}}$

$$\langle X(t)X(t')\rangle \sim \begin{cases} \left(4\frac{tt'}{(t+t')^2}\right)^{\frac{d}{4}}, & t,t' \ll L^2 \\ 1, & t,t' \gg L^2 \end{cases}$$

2. New time variable $T = \log t$, for $t \ll L^2$

$$\langle X(T)X(T')\rangle = \left[\cosh((T-T')/2)\right]^{-d/2}$$

Persistence for a Gaussian stationary process (GSP)

- $X(T)$ is a GSP with correlations

$$\begin{aligned} \langle X(T)X(T')\rangle &= a(T-T') \\ a(T) &= (\cosh(T/2))^{-d/2} \end{aligned}$$

- Persistence probability $\mathcal{P}_0(T)$ (by Slepian's lemma)

For $T \gg 1 \quad a(T) \propto \exp\left(-\frac{d}{2}T\right) \Rightarrow \mathcal{P}_0(T) \propto \exp\left(-\theta(d)T\right)$

- Reverting back to $t = \exp(T)$

$$p_0(t, L) \sim t^{-\theta(d)} \quad 1 \ll t \ll L^2$$

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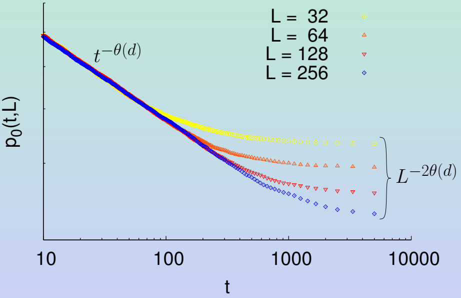

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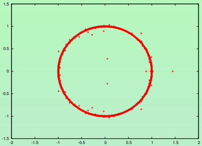
$$p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \qquad h(u) \sim \begin{cases} u \sim u^{-\theta(d)} \, , & u \ll 1 \\ u \sim c^{\rm st} \, , & u \gg 1 \end{cases}$$

Outline

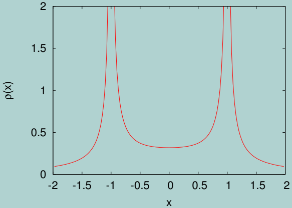
Real roots of generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$



Averaged density of real roots for $n \to \infty$

$$\rho_\infty(x) = \frac{\left(\mathrm{Li}_{-1-d/2}(x^2)(1+\mathrm{Li}_{1-d/2}(x^2)) - \mathrm{Li}^2_{-d/2}(x^2)\right)^{\frac{1}{2}}}{\pi |x| (1+\mathrm{Li}_{1-d/2}(x^2))}$$



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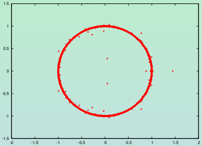

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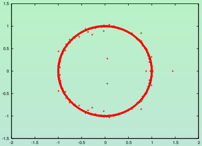
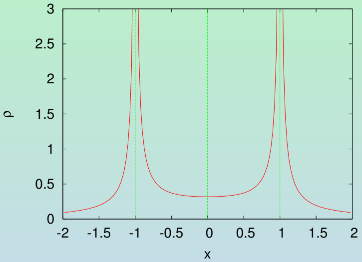


Mean number of real roots in [0, 1] : Kac-Rice formula

$$\langle N_n[0,1] \rangle = \int_0^1 \rho_n(x)\, dx \sim \frac{1}{2\pi}\sqrt{\frac{d}{2}} \log n$$

Probability of no real root for $K_n(x)$



Dembo *et al.* '02

Statistical independence of $K_n(x)$ in the 4 sub-intervals

$\Longrightarrow$ Focus on the interval $[0, 1]$

$$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$$

Probability of no real root for $K_n(x)$

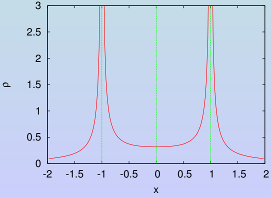
- Two-point correlator

$$C_n(x,y) = \langle K_n(x) K_n(y) \rangle = \sum_{i=0}^{n-1} i^{(d-2)/2} (xy)^i$$

- Normalization

$$\mathcal{C}_n(x,y) = \frac{C_n(x,y)}{(C_n(x,x))^{\frac{1}{2}} (C_n(y,y))^{\frac{1}{2}}}$$

- Change of variable



$$x = 1 - \frac{1}{t} \quad , \quad t \gg 1$$

Probability of no real root for $K_n(x)$

Normalized correlator in the scaling limit

- Scaling limit

$$t \gg 1 \quad , \quad n \gg 1 \quad \text{keeping} \quad \tilde{t} = \frac{t}{n} \quad \text{fixed}$$

- $\mathcal{C}_n(t, t') \to \mathcal{C}(\tilde{t}, \tilde{t}')$ with the asymptotic behaviors

$$\mathcal{C}(\tilde{t}, \tilde{t}') \sim \begin{cases} \left(4 \frac{\tilde{t}\tilde{t}'}{(\tilde{t} + \tilde{t}')^2}\right)^{\frac{d}{4}} , & \tilde{t}, \tilde{t}' \ll 1 \\ 1 , & \tilde{t}, \tilde{t}' \gg 1 \end{cases}$$

Persistence of diffusion equation (reminder)

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x-x')$$

$$\phi(x,t) = \int d^d y G(x-y)\phi(y,0)$$
$$G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

Mapping of $\phi(x,t)$ to a Gaussian stationary process

1. Normalized process $X(t) = \frac{\phi(x,t)}{\langle \phi(x,t)^2\rangle^{1/2}}$

$$\langle X(t)X(t')\rangle \sim \begin{cases} \left(4\frac{tt'}{(t+t')^2}\right)^{\frac{d}{4}}, & t,t' \ll L^2 \\ 1\,, & t,t' \gg L^2 \end{cases}$$

2. Persistence probability $p_0(t,L)$

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

Probability of no real root for $K_n(x)$

$$\mathcal{C}(\tilde{t},\tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}} , & \tilde{t},\tilde{t}' \ll 1 \\ 1 , & \tilde{t},\tilde{t}' \gg 1 \end{cases}$$

$P_0(x,n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0,x]$

Scaling form for $P_0(x,n)$

$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

$$\tilde{h}(u) \sim \begin{cases} c^{st} , & u \ll 1 \\ u^{\theta(d)} , & u \gg 1 \end{cases}$$

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$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) = P(1, n) \sim n^{-\theta(d)}$$

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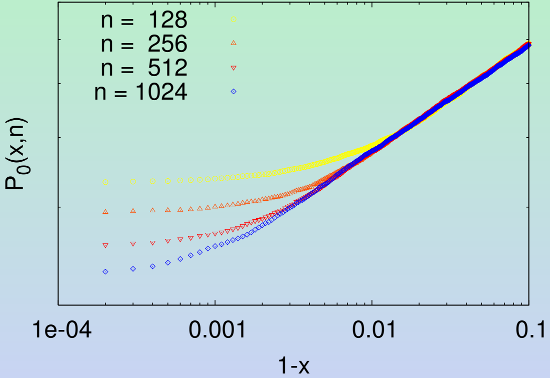
Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$



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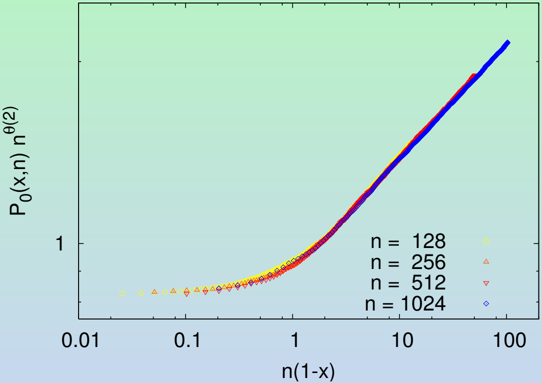


$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

Outline

Conclusion

- A link between diffusion equation and random polynomials

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

$$b(d) = \theta(d)$$

1. Universality see A. Dembo, S. Mukherjee 12
2. Towards exact results for $\theta(d)$, $1/(4\sqrt{3}) \leq \theta(2) \leq 1/4$

   G. Molchan 12 W. Li, Q. M. Shao 02

   see also D. Zaporozhets 06

Outline

Generalization to k zero crossings

- Diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x - x')$$

$p_k(t, L) \equiv$ Proba. that $\phi(x,t)$ crosses zero k times up to t

- Real polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$q_k(n) \equiv$ Proba. that $K_n(x)$ has exactly k real roots

Outline

Mean field approximation for diffusion equation

- For $t \ll L^2$, $p_k(t, L)$ is given by $\mathcal{P}_k(T)$, $T = \log t$

$$\mathcal{P}_k(T) \equiv \text{Proba. that } X(T) \text{ crosses zero } k \text{ times up to } T$$

- $\langle X(T)X(T')\rangle = (\cosh(T - T'))^{-d/2} = 1 - \frac{d}{16}T^2 + o(T^2)$

$$\begin{aligned} \langle \mathcal{N}(T)\rangle &\equiv \text{Mean number of zero crossings in } [0, T] \\ &= \rho T \qquad\qquad \rho = \frac{1}{2\pi}\sqrt{\frac{d}{2}} \end{aligned}$$

- Assuming that the zeros of $X(T)$ are independent

$$\mathcal{P}_k(T) = \frac{(\rho T)^k}{k!} e^{-\rho T}$$

Mean field approximation for diffusion equation

- For $t \ll L^2$, $p_k(t, L)$ is given by $\mathcal{P}_k(T)$, $T = \log t$

$$\mathcal{P}_k(T) \equiv \text{Proba. that } X(T) \text{ crosses zero } k \text{ times up to } T$$

- Assuming that the zeros of $X(T)$ are independent

$$\mathcal{P}_k(T) = \frac{(\rho T)^k}{k!} e^{-\rho T}$$

- For $k \gg 1$, $T \gg 1$, $k/\rho T$ fixed

$$\log \mathcal{P}_k(T) \sim -T\varphi\left(\frac{k}{\rho T}\right) \quad , \quad \varphi(x) = \rho(x \log x - x + 1)$$

Scaling form for a smooth GSP

$X(T)$ is a GSP with $\langle X(T)X(T') = [\mathrm{sech}((T - T')/2)]^{d/2}$

$\mathcal{P}_k(T) \equiv$ Proba. that $X(T)$ crosses 0 exactly k times up to T

$$\log \mathcal{P}_k(T) \sim -T\varphi\left(\frac{k}{\rho T}\right)$$

$\varphi(x)$ is a large deviation function

Outline

A more refined analysis

- Generating function **S.N. Majumdar, A.J. Bray, PRL'98**

$$\hat{\mathcal{P}}(z,T) = \sum_{k=0}^{\infty} z^k \mathcal{P}(k,T) \sim \exp(-\hat{\theta}(z)T)$$

where $\hat{\theta}(z)$ depends continuously on z

Application to the integrated Brownian motion

- Integrated Brownian motion (Random acceleration process)

$$\frac{d^2x(t)}{dt^2} = \zeta(t) \qquad , \qquad \langle \zeta(t)\zeta(t') \rangle = \delta(t - t')$$

- Generating function T.W. Burkhardt '01

$$\hat{\mathcal{P}}(z,T) = \sum_{k=0}^{\infty} z^k \mathcal{P}(k,T) \sim \exp(-\hat{\theta}(z)T)$$

$$\hat{\theta}(z) = \frac{1}{4}\left(1 - \frac{6}{\pi}\mathrm{ArcSin}\left(\frac{z}{2}\right)\right) \quad , \quad 0 \leq z \leq 2$$

Application to the integrated Brownian motion

- Integrated Brownian motion (Random acceleration process)

$$\frac{d^2 x(t)}{dt^2} = \zeta(t) \qquad , \qquad \langle \zeta(t)\zeta(t') \rangle = \delta(t - t')$$

- Exact result G. S., S. N. Majumdar 08

$$p_k(t) \sim t^{-\varphi\left(\frac{k}{\rho \log t}\right)} \; , \; \rho = \frac{\sqrt{3}}{2\pi}$$

$$\varphi(x) = \frac{\sqrt{3}}{2\pi} x \log\left(\frac{2x}{\sqrt{x^2+3}}\right) + \frac{1}{4}\left(1 - \frac{6}{\pi}\mathrm{ArcSin}\left(\frac{x}{\sqrt{x^2+3}}\right)\right)$$

Probability of k-zero crossings for diffusion equation

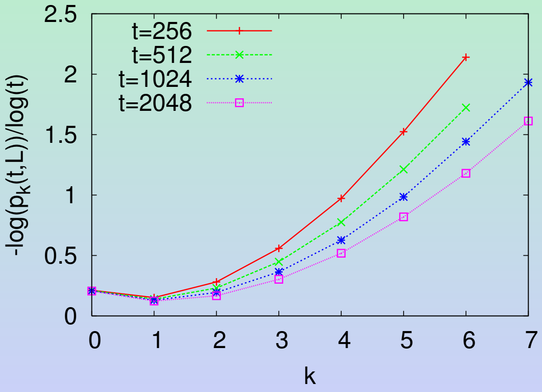
$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x-x')$$

$p_k(t,L) \equiv$ Proba. that $\phi(x,t)$ crosses zero k times up to t

$$p_k(t,L) \sim t^{-\varphi\left(\frac{k}{\log t}\right)} \quad , \quad t \ll L^2$$

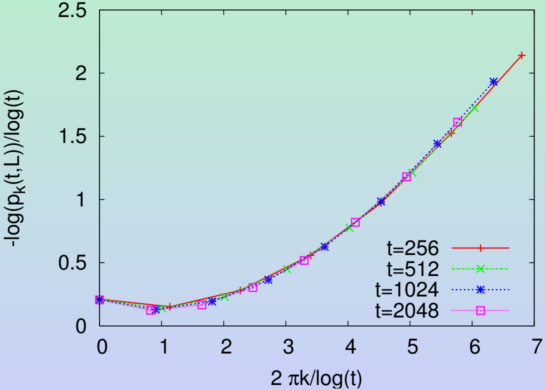
Numerical check

$$-\frac{\log p_k(t,L)}{\log t} \quad \text{as a function of} \quad k$$



Numerical check

$$-\frac{\log p_k(t,L)}{\log t} \quad \text{as a function of} \quad \frac{k}{\log t}$$



Outline

Conclusion

- Large deviation function for random polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$q_k(n) \equiv$ Proba. that $K_n(x)$ has exactly k real roots in $[0, 1]$

$$q_k(n) \propto n^{-\varphi\left(\frac{k}{\log n}\right)}$$

- Extension to other class of polynomials
- Extension to real eigenvalues of real random matrices (Ginibre's matrices)

A heuristic argument

- Diffusion equation

$$\begin{aligned}
\phi(x=0,t) &= (4\pi t)^{-d/2} \int_{0<|\mathbf{x}|<L} d^d\mathbf{x}\ \exp\left(-\frac{\mathbf{x}^2}{4t}\right) \phi(\mathbf{x},0) \\
&= \frac{S_d^{1/2}}{(4\pi t)^{d/2}} \int_0^L dr\ r^{\frac{1}{2}(d-1)} e^{-\frac{r^2}{4t}}\ \Psi(r) \\
\Psi(r) &= S_d^{-1/2} r^{-\frac{1}{2}(d-1)} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \int_{r<|\mathbf{x}|<r+\Delta r} d^d\mathbf{x}\ \phi(\mathbf{x},0) \\
\langle \Psi(r)\Psi(r')\rangle &= \delta(r-r')
\end{aligned}$$

A heuristic argument

- Diffusion equation

$$\begin{aligned}\phi(x=0,t) &\propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u) \\ \langle \tilde{\Psi}(u)\tilde{\Psi}(u')\rangle &= \delta(u-u')\end{aligned}$$

A heuristic argument

- Diffusion equation

$$\begin{aligned}\phi(x=0,t) &\propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u) \\ \langle \tilde{\Psi}(u)\tilde{\Psi}(u')\rangle &= \delta(u-u')\end{aligned}$$

- Random polynomials : $K_n(x) = a_0 + \sum_{i=1}^n a_i i^{\frac{d-2}{4}} x^i$

$$\begin{aligned}K_n(1-1/t) &\sim a_0 + \sum_{i=1}^n i^{\frac{d-2}{4}} e^{-\frac{i}{t}} a_i \\ &\sim \int_0^n du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}} a(u) \\ \langle a(u)a(u')\rangle &= \delta(u-u')\end{aligned}$$