Windings of the stable Kolmogorov process[a]

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[a] Joint work with Christophe Profeta (Evry-Val d'Essonne)

Some background

A classic result by F. Spitzer (1956) states that the angular part $\{\omega(t),\, t \geq 0\}$ of a two-dimensional Brownian motion starting away from the origin satisfies the following limit theorem

$$\frac{2\,\omega(t)}{\log t} \xrightarrow{d} \mathcal{C} \qquad \text{as } t \to +\infty,$$

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where $\mathcal{C}$ denotes the standard Cauchy law. An analogue of this result for planar isotropic $\alpha-$stable Lévy processes was obtained by Bertoin and Werner (1996), who showed that

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where $\mathcal{G}_\alpha$ is a centered Gaussian limit law whose variance depends on α. Notice that both theorems can be obtained as a functional limit theorem in the Skorohod topology. Doney and Vakeroudis (2013) revisit these problems, with further results and an updated bibliography.

The Kolmogorov diffusion

The Kolmogorov diffusion is the two-dimensional process $Z = (A, B)$, where $B = \{B_t, t \geq 0\}$ is a linear Brownian motion and

$$A_t \; = \; \int_0^t B_s \, ds$$

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its primitive. It was introduced by Kolmogorov (1934), who considered the more general process on $\mathbb{R}^n$ made out of the successive primitives of order p of a linear Brownian motion, $0 \leq p \leq n-1$. It is a Feller process with infinitesimal generator

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Nowadays, this process is still an object of research in analysis or in probability, as a toy model for degenerate diffusion processes. See e.g. Kendall et al. (2004) for coupling theorems and Hamel et al. (2014) for Harnack inequalities.

Windings of the Kolmogorov diffusion

McKean (1963) investigated the asymptotic behaviour of the windings of the Kolmogorov diffusion, and showed the following almost sure limit theorem:

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$$T_0^{(1)} = \inf\{t > 0,\ A_t = 0\} \quad \text{and} \quad T_0^{(n)} = \inf\{t > T_0^{(n-1)},\ A_t = 0\},$$

and computed the exponential rate of escape of this sequence towards infinity. These successive passage times have been studied further by Lachal (1997).

The stable Kolmogorov process

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$$\log(\mathbb{E}[e^{\mathrm{i}\lambda L_1}]) \;=\; -(\mathrm{i}\lambda)^{\alpha} e^{-\mathrm{i}\pi\alpha\rho\,\mathrm{sgn}(\lambda)}, \qquad \lambda \in \mathbb{R},$$

where $\alpha \in (0, 2]$ is the self-similarity parameter and $\rho = \mathbb{P}[L_1 \geq 0]$ is the positivity parameter.

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where $\alpha \in (0, 2]$ is the self-similarity parameter and $\rho = \mathbb{P}[L_1 \geq 0]$ is the positivity parameter. When $\alpha = 2$, then necessarily $\rho = 1/2$ and $L = \{\sqrt{2}B_t,\; t \geq 0\}$ is a Brownian motion. When $\alpha = 1$, the process L is a Cauchy process, with or without drift. When $\alpha \in (0, 1) \cup (1, 2)$, the positivity parameter ρ reads

$$\rho \;=\; \frac{1}{2} \;+\; \frac{1}{\pi\alpha}\,\arctan(\beta \tan(\pi\alpha/2)),$$

where β is the skewness parameter of L_1 (Zolotarev's formula).

The case when $|L|$ is a subordinator

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$$(\widehat{Z_0, \mathrm{O}x}) \in (-2\pi, \pi/2),$$

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If $\rho = 0$, the process $t \mapsto -L_t$ is a stable subordinator and the stable Kolmogorov process enters the negative quadrant after a while, $\omega(t)$ converging a.s. to the finite limit

$$(\widehat{Z_0, \mathrm{O}x}) \pm \pi \in (-2\pi, \pi/2).$$

The harmonic measure of the half-planes

We henceforth suppose $\rho \in (0, 1)$, so that the processes L and A oscillate, taking arbitrary large positive or negative values. We also suppose that Z does not start at $(0, 0)$. We partition the punctured plane into

$$\mathcal{P}_- \; = \; \{x < 0\} \cup \{x = 0, y < 0\} \qquad \text{and} \qquad \mathcal{P}_+ \; = \; \{x > 0\} \cup \{x = 0, y > 0\}.$$

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When Z starts in $\mathcal{P}_-$, it eventually enters $\mathcal{P}_+$ through the half-line $\{x = 0, y > 0\}$. The family of probability measures

$$\mathbb{P}_{(x,y)}[L_{T_0} \in .], \qquad (x,y) \in \mathcal{P}_-,$$

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is called the harmonic measure of the left half-plane. Similarly, when Z starts in $\mathcal{P}_+$, it eventually enters $\mathcal{P}_-$ through the half-line $\{x = 0, y < 0\}$ and the family of probability measures $\mathbb{P}_{(x,y)}[|L_{T_0}| \in .], (x,y) \in \mathcal{P}_+$, is the harmonic measure of the right half-plane.

An exact computation

Introduce the two parameters

$$\gamma = \frac{\rho\alpha}{1+\alpha} \in (0, 1/2) \qquad \text{and} \qquad \mu = \frac{\rho\alpha}{1+\alpha(1-\rho)} \in (0, 1).$$

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Then, for every $y < 0$, under $\mathbb{P}_{(0,y)}$ we show that

$$L_{T_0} \stackrel{d}{=} |y|(\mathbf{C}_\mu^{1-\gamma})^{(1)}$$

where $\mathbf{C}_\mu$ is a half-Cauchy random variable with density

$$\frac{\sin(\pi\mu)}{\pi\mu(x^2 + 2\cos(\pi\mu)x + 1)} \mathbf{1}_{\{x \geq 0\}}$$

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and the size bias $X^{(1)}$ of the integrable random variable $X = \mathbf{C}_\mu^{1-\gamma}$ is defined by multiplying its density by x and renormalizing. For $\alpha = 2$, this formula had been obtained by McKean (1963).

Some uniform estimates

The law of L_{T_0} under $\mathbb{P}_{(x,y)}$ with $x < 0$ is more complicated. Its Mellin transform

$$s \mapsto \mathbb{E}_{(x,y)}[L_{T_0}^{s-1}]$$

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$$f_{x,y}(z) \underset{z\to 0}{\sim} c_1\, z^{\mu/\gamma} \qquad \text{and} \qquad f_{x,y}(z) \underset{z\to +\infty}{\sim} c_2\, z^{-\mu-1}.$$

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The same result holds for $(x, y) \in \mathcal{P}_+$, replacing ρ by $1 - \rho$ in the definition of γ and μ. Notice that the rough estimate

$$\mathbb{P}_{(x,y)}[L_{T_0} > z] \asymp z^{-\mu} \quad \text{as } z \to +\infty$$

is crucial to obtain the persistence exponent of A (see next talk).

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Our main result is the following.

Theorem. *Assume* $\rho \in (0,1)$ *and* $(x,y) \neq (0,0)$. *Then, under* $\mathbb{P}_{(x,y)}$, *one has*

$$\frac{\omega(t)}{\log t} \;\stackrel{a.s.}{\longrightarrow}\; -\frac{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})}{\alpha\sin(\pi(\gamma+\overline{\gamma}))} \qquad \text{as } t \to +\infty.$$

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Notice that in the Brownian case $\alpha = 2$, we have $\gamma = \overline{\gamma} = 1/3$, so that

$$\frac{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})}{\alpha\sin(\pi(\gamma+\overline{\gamma}))} \;=\; \frac{\sqrt{3}}{2}$$

and we recover McKean's result.

Elements of proof I

By symmetry, we can suppose $(x, y) \in \mathcal{P}_-$. We consider the Markovian sequence

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By self-similarity, the Markov property and an induction we obtain the identities

$$|L_{T_0^{(2p)}}| \stackrel{d}{=} |L_{T_0}| \times \prod_{k=1}^{p-1} \ell_k^- \times \prod_{k=1}^{p} \ell_k^+ \quad \text{and} \quad |L_{T_0^{(2p-1)}}| \stackrel{d}{=} |L_{T_0}| \times \prod_{k=1}^{p-1} \ell_k^- \times \prod_{k=1}^{p-1} \ell_k^+$$

for all $p \geq 1$, where $\{\ell_k^+, k \geq 1\}$ and $\{\ell_k^-, k \geq 1\}$ are two i.i.d. sequences distributed as L_{T_0} under $\mathbb{P}_{(0,1)}$, resp. $|L_{T_0}|$ under $\mathbb{P}_{(0,-1)}$.

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for all $p \geq 1$, where $\{\ell_k^+, k \geq 1\}$ and $\{\ell_k^-, k \geq 1\}$ are two i.i.d. sequences distributed as L_{T_0} under $\mathbb{P}_{(0,1)}$, resp. $|L_{T_0}|$ under $\mathbb{P}_{(0,-1)}$. The above exact computation shows that $\log(\ell^{\pm})$ has finite exponential moments of both signs and that

$$\mathbb{E}\left[\log(\ell^-)\right] = \pi \cot(\pi\gamma) \qquad \text{and} \qquad \mathbb{E}\left[\log(\ell^+)\right] = \pi \cot(\pi\overline{\gamma}).$$

Elements of proof II

Setting $\theta_0 = \widehat{Z_0 Z_{T_0}}$, we observe the a.s. identifications

$$\{\omega(t) \geq -(n-1)\pi + \theta_0\} \; = \; \{T_0^{(n)} \geq t\}$$

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Hence, the Theorem amounts to show that

$$\frac{1}{n}\log(T_0^{(n)}) \;\xrightarrow{a.s.}\; \frac{\pi\alpha\sin(\pi(\gamma+\overline{\gamma}))}{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})} \;=\; \frac{\pi\alpha}{2}(\cot(\pi\gamma)\;+\;\cot(\pi\overline{\gamma})) \quad \text{as } n \to +\infty.$$

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We set $S_n = T_0^{(n)} - T_0^{(n-1)}$. By self-similarity and the Markov property,

$$S_{2p} \;\stackrel{d}{=}\; |L_{T_0}|^{\alpha} \;\times\; \tau^{+} \;\times\; \left(\prod_{k=1}^{p-1} \ell_k^{-} \;\times\; \ell_k^{+}\right)^{\alpha}$$

with τ^+ distributed as T_0 under $\mathbb{P}_{(0,1)}$.

Elements of proof III

Putting everything together, we deduce from the law of large numbers and an elementary large deviation estimate the required a.s. lower bound:

$$\liminf_{n\to\infty} \frac{1}{n} \log(T_0^{(n)}) \;\geq\; \liminf_{n\to\infty} \frac{1}{2p} \log(S_{2p}) \;\geq\; \frac{\pi\alpha}{2}(\cot(\pi\gamma) + \cot(\pi\overline{\gamma})) \;=\; \kappa_{\alpha,\rho}.$$

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To obtain the upper bound, we write

$$\begin{aligned} \mathbb{P}_{(x,y)}\left[T_0^{(n)} \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon)}\right] &\leq \sum_{k=1}^{n} \mathbb{P}_{(x,y)}\left[S_k \geq n^{-1} e^{n(\kappa_{\alpha,\rho}+\varepsilon)}\right] \\ &\leq \sum_{k=1}^{n} \mathbb{P}_{(x,y)}\left[S_k \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon/2)}\right] \end{aligned}$$

for every $\varepsilon > 0$ and n large enough.

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for every $\varepsilon > 0$ and n large enough. We then proceed similarly with Cramér's theorem, thanks to the analogous identity in law for S_{2p-1}.