Persistence and Spherical Intrinsic Volumes

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Talk outline

1. Motivation: Euclidean intrinsic volumes and Sudakov-Tsirelson result
2. Spherical counterpart and Gaussian persistence

Intrinsic volumes in $\mathbb{R}^n$

Steiner's formula

Let $K \subset \mathbb{R}^n$ be a bounded convex set.

$$\mathrm{Vol}_n(K + rB_n) = \sum_{k=0}^{n} r^{n-k} \kappa_{n-k} V_k(K), \quad r > 0.$$

Notations

- Minkowski sum: $A + B = \{x + y \mid x \in A, y \in B\}$.
- B_n is the unit ball of dimension n.
- $\kappa_k = \mathrm{Vol}_k(B_k)$.

V_1 and the mean width

Kubota's formula

$$V_k(K) = \frac{\binom{n}{k}\kappa_n}{\kappa_k \kappa_{n-k}} \int_{\mathcal{L}_k^n} \mathrm{Vol}_k(K|L)\, \mu_k(dL).$$

In particular,

- $V_n =$ volume,
- $V_{n-1} = \frac{1}{2} \cdot$ surface area,
- ...
- $\frac{2\kappa_{n-1}}{n\kappa_n} V_1 =$ mean width,
- $V_0 = 1$.

Intrinsic volumes in Hilbert space

Definition (Sudakov 1971; Chevet 1976)

Let H be a Hilbert space and $K \subset H$ a convex set. Define

$$V_k(K) = \sup_{K'} V_k(K') \in [0, +\infty],$$

where the supremum is taken over all **finite-dimensional** bounded convex subsets $K' \subset K$.

Sudakov's formula

Notations

- $(\Omega, \mathcal{F}, \mathbb{P})$ – complete probability space
- $H = L_2(\Omega, \mathcal{F}, \mathbb{P})$ – Hilbert space
- $\{X(t)\}_{t\in T} \subset H$ – centered Gaussian process
- $K = \overline{\mathrm{conv}}(\{X(t)\}_{t\in T})$ – convex set in H

Theorem (Sudakov, 1971)

$$V_1(K) = \sqrt{2\pi}\,\mathbb{E}\sup_{t\in T} X(t).$$

Tsirelson's formula

Theorem (Tsirelson '85)

Let $X_1(t), \dots, X_k(t)$ be independent copies of the process $\{X(t)\}$. Define

$$\mathsf{Spec}_k K = \mathrm{conv}(\{(X_1(t), \dots, X_k(t))\colon t \in T\}) \subset \mathbb{R}^k.$$

Then

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbb{E}\mathrm{Vol}_k(\mathsf{Spec}_k K).$$

Wiener spiral

Definition (Kolmogorov, 1940)

The Wiener spiral is a curve in $L^2[0,1]$ defined as

$$W = \{\mathbb{1}_{[0,t]} \colon t \in [0,1]\} \subset L^2[0,1].$$

Theorem (Gao and Vitale, 2003)

$$V_k(\mathrm{conv}(W)) = \frac{\kappa_k}{k!}.$$

Theorem (Eldan, 2012)

Let $\{B(t) \colon t \in [0,1]\}$ be a Brownian motion with values in $\mathbb{R}^k$. Then,

$$\mathbb{E}\,\mathrm{Vol}_k(\mathrm{conv}\{B(t) \colon t \in [0,1]\}) = \frac{\kappa_k^2}{(2\pi)^{k/2}}.$$

Brownian bridge spiral

Definition

The Brownian bridge spiral is a curve in $L^2[0,1]$ given by

$$W_0 = \{\mathbb{1}_{[0,t]}(\cdot) - t\mathbb{1}_{[0,1]}(\cdot) \colon t \in [0,1]\} \subset L^2[0,1].$$

Theorem (Gao, 2003)

$$V_k(\mathrm{conv}(W_0)) = \frac{\kappa_{k+1}}{2k!}.$$

Corollary (Randon-Furling, Majumdar, Comtet, 2009, for k=2)

Let $\{B_0(t) \colon t \in [0,1]\}$ be a Brownian bridge with values in $\mathbb{R}^k$. Then,

$$\mathbb{E}\,\mathrm{Vol}_k(\mathrm{conv}\{B_0(t) \colon t \in [0,1]\}) = \frac{\kappa_k \kappa_{k+1}}{2(2\pi)^{k/2}}.$$

Lipschitz balls

Definition

Consider the following compact subsets of $L^2[0,1]$:

$$L_{BM} = \{f : [0,1] \to \mathbb{R} \colon f(0) = 0, \operatorname{Lip}(f) \leq 1\},$$
$$L_{BB} = \left\{f : [0,1] \to \mathbb{R} \colon \int_0^1 f(t)dt = 0, \operatorname{Lip}(f) \leq 1\right\}.$$

Theorem (Kabluchko, Zaporozhets, 2013)

$$V_k(L_{BM}) = \frac{\pi^{k/2}}{\Gamma\left(\frac{3}{2}k+1\right)}, \quad V_k(L_{BB}) = \frac{\pi^{(k+1)/2}}{2\Gamma\left(\frac{3}{2}k+\frac{3}{2}\right)}.$$

Brownian zonoids

Claim

Let $\{B(t) \colon t \in [0,1]\}$ be the Brownian motion with values in $\mathbb{R}^k$. The spectrum of L_{BM} is given by

$$\mathrm{Spec}_k(L_{BM}) = \left\{ \int_0^1 B(t) g(t) dt \colon -1 \leq g(t) \leq 1 \right\}.$$

It is the zonoid generated by the Brownian motion.

Corollary

The expected volume of the Brownian zonoid in $\mathbb{R}^k$ is

$$\frac{1}{(2\sqrt{2\pi})^k} \binom{\frac{3}{2}k}{k}^{-1}.$$

Wills functional

Definition (Wills '73)

K – convex body in $\mathbb{R}^n$. Wills functional of K:

$$W(K) = \int_{\mathbb{R}^n} e^{-\pi \cdot \mathrm{dist}^2(x,K)}\, dx.$$

Theorem (Hadwiger '75)

$$W(K) = \sum_{k=0}^{n} \left(\frac{1}{\sqrt{2\pi}} \right)^k V_k(K).$$

Corollary

$$\int_{\mathbb{R}^n} e^{-\pi \cdot \mathrm{dist}^2(x,rK)}\, dx = \sum_{k=0}^{n} \left(\frac{r}{\sqrt{2\pi}} \right)^k V_k(K).$$

Infinite dimensional Steiner's formula

$$K = \overline{\mathrm{conv}}(\{X(t)\}_{t\in T}) \subset H = L_2(\Omega, \mathcal{F}, \mathbb{P}).$$

Theorem (Tsirelson '85, Vitale '01)

$$\mathbb{E}\exp\left(sup_{X\in K}\left[rX - \frac{r^2}{2}\mathrm{Var} X \right] \right) = \sum_{k=0}^{\infty} \left(\frac{r}{\sqrt{2\pi}} \right)^k V_k(K).$$

Basic notations

Notations

- $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ – unit sphere
- $d_s(x, y) = \arccos\langle x, y\rangle$ – usual metric in $\mathbb{S}^{n-1}$
- convex cone C: $\lambda(x + y) \in C$ for all $x, y \in C$ and $\lambda > 0$
- spherical convex body $\tilde{C}$ – intersection of $\mathbb{S}^{n-1}$ with convex cone $C \neq \{0\}$ in $\mathbb{R}^n$
- equivalence $\tilde{C} \leftrightarrow C$

Spherical Steiner's formula

Spherical (conic) Steiner's formula (Weil '39)

Let $C \subset \mathbb{R}^n$ be a convex cone.

$$\mu_{n-1}(x \in \mathbb{S}^{n-1} \ : \ d_s(x, \tilde{C}) \leqslant \lambda) = \sum_{k=1}^{n} \beta_{k,n}(\lambda)\nu_k(C), \lambda \in [0,1].$$

$$\nu_0(C) = 1 - \sum_{k=1}^{n} \nu_k(C).$$

Spherical Kubota's formula

Definition (Santaló)

$$U_j(C) = \frac{1}{2} \int_{\mathcal{L}^n_{n-j-1}} \chi(C \cap L)\, \mu_{n-j-1}(dL).$$

Spherical Kubota's formula (Santaló)

$$U_j(C) = \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \nu_{j+2k}(C).$$

Spherical intrinsic volumes in Hilbert space

Definition of U_j

Let H be a Hilbert space and $C \subset H$ a convex cone. Define

$$U_j(C) = \sup_{C'} U_j(C'),$$

where the supremum is taken over all **finite-dimensional** convex cones $C' \subset C$.

Definition of ν_k

$$\nu_k(C) = U_{k+1}(C) - U_{k-1}(C), \quad k = 2, 3, \dots$$

and

$$\nu_0(C) = \frac{1}{2} - U_1(C), \quad \nu_1(C) = \frac{1}{2} - U_2(C).$$

Spherical Tsirelson formula

$$K = \overline{\mathrm{conv}}(\{X(t)\}_{t \in T}) \subset H = L_2(\Omega, \mathcal{F}, \mathbb{P}).$$

Theorem (Kabluchko, Zaporozhets '14)

Let $X_1(t), \ldots, X_k(t)$ be independent copies of the process $\{X(t)\}$. Define

$$\mathrm{Spec}_k K = \mathrm{conv}(\{(X_1(t), \ldots, X_k(t)) \colon t \in T\}) \subset \mathbb{R}^k.$$

Then

$$U_k(\mathrm{cone}(K)) = \frac{1}{2}\mathbb{P}(0 \in \mathrm{Spec}_k K).$$

$k = 1$

$$U_1(\mathrm{cone}(K)) = \frac{1}{2} - \mathbb{P}(\inf_T X(t) > 0),$$

$$\nu_0(\mathrm{cone}(K)) = \mathbb{P}(\inf_T X(t) > 0).$$

Spherical Wills functional

Theorem (McCoy, Tropp '14)

C – convex cone in $\mathbb{R}^n$, g – standard Gaussian vector in $\mathbb{R}^n$.

$$\lambda^n \mathbb{E} \exp\left(\frac{1-\lambda^2}{2} \cdot \mathrm{dist}(g, C)\right) = \sum_{k=0}^{n} \lambda_k \nu_k(C), \quad \lambda > 0.$$

Infinite dimensional spherical Steiner's formula

Theorem (Kabluchko, Zaporozhets '14)

$$\mathbb{E}\exp\left(\frac{1-\lambda^{-2}}{2}\cdot\sup_T{}^{+}\frac{X(t)}{\sqrt{\mathrm{Var}X(t)}}\right)=\sum_{k=0}^{n}\lambda_k\nu_k(\mathrm{cone}(K)).$$

$\lambda = 1$

$$\sum_{k=0}^{n}\nu_k(\mathrm{cone}(K))=1.$$

$\lambda = 0$

$$\nu_0(\mathrm{cone}(K))=\mathbb{P}(\sup_T X(t)<0).$$

Final slide

THANK YOU!