Realizability Models for CZF+¬Pow

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In the late 1970ies Peter Aczel has shown how to provide the predicative constructive set theory CZF with meaning by translating it to MLWU, i.e. Martin-Löf type theory with W-types and one universe U. The key idea of his construction (from now on referred to simply as *Aczel construction*) is to replace the type equation $V = \mathcal{P}(V)$ by the type equation $V_U = \Sigma A : U.V^A$ whose canonical solution is given by the W-type $(WA : U)A$. In other words one (transfinitely) iterates the *enumeration construction* $\Sigma A : U.(-)^A$ instead of the (covariant) powerset construction $\mathcal{P}$. Of course, in case of $V_U = (WA : U)A$ one has to *define* an appropriate notion of equality on V by structural (transfinite) recursion over V_U as follows

$$\begin{aligned}&\sup(A,f) =_V \sup(B,g) \;\equiv\\ &\qquad (\forall a{:}A)(\exists b{:}B)f(a){=}_V g(b) \;\wedge\; (\forall b{:}B)(\exists a{:}A)f(a){=}_V g(b)\end{aligned}$$

based on which elementhood is defined as

$$a \in_V \sup(A,f) \;\equiv\; (\exists a{:}A)\; b{=}_V f(a)$$

in a recursion free way.

There arises the question how to read the quantifiers in the above defintions of equality and elementhood on V_U. As the target language of Aczel's (original) translation was MLWU it was natural for him to read $\forall$ and $\exists$ as dependent product Π and sum Σ, respectively, thus giving rise to a *proof-relevant* model of CZF. This had, in particular, the benefit of having available the unrestricted type-theoretic Axiom of Choice (AC) which is most useful for validating the (Strong) Collection and Subset Collection axioms of CZF. Moreover, this reading of quantifiers (and the other logical conncectives) has the consequence that the set-theoretic Separation scheme can be verified only for *bounded* formulas (in the language of set theory).[1]

However, one might be interested (as we are in this note) in *proof-irrelevant* models for CZF which can be achieved by letting propositions live in a type *Prop* of proof-irrelevant propositions which in case of realizability models may be taken as $\mathcal{P}(\mathcal{A})$ where $\mathcal{A}$ is the underlying partial combinatory algebra (pca).

[1] Even if $x : V_U \vdash \phi(x) : U$ then $(\Pi x{:}V_U)\phi(x)$ and $(\Sigma x{:}V_U)\phi(x)$ are **not** types in U because V_U is not an element of U.

Notice that under this interpretation of *Prop* the Axiom of Choice will not hold in general anymore. Nevertheless, we may perform the Aczel construction $V_U = (WA : U)A$ for universes U as given by $\mathbf{Mod}(\mathcal{A})$ or $\mathbf{Asm}_\kappa(\mathcal{A})$ (i.e. the assemblies over $\mathcal{A}$ whose underlying set has cardinality < some strongly inaccessible cardinal κ). It is well known that $\mathbf{Asm}(\mathcal{A})$ is a model for Z. Luo's Extended Calculus of Constructions (ECC) where the n-th universe $\mathit{Type}(n)$ is interpreted as $\mathbf{Asm}_{\kappa_n}(\mathcal{A})$ for a strictly increasing sequence $(\kappa_n)_{n\in\mathbb{N}}$ of strongly inaccessible cardinals and *Prop* is given by subterminal modest sets, i.e. $\mathcal{P}(\mathcal{A})$. In ECC one verifies without (too much) pain that $V_U = (WA : U)A$ validates all axioms of CZF but Strong Collection and Subset Collection. Moreover, if $\mathit{Prop} \in U$ then it is also easy to verify the Powerset Axiom Pow. Thus, via the Aczel construction we get models for CZ and IZ when performing it for $U = \mathbf{Asm}_\kappa(\mathcal{A})$ and $U = \mathbf{Mod}(\mathcal{A})$, respectively.

So the main problem to solve is how to validate Strong Collection and Subset Collection without having AC available in ECC. Though not every object of U is projective, i.e. satisfies Choice, it suffices that every object can be covered by a projective object which typically holds in all realizability models even *internally* in the sense that

$$\text{(PCA)} \qquad (\forall A{:}U)(\exists C{:}U)(\exists e{:}A^C)\ \mathrm{Proj}(C) \wedge \mathrm{Surj}(e)$$

holds in the internal logic of $\mathbf{Asm}(\mathcal{A})$ where $\mathrm{Surj}(e)$ is an abbreviation for $(\forall a{:}A)(\exists c{:}C)\ a = f(c)$ and

$$\mathrm{Proj}(C) \equiv (\forall X, Y{:}U_1)(\forall e{:}X^Y)\ \mathrm{Surj}(e) \Rightarrow (\forall f{:}X^C)(\exists g{:}Y^C)\ e \circ g = f$$

where U_1 is a universe with $U \in U_1$ and $U \subseteq U_1$, namely $U_1 = \mathbf{Asm}_{\kappa_1}(\mathcal{A})$ where κ_1 is a strongly inaccessible cardinal with $U \in V_{\kappa_1}$.

If $U = \mathbf{Asm}_\kappa(\mathcal{A})$ for some strongly inaccessible cardinal then V_U (interpreted in $\mathbf{Asm}(\mathcal{A})$) is a model for Intuitionistic Zermelo-Fraenkel set theory (IZF). On the other hand if $U = \mathbf{Mod}(\mathcal{A})$ then V_U is a model[2] for CZF which, moreover, validates $\neg$Pow for $\mathcal{A} = K_1$ (number realizability) as then from the external

[2]This argument can be fully formalized in type theory augmented by the "non-logical" axiom PCA. An alternative to PCA for verifying the Collection Axiom is the following *Type Theoretic Collection Axiom* introduced by Joyal and Moerdijk in their book on *Algebraic Set Theory*

$$\text{(TTCA)} \quad (\forall A{:}U)(\forall X{:}U_1)(\forall e{:}A^X)\ \mathrm{Surj}(e) \Rightarrow (\exists C{:}U)(\exists f{:}X^C)\ \mathrm{Surj}(e{\circ}f)$$

which has the advantage that it holds not only in realizability but also in sheaf models! However, when the Powerset Axiom is not available for validating Subset Collection one needs the following somewhat stronger axiom

$$\text{(TTCA}_f\text{)} \quad (\forall A{:}U)(\exists I{:}U)(\exists C{:}U^I) \\ (\forall X{:}U_1)(\forall e{:}A^X)\ \mathrm{Surj}(e) \Rightarrow (\exists i{:}I)(\exists f{:}X^{C(i)})\ \mathrm{Surj}(e{\circ}f)$$

in the ambient type theory.

For LEGO files containing such formal verifications see the files `ast.l.gz`, `astp.l.gz`, `czf.l.gz` and `izf.l.gz` which can be found in the directory `www.mathematik.tu-darmstadt.de/~streicher/CIZF`.

point of view $\omega \in V_U$ has uncountably many subsets whereas every set in V_U has just countably many elements from which it follows that $\mathcal{P}(\omega)$ does not exist in V_U because V_U validates the axiom of subcountability, saying that every set can be enumerated by a subset of ω. Actually, one need not restrict $\mathcal{A}$ to be countable in order to refute the power set axiom as if $|\mathcal{A}| < \beth_\omega$ then $\mathcal{P}^n(\omega)$ does not exist in the model V_U with $U = \mathbf{Mod}(\mathcal{A})$ where n is the least natural number with $|\mathcal{A}| \leq \beth_n$. As most pca's of interest have cardinality strictly less than $\beth_1$ this is no real restriction in practice.

Unfortunately, however, the model still validates the unrestricted separation axiom Sep. Notice that CZF+Sep is (at least) as strong as PA_2 because quantification over all sets allows one in particular to quantify over subsets of ω and Sep guarantees that subclasses of ω defined this way are actually sets. Accordingly, one cannot consider models of CZF+Sep as genuinely predicative.

An alternative model for $\mathsf{CZF}+\neg\mathsf{Pow}$ is given by the topos $\widehat{\mathsf{Ord}}$ of **Set**-valued presheaves on the large poset Ord of all ordinals in **Set**. Alas, it also validates Sep at least if one assumes that subclasses of sets are again sets (as usual in GBN class theory).

Thus, there are plenty of categorical models for $\mathsf{CZF} + \neg\mathsf{Pow} + \mathsf{Sep}$ which can be constructed in a *syntax-free way* and without *restricting the logic on the meta-level to be predicative.* However, in the moment[3] we have to leave it as an open question whether one can find similarly "purely semantic" model constructions refuting also the full separation schema.

[3]To get out of this problem one might feel tempted to consider models of CZF within $\mathbf{Asm}(\mathbf{ALat})$, the category of assemblies over the typed pca $\mathbf{ALat}$ of algebraic lattices and Scott continuous functions between them, because $U = \mathbf{Mod}(\mathcal{P}\omega)$ gives rise to a *genuinely predicative* universe within $\mathbf{Asm}(\mathbf{ALat})$ which can be seen as follows.

It is pretty straightforward to see that $U = \mathbf{Mod}(\mathcal{P}\omega)$ satisfies the closure properties required for a predicative universe. The point, however, is that it is *genuinely predicative* in the sense that U is not closed under arbitrary dependent products. For this purpose consider the type $A = \Delta(\mathcal{P}\kappa)$ where κ is some (infinite) cardinal strictly greater than $\aleph_0$ (e.g. $\kappa = \beth_1 = 2^{\aleph_0}$). Now consider the (constant) family $B : A \to U : x \mapsto \Delta(\mathcal{P}\omega)$ whose (actually non-)dependent product is $\Delta(\mathcal{P}\omega)^{\Delta(\mathcal{P}\kappa)} \cong \Delta(\mathcal{P}\omega^{\mathcal{P}\kappa})$ which (for obvious cardinality reasons) is not isomorphic to a modest set over $\mathcal{P}\omega$, i.e. not isomorphic to a type in U.

Now performing the Aczel construction in $\mathbf{Asm}(\mathbf{ALat})$ for the *genuinely predicative* universe $U = \mathbf{Mod}(\mathcal{P}\omega)$ gives rise to model of CZF not validating $\neg\mathsf{Pow}$ as, obviously, all sets in $V_{\mathbf{Mod}(\mathcal{P}\omega)}$ contain at most as many elements as $\mathcal{P}\omega$ from which it follows that $\mathcal{P}^2(\omega)$ does not exist in $V_{\mathbf{Mod}(\mathcal{P}\omega)}$. Alas, it is very unlikely that the types of U are not closed under subobjects which would be necessary for $V_{\mathbf{Mod}(\mathcal{P}\omega)}$ refuting Sep.