One-sided boundary crossing for random walks

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(joint work with Denis Denisov)

Let S_n be a random walks with independent identically distributed increments X_i. Let T_x denote the first time when S_n crosses $-x$, that is,

$$T_x := \inf\{n \ge 1 : S_n < -x\}.$$

It is well known that if $\mathbf{P}(S_n > 0) \to \rho \in (0, 1)$ then

$$\mathbf{P}(T_x > n) \sim h(x) n^{\rho-1} L(n),$$

where h is the renewal function corresponding to strict decreasing ladder heights of our random walk.

(The proof of this relation is based on the Wiener-Hopf factorisation.)

Let $g(t)$ be an increasing function and consider

$$T_g := \inf\{n \geq 1 : S_n < -g(n)\}.$$

Wiener-Hopf identity for T_g is known under additional restrictions on $g(t)$. Moreover, this identity for curved boundaries is not very useful.

Brownian motion.

Uchiyama (1980), Novikov (1981) and Gärtner (1982):

$$0 < \mathbf{E}[|B(T_g^{(bm)})|] < \infty \iff \int_1^\infty g(t)t^{-3/2}dt < \infty$$

and, moreover,

$$\mathbf{P}(T_g^{(bm)} > t) \sim \mathbf{E}[B(T_g^{(bm)})]\mathbf{P}(T_1^{(bm)} > t) \text{ as } t \to \infty.$$

Finiteness of the integral is also necessary: if $\int_1^\infty g(t)t^{-3/2}dt = \infty$ then

$$\mathbf{P}\{T_g^{(bm)} > t\} \gg \mathbf{P}\{T_1^{(bm)} > t\} \quad \text{as } t \to \infty.$$

Random walks.

Novikov (1981), Novikov (1982) and Novikov and Greenwood (1986): If g is concave and $\mathbf{E}g(T_0) < \infty$ then there exists $R_g \in (0, \infty)$ such that

$$\mathbf{P}(T_g > n) \sim R_g \mathbf{P}(T_0 > n)$$

Since $\mathbf{P}(T_0 > n) \sim n^{\rho-1} L(n)$,

$$\mathbf{E}g(T_0) < \infty \quad \Leftrightarrow \quad \int_1^\infty g(t) \frac{L(t)}{t^{2-\rho}} dt < \infty.$$

Therefore, one can not take $g(t) \geq t^{1-\rho+\varepsilon}$.

Aurzada, Kramm and Savov (2012), Aurzada and Kramm (2013):

If S_n is asymptotically stable and $g(t) = t^\gamma$ with some $\gamma < 1/\alpha$ then

$$\mathbf{P}(T_g > n) = n^{\rho-1+o(1)}.$$

This relation gives strong grounds to expect that $\mathbf{P}(T_g > n)$ and $\mathbf{P}(T_0 > n)$ are asymptotically equivalent under a weaker than $\mathbf{E}g(T_0) < \infty$ condition.

Theorem 1. Assume that $\mathbf{P}(S_n > 0) \to \rho \in (0, 1)$. If $h(g(x))$ is subadditive and $\mathbf{E}h(g(T_0)) < \infty$ then there exists $V(g) \in (0, \infty)$ such that

$$\mathbf{P}(T_g > n) \sim V(g)\mathbf{P}(T_0 > n).$$

Remarks

- If S_n is asymptotically stable then

$$\mathbf{E}h(g(T_0)) < \infty \quad \Leftrightarrow \quad \int_1^\infty \frac{h(g(t))}{th(c(t))}dt < \infty.$$

- The theorem is valid if $h(g(x)$ posesses a subadditive majorant $r(x)$ such that $\mathbf{E}r(T_0) < \infty$. In particular,Theorem 1 is applicable to all functions $g(t) \leq t^\gamma$ with some $\gamma < 1/\alpha$.

- Since every renewal function is subadditive, concavity of $g(x)$ implies that $h(g(x))$ is subadditive. Therefore, our condition is weaker than that in Greenwood and Novikov (1986).

- Mogulskii and Pecherskii (1979): If g is superadditive, i.e., $g(x+y) \geq g(x) + g(y)$, then there exists a sequence of events E_n such that

$$\sum_{n=0}^{\infty} z^n \mathbf{P}(T_g > n) = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}(E_n)\right\}.$$

Moreover,

$$E_n \subseteq \{S_n \geq -g(n)\} \quad \text{for all } n \geq 1$$

and

$$E_n = \{S_n \geq -g(n)\} \quad \text{for linear } g(t).$$

From the upper bound for E_n we get

$$\mathbf{P}(T_g > n) \leq q_n,$$

where q_n is determined by

$$\sum_{n=0}^{\infty} z^n q_n = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}(S_n \geq -g(n))\right\}.$$

Assume that function g satisfies

$$\int_1^\infty \frac{g(t)}{tc(t)}dt < \infty.$$

Then applying the estimate

$$\mathbf{P}(S_n \in [x, x+1)) \le \frac{C}{c(n)},$$

we conclude that coefficients of

$$R(z) := \exp\left\{\sum_{n=1}^\infty \frac{z^n}{n}\mathbf{P}(S_n \in [-g(n), 0]\right\}$$

are summable, i.e., $R(1) < \infty$. Noting now that

$$\sum_{n=0}^\infty z^n q_n = \left(\sum_{n=0}^\infty z^n \mathbf{P}(T_0 > n)\right) R(z),$$

we arrive at the relation

$$q_n \sim R(1)\mathbf{P}(T_0 > n)$$

and, consequently,

$$1 \le \liminf_{n\to\infty} \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(T_0 > n)} \le \liminf_{n\to\infty} \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(T_0 > n)} \le R(1).$$

Note also that in order to obtain the relation $\mathbf{P}(T_g > n) \sim C\mathbf{P}(T_0 > n)$ it suffices to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|\mathbf{P}(E_n) - \mathbf{P}(S_n > 0)\right| < \infty.$$

The starting point of the proof of Theorem 1 is the following simple observation: T_g coincides with one of strict descending ladder epochs of S_n. Let (τ_k, χ_k) be independent copies of $(T_0, -S_{T_0})$. Then

$$T_g = \sum_{k=1}^{\nu} \tau_k,$$

where

$$\nu := \min\{k \geq 1 : \chi_1 + \cdots + \chi_k > g(\tau_1 + \cdots + \tau_k)\}.$$

Since the tail distribution function of τ's is regularly varying with index $\rho - 1 \in (-1, 0)$, we prove that, for any increasing function g,

$$\lim_{n\to\infty} \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(T_0 > n)} = \mathbf{E}\nu \in [1, \infty].$$

Thus, to prove Theorem 1 it suffices to show that $\mathbf{E}\nu$.

By the subadditivity of $h(g(x))$,

$$\nu \le \mu := \min\{k \ge 1 : h(\chi_1 + \ldots + \chi_k) > h(g(\tau_1)) + \ldots + h(g(\tau_k))\}.$$

The random walk on LHS has finite expectation. Thus, the finiteness of $\mathbf{E}\mu$ will follow from the relation $\mathbf{E}h(\chi) = \infty$. But the latter is equivalent to $\mathbf{E}\chi = \infty$.

In case of finite $\mathbf{E}\chi$ we use a bit different upper bound for ν: Fix some $k \ge 1$ and consider

$$Z^{(k)} := \chi_1 + \chi_2 + \ldots + \chi_k - g(\tau_1 + \tau_2 + ... + \tau_k)$$

and let $Z_i^{(k)}$ be independent copies of Z. Using the subadditivity of g once again, one can easily show that

$$\nu \le k\mu^{(k)},$$

where

$$\mu^{(k)} := \min\{n \ge 1 : Z_1^{(k)} + Z_2^{(k)} + \ldots + Z_n^{(k)} > 0\}.$$

The subadditivity assumption in our Theorem 1 seems to be purely technical.

How can one obtain the same asymptotic behaviour without this assumption?

Theorem 2. Assume that S_n is asymptotically stable. If

$$\int_0^\infty \frac{h(g(t))}{th(c(t/\log t))}dt < \infty$$

then

$$\mathbf{P}(T_g > n) \sim V(g)\mathbf{P}(T_0 > n)$$

and

$$V(g) = \lim_{n\to\infty} \mathbf{E}[h(S_n + g(n)); T_g > n] < \infty.$$

“Proof” of Theorem 2: Set

$$\nu_n := \min\{k \geq 1 : |S_k| > c(\varepsilon_n n)\},$$

where ε_n is such that $c(\varepsilon_n n) > g(n)$. Then

$$\mathbf{P}(T_g > n) = \mathbf{P}(T_g > n, \nu_n \leq \delta_n n) + \mathbf{P}(T_g > n, \nu_n > \delta_n n).$$

For the second probability we have

$$\mathbf{P}(T_g > n, \nu_n > \delta_n n) \leq \mathbf{P}(\nu_n > \delta_n n) \leq e^{-c\delta_n/\varepsilon_n}.$$

By the Markov property,

$$\mathbf{P}(T_g > n, \nu_n \leq \delta_n n) \sim \mathbf{P}(T_0 > n)\mathbf{E}[h(S_{\nu_n} + g(\nu_n)); T_g > \nu_n, \nu_n \leq n\delta_n].$$

Next we show that

$$\begin{aligned}\mathbf{E}[h(S_{\nu_n} + g(\nu_n)); T_g > \nu_n, \nu_n \leq n\delta_n] &\sim \mathbf{E}[h(S_{\nu_n} + g(n)); T_g > \nu_n] \\ &\sim \lim_{n\to\infty} \mathbf{E}[h(S_n + g(n)); T_g > n] = V(g).\end{aligned}$$

Define

$$\widehat{T}_g := \min\{n \geq 1 : S_n < g(n)\}.$$

If g is positive then

$$\frac{\mathbf{P}(\widehat{T}_g > n)}{\mathbf{P}(T_0 > n)} = \mathbf{P}(\widehat{T}_g > n | T_0 > n).$$

and one can try to represent the limit of this conditional probability as a functional of $\{S_n\}$ conditioned to stay nonnegative.

It is well-known that $h(x)$ is a positive harmonic function for $\{S_n\}$ killed at leaving $[0, \infty)$, that is,

$$\mathbf{E}[h(x + X), x + X > 0] = h(x), \quad x \geq 0.$$

We denote by $\mathbf{P}^h$ the Doob transform of $\mathbf{P}$ by the function h. More precisely, $\mathbf{P}^h$ corresponds to the transition function

$$p^h(x, dy) = \frac{h(y)}{h(x)} \mathbf{P}(x + X \in dy), \quad x, y \geq 0.$$

Afanasyev, Geiger, Kersting and Vatutin (2005):

$$\lim_{n\to\infty} \frac{\mathbf{P}(\widehat{T}_g > n)}{\mathbf{P}(T_0 > n)} = \mathbf{P}^h(\widehat{T}_g = \infty).$$

Hambly, Kersting and Kyprianou (2003): For driftless random walks with finite variance,

$$\mathbf{P}^h(\widehat{T}_g = \infty) > 0 \quad \Leftrightarrow \quad \int_1^\infty \frac{g(x)}{x^{3/2}} dx < \infty.$$

Theorem 3. Assume that S_n is asymptotically stable. If

$$\int_0^\infty \frac{h(g(t))}{th(c(t/\log t))}dt < \infty$$

then

$$\mathbf{P}(\widehat{T}_g > n) \sim \widehat{V}(g)\mathbf{P}(T_0 > n)$$

and

$$\widehat{V}(g) = \lim_{n\to\infty} \mathbf{E}[h(S_n - g(n)); \widehat{T}_g > n] > 0.$$