A Semantic Version of the Diller-Nahm Variant of Gödel's Dialectica Interpretation

Thomas Streicher

September 2000 – September 2006

Following a suggestion by Martin Hyland from the early 1980ies [Hy] we define triposes reflecting as much as possible the original idea of the Diller-Nahm variant of Gödel's Dialectica Interpretation.

1 Diller-Nahm Dialectica Tripos

According to Gödel's original Dialectica Interpretation from 1958 a proposition is a pair of types X and Y together with a *decidable* relation $R \subseteq X \times Y$ between them. For our purposes we assume that types are subsets of $\mathbb{N}$ and (constructive) functionals between them are total recursive functions between sets of natural numbers. Entailment between propositions (X, Y, R) and (U, V, S) is given by a pair of (constructive) functionals $f : X \to U$ and $g : X \times V \to Y$ such that

$$\forall x \in X. \forall v \in V.\ R(x, g(x, v)) \Rightarrow S(f(x), v)\ .$$

However, if relations are not required to be decidable[1] one usually[2] considers the *Diller-Nahm variant* of the Dialectica Interpretation where entailment

[1] Which is impossible as for quantification we have to take into account *arbitrary* unions and intersections of relations.

[2] For an alternative way to cope with undecidability of relations *cf.* the appendix on the Dialectica Tripos considered in [BiRo].

is defined in a somewhat different way: $(X, Y, R) \vdash (U, V, S)$ iff there are (constructive) functionals $f : X \to U$ and $g : X \times V \to \mathsf{P_f}(Y)$ such that

$$\forall x \in X. \forall v \in V.\ [\forall y \in g(x, v).\, R(x, y)] \Rightarrow S(f(x), v)\ .$$

Here $m \in n$ stands for $m \in e_n$ (where e is some standard Gödel numbering of finite sets of natural numbers) and $\mathsf{P_f}(Y)$ is a shorthand for $\{n \in \mathbb{N} \mid e_n \subseteq Y\}$.

Motivated by these considerations we are now going to define the Diller-Nahm Dialectica tripos DN over **Set**. We will assume in the following that $\langle 0, 0\rangle = 0$ and $0 \cdot n = 0$ for all $n \in \mathbb{N}$. Let

$$\Sigma_{\mathsf{DN}} = \{(X, Y, R) \in \mathcal{P}(\mathbb{N})^2 \times \mathcal{P}(\mathbb{N} \times \mathbb{N}) \mid R \subseteq X \times Y\}$$

be the "set of truth values of DN". If $p = (X, Y, R) \in \Sigma_{\mathsf{DN}}$ we write p^+, p^-, $p(a, b)$ for X, Y, $R(a, b)$, respectively. For $I \in \mathbf{Set}$ the fibre DN^I is defined as the preorder $({\Sigma_{\mathsf{DN}}}^I, \vdash_I)$, where ${\Sigma_{\mathsf{DN}}}^I$ is the set of all functions from I to Σ_{DN} and $\varphi \vdash_I \psi$ iff there exist

$$e^+ \in \bigcap_{i \in I} [\varphi_i^+ \to \psi_i^+] \quad \text{and} \quad e^- \in \bigcap_{i \in I} [\varphi_i^+ \times \psi_i^- \to \mathsf{P_f}(\varphi_i^-)]$$

such that

$$\forall i \in I. \forall a \in \varphi_i^+, b \in \psi_i^-.\, [\forall c \in e^- \langle a, b\rangle.\, \varphi_i(a, c)] \Rightarrow \psi_i(e^+ a, b)\ \ .$$

All fibres of DN are cartesian closed and this structure is preserved by reindexing since finite products and exponentials are constructed componentwise.

Terminal Object
is given by $\top = (\{0\}, \emptyset, \emptyset)$. Notice, that $\top \cong (\{0\}, \{0\}, \{\langle 0, 0\rangle\})$.

Products
The conjunction $p \wedge q$ is given by

(1) $(p \wedge q)^+ = p^+ \times q^+$

(2) $(p \wedge q)^- = p^- + q^-$

(3) $(p \wedge q)(\langle n, m\rangle, \langle i, k\rangle) \Leftrightarrow (i = 0 \wedge p(n, k)) \vee (i = 1 \wedge q(m, k))$.

Exponentials
The implication $p \to q$ is given by

(1) $(p \to q)^+ = (p^+ \to q^+) \times (p^+ \times q^- \to \mathsf{P_f}(p^-))$

(2) $(p \to q)^- = p^+ \times q^-$

(3) $(p \to q)(\langle e^+, e^- \rangle, \langle a, b \rangle) \Leftrightarrow [\forall c \in e^- \langle a, b \rangle .\, p(a, c)] \Rightarrow q(e^+ a, b)$.

Next we consider quantification in DN. In the following we write $[i = j]$ for $\{0 \mid i = j\}$.

Universal Quantification
For $u : I \to J$ and $\varphi \in \mathsf{DN}^I$ we construct $\forall_u(\varphi) \in \mathsf{DN}^J$ as follows

(1) $\forall_u(\varphi)^+_j = \bigcap_{i \in I} [u(i) = j] \to \varphi^+_i$

(2) $\forall_u(\varphi)^-_j = \bigcup_{i \in u^{-1}(j)} \varphi^-_i$

(3) $\forall_u(\varphi)_j(a, b) \Leftrightarrow \forall i \in u^{-1}(j).\, \big(b \in \varphi^-_i \Rightarrow \varphi_i(a \cdot 0, b)\big)$.

Lemma 1.1 *In* DN *it holds for all functions* $u : I \to J$ *that* $u^* \dashv \forall_u$.

Proof. First we show that for all $\varphi \in \mathsf{DN}^I$ and $\psi \in \mathsf{DN}^J$ it holds that

$$u^*\psi \vdash_I \varphi \quad \text{iff} \quad \psi \vdash_J \forall_u(\varphi)$$

i.e. that $\forall_u$ is right adjoint to u^* w.r.t. DN.

The pair (e^+, e^-) realises $\psi \vdash_J \forall_u(\varphi)$ iff

$$\begin{aligned} e^+ \in & \textstyle\bigcap_{j \in J} (\psi^+_j \to \bigcap_{i \in I} [u(i) = j] \to \varphi^+_i) = \\ & = \textstyle\bigcap_{j \in J} \bigcap_{i \in u^{-1}(j)} (\psi^+_j \to \{0\} \to \varphi^+_i) \end{aligned}$$

and

$$\begin{aligned} e^- \in & \textstyle\bigcap_{j \in J} ((\psi^+_j \times \bigcup_{i \in u^{-1}(j)} \varphi^-_i) \to \mathsf{P_f}(\psi^-_j)) = \\ & = \textstyle\bigcap_{j \in J} \bigcup_{i \in u^{-1}(j)} ((\psi^+_j \times \varphi^-_i) \to \mathsf{P_f}(\psi^-_j)) \end{aligned}$$

such that

$$\forall j \in J.\, \forall a \in \psi_j^+.\, \forall b \in \bigcup_{i \in u^{-1}(j)} \varphi_i^-.$$
$$[\forall c \in e^-\langle a, b\rangle.\, \psi_j(a, c)] \Rightarrow \forall i \in u^{-1}(j).\, b \in \varphi_i^- \Rightarrow \varphi_i(e^+ a0, b)$$

which is equivalent to

$$\forall j \in J.\, \forall a \in \psi_j^+.\, \forall i \in u^{-1}(j).\, \forall b \in \varphi_i^-.\, [\forall c \in e^-\langle a, b\rangle.\, \psi_j(a, c)] \Rightarrow \varphi_i(e^+ a0, b)$$

which in turn is equivalent to

$$\forall i \in J.\, \forall a \in \psi_{u(i)}^+.\, \forall b \in \varphi_i^-.[\forall c \in e^-\langle a, b\rangle.\, \psi_j(a, c)] \Rightarrow \varphi_i(e^+ a0, b) \ \ .$$

The pair (f^+, f^-) realises $u^*\psi \vdash_I \varphi$ iff

$$f^+ \in \bigcap_{i \in I} (\psi_{u(i)}^+ \to \varphi_i^+)$$

and

$$f^- \in \bigcap_{i \in I} ((\psi_{u(i)}^+ \times \varphi_i^-) \to \mathsf{P_f}(\psi_{u(i)}^-))$$

such that

$$\forall i \in J.\, \forall a \in \psi_{u(i)}^+.\, \forall b \in \varphi_i^-.[\forall c \in f^-\langle a, b\rangle.\, \psi_j(a, c)] \Rightarrow \varphi_i(f^+ a, b) \ \ .$$

Using these explicitations one gets that

(1) If (e^+, e^-) realises $\psi \vdash_J \forall_u(\varphi)$ then putting $f^+ := \Lambda a.e^+ a0$ and $f^- := e^-$ the pair (f^+, f^-) realises $u^*\psi \vdash_I \varphi$.

(2) If (f^+, f^-) realises $u^*\psi \vdash_I \varphi$ then putting $e^+ := \Lambda a.\Lambda n.f^+ a$ and $e^- := f^-$ the pair (e^+, e^-) realises $\psi \vdash_J \forall_u(\varphi)$.

as is easily shown by standard logical manipulations. $\square$

It is obvious from the construction that universal quantification is preserved by substitution, i.e. that BCC holds.

Notice that in case $u : I \to J$ is onto one may simplify the construction of $\forall_u(\varphi)$ by putting $\forall_u(\varphi)^+_j = \bigcap_{i \in u^{-1}(j)} \varphi^+_i$ and

$$\forall_u(\varphi)_j(a,b) \Leftrightarrow \forall i \in u^{-1}(j).\,\big(b \in \varphi^-_i \Rightarrow \varphi_i(a,b)\big) \quad .$$

Existential Quantification
For $u : I \to J$ and $\varphi \in \mathsf{DN}^I$ existential quantification is given by

(1) $\exists_u(\varphi)^+_j = \bigcup_{i \in u^{-1}(j)} \varphi^+_i$

(2) $\exists_u(\varphi)^-_j = \bigcap_{i \in u^{-1}(j)} [\varphi^+_i \to \mathsf{P_f}(\varphi^-_i)]$

(3) $\exists_u(\varphi)_j(a,b) \Leftrightarrow \exists i \in u^{-1}(j).\,\big(a \in \varphi^+_i \wedge \forall c \in ba.\, \varphi_i(a,c)\big)$.

Lemma 1.2 *In* DN *it holds for every function $u : I \to J$ that $\exists_u \dashv u^*$.*

Proof. Let $\varphi \in \Sigma^I$ and $\psi \in \Sigma^J$. We have to show the equivalence of the following two statements

(a) $\exists_u(\varphi) \dashv_J \psi$, i.e. there exist

$$e^+_1 \in \bigcap_{j \in J} \Big[\exists_u(\varphi)^+_j \to \psi^+_j\Big] \quad \text{and} \quad e^-_1 \in \bigcap_{j \in J} \Big[\exists_u(\varphi)^+_j \times \psi^-_j \to \mathsf{P_f}(\exists_u(\varphi)^-_j)\Big]$$

such that

(1) $\forall j{\in}J.\forall a{\in}\exists_u(\varphi)^+_j.\forall b{\in}\psi^-_j.[\forall c{\in}e^-_1(a,b).\exists_u(\varphi)_j(a,c)] \Rightarrow \psi_j(e^+_1 a, b)$

(b) $\varphi \vdash_I u^*\psi$, i.e. there exist

$$e^+_2 \in \bigcap_{i \in I} \Big[\varphi^+_i \to \psi^+_{u(i)}\Big] \quad \text{and} \quad e^-_2 \in \bigcap_{i \in I} \Big[\varphi^+_i \times \psi^-_{u(i)} \to \mathsf{P_f}(\varphi^-_i)\Big]$$

such that

(2) $\forall i{\in}I.\forall a{\in}\varphi^+_i.\forall b{\in}\psi^-_{u(i)}.\ [\forall c{\in}e^-_2(a,b).\varphi_i(a,c))] \Rightarrow \psi_{u(i)}(e^+_2 a, b)$.

Suppose (a), i.e. there exists e_1^+ and e_1^- satisfying (1). We have to show that (b). For this purpose put $e_2^+ = e_1^+$ and e_2^- in such a way such that $e_2^-(a,b)$ is a code for the finite set $\bigcup_{c \in e_1^-(a,b)} \varepsilon_{c \cdot a}$. For showing that e_2^+ and e_2^- validate (2) suppose $i \in I$, $a \in \varphi_i^+$, $b \in \psi_{u(i)}^-$ satisfying

(3) $\forall c \in e_2^-(a,b).\, \varphi_i(a,c)$.

Then from assumption (1) it follows that

(4) $[\forall c{\in}e_1^-(a,b).\exists_u(\varphi)_{u(i)}(a,c)] \Rightarrow \psi_{u(i)}(e_1^+ a,b)$

Since $e_1^+ = e_2^+$ for showing the desired $\psi_{u(i)}(e_2^+(a,b)$ it suffices to show the premiss of (4), i.e. more explicitly that

(5) $\forall c \in e_1^-(a,b).\, \exists i' \in u^{-1}(u(i)).\, a \in \varphi_{i'}^+ \wedge \forall d \in c{\cdot}a.\, \varphi_{i'}(a,d)$

For that purpose suppose that $c \in e_1^-(a,b)$. Put $i' = i$. By assumption we have $a \in \varphi_i^+$. Suppose $d \in c{\cdot}a$. Then also $d \in e_2^-(a,b)$ from which it follows by (3) that $\varphi_i(a,d)$ and we are done.

Suppose (b), i.e. there exists e_2^+ and e_2^- satisfying (2). We have to show that (a). For this purpose put $e_1^+ = e_2^+$ and choose e_1^- in such a way that $e_1^-(a,b)$ is a code for the set $\{\Lambda x.e_2^-(a,b)\}$. For showing that e_1^+ and e_1^- validate (1) suppose $j \in J$, $a \in \exists_u(\varphi)_j^+$, $b \in \psi_j^-$ satisfying the condition $\forall c \in e_1^-(a,b).\, \exists_u(\varphi)_j(a,c)$, i.e. $\exists_u(\varphi)_j(a, \Lambda x.e_2^-(a,b))$ by definition of e_1^-. Thus, by definition of $\exists_u(\varphi)_j$ and since $(\Lambda x.e_2^-(a,b))a = e_2^-(a,b)$ we get that

(6) $\exists i \in u^{-1}(j).a \in \varphi_i^+ \wedge \forall c \in e_2^-(a,b).\, \varphi_i(a,c)$

Now let $i \in u^{-1}(j)$ such that

(7) $a \in \varphi_i^+ \wedge \forall c \in e_2^-(a,b).\, \varphi_i(a,c)$

Then it follows by assumption (2) that $\psi_{u(i)}(e_2^+ a,b))$ and thus since $j = u(i)$ and $e_1^+ = e_s^+$ also that $\psi_j(e_2^+(a,b)$ as desired. $\square$

It is obvious from the construction that existential quantification is preserved by substitution, i.e. that BCC holds.

Notice that in case $u : I \to J$ is onto one may simplify the construction of $\exists_u(\varphi)$ by putting $\exists_u(\varphi)_j^- = \bigcap_{i \in u^{-1}(j)} \mathsf{P_f}(\varphi_i^-)$ and

$$\exists_u(\varphi)_j(a,b) \Leftrightarrow \exists i \in u^{-1}(j).\, \big(a \in \varphi_i^+ \wedge \forall c \in b.\, \varphi_i(a,c)\big) \quad .$$

Generic Predicate
is given by the identity function on Σ_{DN} considered as an element of DN^{Σ}.

The structure exhibited so far guarantees DN to be a tripos. Disjunction can be defined impredicatively à la Prawitz as usual. We give a more direct construction of $\bot$, negation and (Lawvere) equality.

Falsity
is given by $\bot = (\emptyset, \emptyset, \emptyset)$. Notice that $\bot \cong (\emptyset, \{0\}, \emptyset)$.

Negation
For a proposition p its negation $\neg p$ is given by

$$\neg p = \begin{cases} (\mathbb{N}, \emptyset, \emptyset) & \text{if } p^+ = \emptyset \\ (\emptyset, \emptyset, \emptyset) & \text{otherwise.} \end{cases}$$

Accordingly, double negation is given by

$$\neg\neg p = \begin{cases} (\emptyset, \emptyset, \emptyset) & \text{if } p^+ = \emptyset \\ (\mathbb{N}, \emptyset, \emptyset) & \text{otherwise.} \end{cases}$$

As $(\mathbb{N}, \emptyset, \emptyset) \cong \top$ we have that $\neg p$ is $\top$ if $p^+ = \emptyset$ and $\bot$ otherwise.

Lawvere Equality
For a set I the equality predicate $\mathsf{eq}_I \in \mathsf{DN}^{I \times I}$ is given by

$$\mathsf{eq}_I(i,j)^+ = \{0 \mid i = j\} \quad \text{and} \quad \mathsf{eq}_I(i,j)^- = \emptyset$$

for $i, j \in I$. One easily shows that $\mathsf{eq}_I \dashv\vdash \exists_{\delta(I)}(\top_I)$, i.e. that eq_I coincides with Lawvere's notion of equality.

As DN is a tripos one may consider the associated "Dialectica topos" $\mathbf{DN} = \mathbf{Set}[\mathsf{DN}]$ obtained by the tripos-to-topos construction. Due to the particular nature of (double) negation the $\neg\neg$–sheaves of $\mathbf{DN}$ are equivalent to $\mathbf{Set}$. However, the sheafification functor for the $\neg\neg$-topology is not given by the global sections functor. The reason is that DN lacks the $\exists$-property as can be seen from the following counterexample. Consider $p \in \mathsf{DN}^{\mathbb{N}}$ as given by $p_n^+ = (\{0\}, \{0, n+1\}, \{\langle 0, 0\rangle\})$ all whose items p_n are not valid whereas $\exists_{\mathbb{N}}(p) = (\{0\}, \{0\} \to \mathsf{P_f}(\{0\}), \{0\} \times (\{0\} \to \mathsf{P_f}(\{0\})))$ is valid. Anyway, due to the pleasantly simple nature of double negation a useful notion of "assembly" seems to be available via $\neg\neg$-separated objects as usual.

2 Relation to Number Realizability

Next we will show that DN contains as subtripos the number realizability tripos R from which the effective topos $\mathcal{E}\!f\!f = \mathbf{Set}[\mathsf{R}]$ arises via the tripos-to-topos construction. Recall that $\Sigma_{\mathsf{R}} = \mathcal{P}(\mathbb{N})$ and $\varphi \vdash_I \psi$ iff $\bigcap_{i\in I}[\varphi_i {\rightarrow} \psi_i]$ is inhabited.

Theorem 2.1 *There is an injective geometric morphism $i : \mathsf{R} \to \mathsf{DN}$ and, accordingly, $\mathcal{E}\!f\!f$ is a subtopos of $\mathbf{DN}$. This geometric inclusion arises from the topology $u \to (-)$ on $\mathbf{DN}$ where $u = (\{0\},\{0\},\emptyset)$. Moreover, one obtains $\mathbf{Set}$ as the subtopos of $\neg\neg$-sheaves of $\mathbf{DN}$.*

Proof. We define $i : \mathsf{R} \to \mathsf{DN}$ as the fibred adjunction

$$i^*(X,Y,R) = X \qquad \text{and} \qquad i_*(X) = (X,\emptyset,\emptyset)$$

and one easily checks that i^* preserves $\top$ and also conjunction in each fibre (if conjuction is constructed à la Gödel 1958).

One easily checks that $i_*i^*(X,Y,R) = (X,\emptyset,\emptyset) \cong u \to (X,Y,R)$ from which it follows that the geometric inclusion i is induced by the topology $u \to (-)$ on $\mathbf{DN}$.

It is well known that $\mathbf{Set} \simeq \mathcal{E}\!f\!f_{\neg\neg}$. Thus, since $\neg\neg(u \to (X,Y,R)) = \neg\neg(X,Y,R)$ it follows that $\mathbf{Set} \simeq \mathbf{DN}_{\neg\neg}$. $\square$

It might be worthwhile to further investigate the subtopos of $\mathbf{DN}$ which is the complement of $\mathcal{E}\!f\!f$, i.e. the subtopos of $\mathbf{DN}$ induced by the topology $u \vee (-)$. It arises via the tripos-to-topos construction from the subtripos of DN consisting of those predicates $(\varphi_i)_{i\in I}$ where $\bigcap_{i\in I}\varphi_i^+$ is nonempty.

Since direct image parts of injective geometric morphisms preserve finite limits and exponentials via the geometric inclusion $i : \mathcal{E}\!f\!f \to \mathbf{DN}$ of Theorem2.1 we obtain an embedding of $\mathbf{Asm}(\mathbb{N})$ into $\mathbf{DN}$ which preserves finite limits and exponentials. Recall that $i_*(X,E) = (X, i_*{\circ}E)$.

In general direct image parts of geometric morphisms do not preserve nno's. But in the particluar case $i : \mathcal{E}\!f\!f \to \mathbf{DN}$ one readily checks that $i_*(N) = (\mathbb{N}, E_N)$ with $E_N(n,m) = (\{k \in \mathbb{N} \mid n = k = m\},\emptyset,\emptyset)$ happens to be a nno in $\mathbf{DN}$. Thus i_* restricts to a structure preserving embedding of HEO into $\mathbf{DN}$.

3 Modified Diller-Nahm Dialectica Tripos

The set of truth values of the *Modified Diller-Nahm Dialectica tripos* $\mathsf{DN_m}$ over **Set** is

$$\Sigma = \{(X, Y, R) \in \mathcal{P}_0(\mathbb{N})^2 \times \mathcal{P}(\mathbb{N} \times \mathbb{N}) \mid R \subseteq X \times Y\}$$

where $\mathcal{P}_0(\mathbb{N})$ is the set of all subsets of $\mathbb{N}$ containing 0 as an element. Again we tacitly assume that $\langle 0, 0\rangle = 0 = 0 \cdot n$ and $e_0 = \emptyset$.

For $p = (X, Y, R) \in \Sigma$ we write p^+, p^-, $p(a, b)$ for X, Y, $R(a, b)$, respectively. For $I \in \mathbf{Set}$ the fibre $\mathsf{DN_m}^I$ is defined as the preorder $(\Sigma^I, \vdash_I)$, where Σ^I the set of all functions from I to Σ and $\varphi \vdash_I \psi$ iff there exist $e^+ \in \bigcap_{i\in I}[\varphi_i^+ \to \psi_i^+]$ and $e^- \in \bigcap_{i\in I}[\varphi_i^+ \times \psi_i^- \to \mathsf{P_f}(\varphi_i^-)]$ such that

$$\forall i \in I. \forall a \in \varphi_i^+, b \in \psi_i^-. [\forall c \in e^-\langle a, b\rangle. \varphi_i(a, c)] \Rightarrow \psi_i(e^+a, b) \enspace .$$

In many cases the verification of the tripos requirement is the same as for DN together with the observation that 0 shows up in the positive and negative part of propositions. Sometimes, however, the 0 has to be added after "shifting by 1".

All fibres of $\mathsf{DN_m}$ are cartesian closed and this structure is preserved by reindexing as finite products and exponentials are constructed componentwise.

Terminal Object
is given by $\top = (\{0\}, \{0\}, \{\langle 0, 0\rangle\})$.

Products
The conjunction $p \wedge q$ is given by

(1) $(p \wedge q)^+ = p^+ \times q^+$

(2) $(p \wedge q)^- = p^- + q^-$

(3) $(p \wedge q)(\langle n, m\rangle, \langle i, k\rangle) \Leftrightarrow (i = 0 \wedge p(n, k)) \vee (i = 1 \wedge q(m, k))$.

Notice that $0 \in (p \wedge q)^+$ as $0 \in p^+$ and $0 \in q^+$ and $0 \in (p \wedge q)^-$ as $0 \in p^-$.

Notice, however, that $p \wedge q$ may be constructed also in a different way following more closely Gödel 1958. Put $(p \wedge q)^+ = p^+ \times q^+$, $(p \wedge q)^- = p^- \times q^-$ and

$$(p \wedge q)(\langle n, m\rangle, \langle k, \ell\rangle) \Leftrightarrow p^+(n, k) \wedge q^+(m, \ell) \enspace .$$

We have $0 = \langle 0,0\rangle \in p^+ \times q^+ = (p \wedge q)^+$ and $0 = \langle 0,0\rangle \in p^- \times q^- = (p \wedge q)^-$ since $0 \in p^+, p^-, q^+, q^-$ and thus the so defined $p \wedge q$ is actually in $\Sigma_{\mathsf{DN_m}}$. Notice that if $r \vdash p$ is realized by f and F and $r \vdash q$ is realized by g and G then $r \vdash p \wedge q$ is realized by $c \mapsto \langle f(c), g(c)\rangle$ and $\langle a,b\rangle \mapsto F(c,a) \cup G(c,b)$. For realizing the projections it is essential to have a "dummy" element 0 available in p^- and q, respectively. That is the reason why this construction of conjunction does not work for DN.

Exponentials
The implication $p \to q$ is given by

(1) $(p \to q)^+ = (p^+ \to q^+) \times (p^+ \times q^- \to \mathsf{P_f}(p^-))$

(2) $(p \to q)^- = p^+ \times q^-$

(3) $(p \to q)(\langle e^+, e^-\rangle, \langle a,b\rangle) \Leftrightarrow ([\forall c \in e^-\langle a,b\rangle.\, p(a,c)] \Rightarrow q(e^+a, b))$.

Notice that $0 \in (p \to q)^+$ as $0 \in q^+$ and $0 \in \mathsf{P_f}(p^-)$ (because $e_0 = \emptyset$) and $0 \in (p \to q)^-$ as $0 \in p^+$ and $0 \in q^-$.

Next we consider quantification in $\mathsf{DN_m}$. The following notation will be useful: $\mathsf{succ}(X) := \{n+1 \mid n \in X\}$ for $X \subseteq \mathbb{N}$. Again we write $[i = j]$ for $\{0 \mid i = j\}$.

Universal Quantification
For $u : I \to J$ and $\varphi \in \mathsf{DN_m}_I$ universal quantification is given by

(1) $\forall_u(\varphi)^+_j = \bigcap_{i \in I} [u(i) = j] \to \varphi^+_i$

(2) $\forall_u(\varphi)^-_j = \{0\} \cup \mathsf{succ}(\bigcup_{i \in u^{-1}(j)} \varphi^-_i)$

(3) $\forall_u(\varphi)_j(a, b+1) \Leftrightarrow \forall i \in u^{-1}(j).\, \big(b \in \varphi^-_i \Rightarrow \varphi_i(a0, b)\big)$
and $\forall_u(\varphi)_j(a, 0)$ always holds.

Notice that $\forall_u(\varphi)^+_j$ contains 0 as all φ^+_i contain 0. As $\bigcup_{i \in u^{-1}(j)} \varphi^-_i$ does not contain 0 if $u^{-1}(j)$ is empty we have added 0 to this union and shifted $\bigcup_{i \in u^{-1}(j)} \varphi^-_i$ by 1 in order to enforce that $\forall_f(\varphi)^-_j$ always contains 0.

Lemma 3.1 *In $\mathsf{DN_m}$ it holds for all functions $u : I \to J$ that $u^* \dashv \forall_u$.*

Proof. Essentially like the proof of Lemma 3.1.

The difference is only that for $b \in \forall_u(\varphi)_j$ we to consider the cases $b = 0$ and $b > 0$. In the second case the argument is like in the proof of Lemma 3.1 and for $b = 0$ there is nothing to show as we have defined $\forall_u(\varphi)_j(a,0)$ to hold anyway. Enforced by this additional case analysis we have to show instead

(1) If (e^+, e^-) realises $\psi \vdash_J \forall_u(\varphi)$ then putting $f^+ := \Lambda a.e^+a0$ and $f^- := \Lambda n.e^-(n+1)$ the pair (f^+, f^-) realises $u^*\psi \vdash_I \varphi$.

(2) If (f^+, f^-) realises $u^*\psi \vdash_I \varphi$ then putting $e^+ := \Lambda a.\Lambda n.f^+a$ and $e^- := \Lambda n.\textbf{if } n > 0 \textbf{ then } f^-(n-1) \textbf{ else } 0$ the pair (e^+, e^-) realises $\psi \vdash_J \forall_u(\varphi)$.

which again is verified by straightforward logical manipulation. $\square$

It is obvious from the construction that universal qantification is preserved by substitution, i.e. that BCC holds.

Notice that in case $u : I \to J$ is onto the construction of $\forall_u(\varphi)$ can be simplified by putting $\forall_u(\varphi)^+_J = \bigcap_{i \in u^{-1}(j)} \varphi^+_i$ and

$$\forall(\varphi)_j(a, b+1) \Leftrightarrow \forall i \in u^{-1}(j).\, \big(b \in \varphi^-_i \Rightarrow \varphi_i(a,b)\big) \quad .$$

Existential Quantification

For $u : I \to J$ and $\varphi \in \mathsf{DN_m}^I$ existential quantification is given by

(1) $\exists_u(\varphi)^+_j = \{0\} \cup \mathsf{succ}(\bigcup_{i \in u^{-1}(j)} \varphi^+_i)$

(2) $\exists_u(\varphi)^-_j = \bigcap_{i \in u^{-1}(j)} \big[\varphi^+_i {\to} \mathsf{P_f}(\varphi^-_i)\big]$

(3) $\exists_u(\varphi)_j(a+1, b) \Leftrightarrow \exists i \in u^{-1}(j).\, \big(a \in \varphi^+_i \wedge \forall c \in ba.\, \varphi_i(a,c)\big)$ and $\exists_u(\varphi)_j(0, b)$ never holds.

Notice that $\exists_u(\varphi)^+_j$ contains 0 by construction and $\exists_u(\varphi)^-_j$ contains 0 as all $\mathsf{P_f}(\varphi^-_i)$ and thus all $[\varphi^+_i {\to} \mathsf{P_f}(\varphi^-_i)]$ contain 0.

It is obvious from the construction that existential qantification is preserved by substitution, i.e. that BCC holds.

Notice that in case $u : I \to J$ is onto the construction of $\exists_u(\varphi)$ can be simplified by putting $\exists_u(\varphi)^+_j = \bigcap_{i \in u^{-1}(j)} \mathsf{P_f}(\varphi^-_i)$ and

$$\exists_u(\varphi)_j(a+1, b) \Leftrightarrow \exists i \in u^{-1}(j).\, \big(a \in \varphi^+_i \wedge \forall c \in b.\, \varphi_i(a,c)\big) \quad .$$

Generic Predicate
is given by the identity function on Σ considered as an element of ${\mathsf{DN_m}}^\Sigma$.

The structure exhibited so far guarantees DN to be tripos. Disjunction can de defined impredicatively à la Prawitz as usual. However, we give a more direct construction of $\bot$, negation and (Lawvere) equality.

Falsity
is given by $\bot = (\{0\}, \{0\}, \emptyset)$.

Negation
For a proposition p its negation $\neg p$ is given by

$$(\neg p)^+ = p^+ \to \mathsf{P_f}(p^-) \qquad\qquad (\neg p)^- = p^+$$

$$(\neg p)(a,b) \quad\Leftrightarrow\quad \exists c \in ab.\, \neg p(b,c)$$

Accordingly, double negation of p is given by

$$(\neg\neg p)^+ = [p^+ \to \mathsf{P_f}(p^-)] \to \mathsf{P_f}(p^+) \qquad\qquad (\neg\neg p)^- = p^+ \to \mathsf{P_f}(p^-)$$

$$(\neg\neg p)(a,b) \quad\Leftrightarrow\quad \exists c \in ab.\, \forall d \in bc.\, p(c,d)$$

because $(\neg\neg p)(a,b) \Leftrightarrow \exists c \in ab.\, \neg(\neg p)(b,c) \Leftrightarrow \exists c \in ab.\, \neg(\exists d \in bc.\, \neg p(c,d))$.

Equality
For a set I the equality predicate $\mathsf{eq}_I \in {\mathsf{DN_m}}^{I\times I}$ is given by

$$\mathsf{eq}_I(i,j)^+ = \{0\} \cup \{1 \mid i = j\} \qquad \mathsf{eq}_I(i,j)^- = \{0\}$$

$$\mathsf{eq}_I(i,j)(a,0) \quad\Leftrightarrow\quad (a = 1 \wedge i = j)$$

for all $i,j \in I$ as one easily shows that $\mathsf{eq}_I \dashv\vdash \exists_{\delta(I)}$.
Notice that, alternatively, one may define eq_I as $\mathsf{eq}_I(i,j)^+ = \{0\} = \mathsf{eq}_I(i,j)^-$ and $\mathsf{eq}_I(i,j) \equiv i = j$.

Now we can define our central notion.

Definition 3.1 *The* modified Dialectica Topos $\mathbf{DN_m}$ *is defined as* $\mathbf{Set}[\mathsf{DN_m}]$, *the topos obtained from* $\mathsf{DN_m}$ *by the tripos-to-topos construction.*

As we almost never consider **DN** we often omit the epitheton "modified" when speaking of $\mathbf{DN_m}$ and call it simply the Dialectica topos.

In contrast to the usual realizability toposes (and also the modified realizability topos) $\mathbf{DN_m}$ is not 2-valued.

Lemma 3.2 *The tripos* $\mathsf{DN_m}$ *is not 2-valued, i.e. there are propositions which are neither true nor false. Consequently, the topos* $\mathbf{DN_m}$ *is not 2–valued either.*

Proof. A proposition p is true iff $\exists a \in p^+.\forall b \in p^-.\, p(a,b)$ and p is false iff $\exists e \in p^+ {\rightarrow} \mathsf{P_f}(p^-).\forall a \in p^+.\exists b \in e \cdot a.\, \neg p(a,b)$.

Let f be a function growing faster than any total recursive function[3] Obviously, the proposition $p = (\mathbb{N}, \mathbb{N}, \{\langle n,m\rangle \mid f(n) \neq m\})$ is neither true nor false. The latter follows as $e \cdot a \geq f(a)$ for all $a \in \mathbb{N}$. $\square$

Obviously, the argument goes through for DN and **DN**, too. We do not consider the failure of 2-valuedness as a disadvantage as in the case of **DN** the $\neg\neg$-sheaves are still equivalent to **Set** and for this reason the $\neg\neg$-separated objects of **DN** are well-behaved and provide a good notion of "assembly".

However, for $\mathbf{DN_m}$ this is *not* the case as its $\neg\neg$-sheaves are not equivalent to **Set**. Actually, it is not clear how to provide a sufficiently simple characterisation of the $\neg\neg$–sheaves of $\mathbf{DN_m}$ avoiding the intricacies of double negation.

4 Relation to Modified Realizability

Already in [Hy] Martin Hyland observed that there is a surjective localic geometric morphism from **DN** to $\mathcal{E\!f\!f}_2$, the effective topos built over the Sierpinski topos, induced by a surjective geometric morphism between the triposes from which **DN** to $\mathcal{E\!f\!f}_2$ originate. The aim of this section is to show an analogous result for $\mathbf{DN_m}$ and **Mod**, the modified realizability topos.

For this purpose we first briefly recall the definition of M, the modified realizability tripos, from which the modified realizability topos **Mod** is obtained via the tripos-to-topos construction. The modified realizability topos

[3]Let $(f_n \mid n \in \mathbb{N})$ be some enumeration of all total recursive functions and define $f(n) = 1 + \mathsf{max}_{i \leq n} f(i)$.

was originally introduced by Grayson in [Gr] and further investigated by vanOosten in [vO].

The propositions of M are given by

$$\Sigma_{\mathsf{M}} = \{(A^a, A^p) \in \mathcal{P}(\mathbb{N})^2 \mid A^a \subseteq A^p \ni 0\}$$

i.e. a proposition is given by a set A^p of potential realizers containing 0 and a subset $A^a \subseteq A^p$ of actual realizers. For $I \in \mathbf{Set}$ the fibre $\mathsf{M}^I = (\Sigma_{\mathsf{M}}^I, \vdash_I)$ where Σ_{M}^I is the set of all functions from I to Σ_{M} and $\varphi \vdash_I \psi$ iff there is an

$$e \in \bigcap_{i \in I} (\varphi_i^a \to \psi_i^a) \cap (\varphi_i^p \to \psi_i^p)$$

for $\varphi, \psi \in \mathsf{M}^I$. The propositional logical structure is given by componentwise application of the following operations on Σ_{M}

$$\begin{array}{rcl}
\top & = & (\{0\}, \{0\}) \\
\bot & = & (\emptyset, \{0\}) \\
A \to B & = & ((A^a \to B^a) \cap (A^p \to B^p), A^p \to B^p) \\
A \wedge B & = & (A^a \times B^a, A^p \times B^p) \\
A \vee B & = & (A^a + B^a, A^p + B^p) \ .
\end{array}$$

The quantificational structure of M is given by

$$\begin{array}{l}
\forall_u(\varphi)_j = (\ \bigcap_{i \in u^{-1}(j)} \{0\} \to \varphi_i^a \ ,\ \bigcap_{i \in u^{-1}(j)} \{0\} \to \varphi_i^p\ ) \\
\exists_u(\varphi)_j = (\ \mathsf{succ}(\bigcup_{i \in u^{-1}(j)} \varphi_i^a)\ ,\ \{0\} \cup \mathsf{succ}(\bigcup_{i \in u^{-1}(j)} \varphi_i^p)\ )
\end{array}$$

for $u : I \to J$ and $\varphi \in \mathsf{M}^I$. Notice that this definition of quantifiers is easily seen to be equivalent to the one of [vO].

Now we will define a geometric morphism from $\mathsf{DN_m}$ to M.

Definition 4.1 *Let the maps* $q^* : \Sigma_{\mathsf{M}} \to \Sigma_{\mathsf{DN_m}}$ *and* $q_* : \Sigma_{\mathsf{DN_m}} \to \Sigma_{\mathsf{M}}$ *be defined as*

$$\begin{array}{l}
q^*(A^a, A^p) = (A^p, \{0\}, A^a \times \{0\}) \\
q_*(X, Y, R) = (\ \{x \in X \mid \forall y \in Y.\, R(x,y)\}\ ,\ X\ ) \ .
\end{array}$$

We also write $q^* : \mathsf{M} \to \mathsf{DN_m}$ *and* $q_* : \mathsf{DN_m} \to \mathsf{M}$ *for the morphisms of triposes induced by componentwise application.*

That q^* and q_* are tripos morphisms is immediate from the fact that they respect logical entailment in each fibre and commute with reindexing as they are defined componentwise.

Theorem 4.1 *The tripos morphisms of Definition 4.1 give rise to a connected geometric morphism $q : \mathsf{DN_m} \to \mathsf{M}$ as given by the adjunction $q^* \dashv q_*$ where q^* preserves finite limits and is full and faithful.*

But q_* does not have a right adjoint as q_* does not preserve existential quantification.

Proof. We first check that $q^* \dashv q_*$. Let $A \in \mathsf{M}^I$ and $p \in \mathsf{DN_m}^I$.

Suppose $q^*A \vdash_I p$. Then there exist

$$e^+ \in \bigcap_{i \in I} (A_i^p \to p_i^+) \qquad \text{and} \qquad e^- \in \bigcap_{i \in I} (A_i^p \times p_i^- \to \mathsf{P_f}(\{0\}))$$

such that

$$\forall i \in I. \forall a \in A_i^p, b \in p_i^-. (\forall c \in e^-\langle a, b\rangle. a \in A_i^a) \Rightarrow p_i(e^+a, b)$$

from which it follows that

$$\forall i \in I. \forall a \in A_i^p. a \in A_i^a \Rightarrow \forall b \in p_i^-. p_i(e^+a, b)$$

which is equivalent to

$$\forall i \in I. \forall a \in A_i^p. a \in A_i^a \Rightarrow (q_*p)_i^a$$

and, therefore, it holds that $A \vdash_I q_*p$.

Suppose that $A \vdash_I q_*p$. Then there exists an

$$f \in \bigcap_{i \in I} (A_i^p \to p_i^+)$$

with $\forall i \in I. \forall a \in A_i^p. a \in A_i^a \Rightarrow (q_*p)_i^a$, i.e.

$$\forall i \in I. \forall a \in A_i^p. a \in A_i^a \Rightarrow \forall b \in p_i^-. p_i(fa, b) \ .$$

But then putting $e^+ := f$ and $e^- = \Lambda a.\Lambda b.c_0$ with $e_{c_0} = \{0\}$ we get that $e^+ \in \bigcap_{i \in I} (A_i^p \to p_i^+)$ and $e^- \in \bigcap_{i \in I} (A_i^p \times p_i^- \to \mathsf{P_f}(\{0\}))$ satisfying

$$\forall i \in I. \forall a \in A_i^p, b \in p_i^-. (\forall c \in e^-\langle a, b\rangle. a \in A_i^a) \Rightarrow p_i(e^+a, b)$$

and, therfore, it holds that $q^*A \vdash_I p$.

It is a straightforward exercise to show that q^* preserves $\top$ and $\wedge$ and that q^* is full (it is faithful anyway!).

That q_* does not preserve existential quantification can be seen from the following counterexample. Let $p \in \mathsf{DN_m}^{\mathbb{N}}$ be defined as

$$p_n^+ = \{0\} \qquad p_n^+ = \{0, n+1\} \qquad p_n = \{\langle 0, 0\rangle\}$$

for $n \in \mathbb{N}$. We write $\exists_N$ for existential quantification along the terminal projection $\mathbb{N} \to 1$. We have $\exists_N p = (\{0,1\}, \{0,1\}{\to}\{0\}, \{1\}{\times}(\{0,1\}{\to}\{0\}))$ and, accordingly, $q_*\exists_N p = (\{1\}, \{0,1\})$. On the other hand $(q_*p)_n = q_*(p_n) = (\emptyset, \{0\})$ for all $n \in \mathbb{N}$ and accordingly $\exists_N q_* p = (\emptyset, \{0,1\})$. Thus, $q_*\exists_N p$ and $\exists_N q_* p$ are not equivalent as the former is true and the latter is false. $\square$

5 Markov's Principle in $\mathbf{DN_m}$

We next prove a lemma from which it follows that $\mathbf{DN_m}$ validates Markov's principle.

Lemma 5.1 *Let $p = (X, U, R)$ be a proposition with $U = \{0\}$ and R a decidable predicate on $X \times \{0\}$. Then $\neg\neg p \vdash p$ holds in $\mathsf{DN_m}$.*

Proof. The proposition $\neg\neg p$ looks as follows. Its positive underlying set is $(\neg\neg p)^+ = (\neg\neg p)^- \to \mathsf{P_f}(p^+)$ where $(\neg\neg p)^- = p^+ \to \mathsf{P_f}(\{0\})$ is the set of (Gödel numbers of) decidable predicates on p^+ as $\mathsf{P_f}(\{0\}) = \{0,1\}$ (recall that $e_0 = \emptyset$ and $e_1 = \{0\}$). The underlying relation of $\neg\neg p$ is given by

$$(\neg\neg p)(a,b) \equiv \exists c \in ab. \forall d \in bc. p(c,d) \Leftrightarrow \exists c \in ab. (bc = 1) \to p(c,0) \quad .$$

Now the entailment $\neg\neg p \vdash p$ is realized by

$$e^+ a = \mu c \in a(\Lambda n.1).\, p(c,0) \qquad \text{and} \qquad e^- \langle a, 0\rangle = \Lambda n.1$$

as it holds that

$$\forall a \in (\neg\neg p)^+.\ [\exists c \in a(\Lambda n.1).\, p(c,0)] \Rightarrow p(e^+a, 0)$$

because if $\exists c \in a(\Lambda n.1).\, p(c,0)$ then by definition of e^+ it holds that $e^+ a \in a(\Lambda n.1)$ and $p(e^+a, 0)$. $\square$

Since inverse image parts of geometric morphisms preserve nno's we know that a nno in $\mathbf{DN_m}$ is given by $q^*(N)$ where N is a nno in **Mod**. In [vO] one finds the following description of a nno in **Mod**, namely as $N = (\mathbb{N}, E_N)$ where $E_N(n,m) = (\emptyset, \{0\})$ if $n \neq m$ and $E_N(n,n) = (\{n{+}1\}, \{0, n{+}1\})$ otherwise. Thus, a nno in $\mathbf{DN_m}$ is given by $(\mathbb{N}, E_n)$ where $E_N(n,m) = (\{0\}, \{0\}, \emptyset)$ if $n \neq m$ and $E_N(n,n) = (\{0, n{+}1\}, \{0\}, \{\langle n{+}1, 0\rangle\})$ otherwise.

An alternative construction of nno in $\mathbf{DN_m}$ which is often easier to compute with looks as follows: again the underlyings set is $\mathbb{N}$ but the equality predicate is given by $E_N(n,m) = (\mathbb{N}, \{0\}, \{\langle k, 0\rangle \mid n = k = m\})$.

Notice that for any of these constructions the proposition $E_n(n,m)$ satifies the assumption of Lemma 5.1.

Theorem 5.2 *In* $\mathbf{DN_m}$ *the Markov principle*

$$\text{MP} \qquad \neg\neg\exists x{:}N.P(x) \to \exists x{:}N.P(x)$$

holds for every quantifier-free formula $P(x)$*.*

Proof. Every quantifierfree predicate $P(x)$ is decidable and thus the negative part of the interpretation of P can always be chosen as $\{0\}$. We have already observed that the negative part of the existence predicate E_n can always be chosen as $\{0\}$. Thus, we can always choose $[\![\exists x{:}N.P(x)]\!]_{\mathbf{DN_m}}$ in such a way that its negative part is $\{0\}$ from which it follows by Lemma 5.1 that $[\![\exists x{:}N.P(x)]\!]_{\mathbf{DN_m}}$ is $\neg\neg$-stable in $\mathbf{DN_m}$ as claimed by MP. $\square$

It is a distinguishing feature of Gödel's original functional interpretation that it validates a generalized Markov's principle

$$\text{MP}_\sigma \qquad \neg\neg\exists x{:}\sigma.P(x) \to \exists x{:}\sigma.P(x)$$

for all finite types σ over N. This presumably is wrong for $\mathbf{DN_m}$ since existence predicate for arbitrary finite types σ will not satisfy the assumption of Lemma 5.1. However, when performing the modified Diller-Nahm construction over the *typed* pca HRO one can show the validity of MP_σ with an argument like in the proof of Theorem 5.2. The reason is that typed realizability over HRO renders the existence predicates E_σ "discrete" in the sense that $E_\sigma(u,v) = (\sigma, 1, \{\langle w, *\rangle \mid u = w = v\})$ (where 1 is the terminal type containing just the element $*$).

Acknowledgement I thank Bodil Biering for pointing out mistakes in a previous version.

References

[BiRo] L. Birkedal, G. Rosolini *Dialectica Tripos* private communication, June 2000.

[Gr] R. Grayson *Modified Realisability Toposes* handwritten notes, Universität Münster, 1981.

[Hy] M. Hyland *Lectures on Realizability* handwritten notes of a talk given in the Netherlands in the first half of the 1980ies.

[vO] J. vanOosten *The Modified Realizability Topos* Journal of Pure and Applied Algebra, 1997.

Appendix A Dialectica Tripos

In [BiRo] L. Birkedal and G. Rosolini have shown that implication exists in Gödel's original setting even if relations are not decidable. However, this implication is different from Gödel's original one which rather forms part of a monoidal closed structure different from the cartesian one.

The (modified) Dialectica tripos $\mathsf{D_m}$ is defined as follows. Its underlying set of propositions is given by

$$\Sigma_{\mathsf{D_m}} = \{(X, U, R) \in \mathcal{P}_0(\mathbb{N})^2 \times \mathcal{P}(\mathbb{N}^2) \mid R \subseteq X \times U\} \; .$$

For $I \in \mathbf{Set}$ the fibre ${\mathsf{D_m}}^I$ is the preorder whose underlying set of elements is the set of all arbitrary set-theoretic functions from I to $\Sigma_{\mathsf{D_m}}$ and where $\varphi \vdash_I \psi$ iff there exist

$$e^+ \in \bigcap_{i \in I} (\varphi_i^+ \to \psi_i^+) \qquad e^- \in \bigcap_{i \in I} (\varphi_i^+ \times \psi_i^- \to \varphi_i^+)$$

such that

$$\forall i \in I.\, \forall a \in \varphi_i^+.\, \forall b \in \psi_i^+.\, \varphi_i(a, e^-\langle a, b\rangle) \Rightarrow \psi_i(e^+ a, b) \; .$$

Terminal objects, products, universal and existential quantification are constructed as for $\mathsf{DN_m}$ in section 3. However, the construction of exponentials differs considerably from the orginal one given by Gödel in 1958.

Exponentials
The implication $p \to q$ is constructed as follows

(1) $(p \to q)^+$ consists of all

$$\langle e^+, e^-\rangle \in (p^+ \to q^+) \times (p^+ \times q^- \to \{0\} + p^-)$$

such that

$$\forall a \in p^+.\, \forall b \in q^-.\, \pi_1(e^-\langle a, b\rangle) = 1 \Rightarrow p(a, \pi_2(e^-\langle a, b\rangle)) \Rightarrow q(e^+ a, b)$$

where $\pi_i(\langle n_1, n_2\rangle) = n_i$ for $i = 1, 2$.

(2) $(p \to q)^- = p^+ \times q^-$

(3) $(p \to q)(\langle e^+, e^- \rangle, \langle a, b \rangle) \Leftrightarrow (\pi_1(e^- \langle a, b \rangle) = 0 \Rightarrow q(e^+ a, b))$.

Notice that the restriction to propositions whose positive and negative underlying sets both contain 0 is intrinsic for showing that $\theta \vdash \varphi \to \psi$ implies $\theta \wedge \varphi \vdash \psi$.

Negation
For a proposition p its negation $\neg p$ up to isomorphism looks as follows. The positive part $(\neg p)^+$ consists of all $e : p^+ \to \{0\} + p^-$ such that

$$\forall a \in p^+ .\, \pi_1(e \cdot a) = 1 \Rightarrow \neg p(a, \pi_2(e \cdot a)) \quad .$$

The negative part $(\neg p)^- = p^+$ and $(\neg p)(e, a) \equiv \pi_1(e \cdot a) = 1$.

Again as for $\mathsf{DN_m}$ we have

Lemma 5.3 *Let $p = (X, U, R)$ be a proposition with $U = \{0\}$ and R a decidable predicate on $X \times \{0\}$. Then $\neg\neg p \vdash p$*

Proof. Due to the above explicitation of negation we have that $(\neg p)^+$ consists $e : p^+ \to \{0\} + \{0\}$ such that $\forall a \in p^+ .\, \pi_1(ea) = 1 \Rightarrow \neg p(a, 0)$, $(\neg p)^- = p^+$ and $(\neg p)(e, a) \equiv \pi_1(ea) = 1$. Accordingly, $(\neg\neg p)^+$ consists of all $f \in (\neg p)^+ \to \{0\} + p^+$ such that $\forall e \in (\neg p)^+ .\, \pi_1(fe) = 1 \Rightarrow \pi_1(e \cdot \pi_2(f \cdot e)) = 0$, $(\neg\neg p)^+ = (\neg p)^+$ and $(\neg\neg p)(f, e) \equiv \pi_1(f \cdot e) = 1$.

Let $n_0 \in p^+ \to \{0\} + \{0\}$ with $\pi_1(n_0 \cdot a) = 1$ iff $p(a, 0)$ for all $a \in p^+$. Such an n_0 exists as $p(\_, 0)$ is decidable by assumption and, obviously, it is an element of $(\neg p)^+$ by construction. Now $\neg\neg p \vdash p$ is realized by

$$e^+ = \Lambda f.\, \pi_2(f \cdot n_0) \qquad \text{and} \quad e^- = \Lambda z.\, n_0$$

as it holds that

$$\forall f \in (\neg\neg p)^+ .\, \pi_1(f \cdot n_0) = 1 \Rightarrow p(\pi_2(f \cdot n_0), 0)$$

which follows immediately from the expansions of the statements $f \in (\neg\neg p)^+$ and $n_0 \in (\neg p)^+$. $\square$

Finally, we observe that there is a connected geometric morphism from $\mathsf{D_m}$ to M which is defined in the same way as in Theorem 4.1.

Appendix B
Assemblies and Discrete Sets in DN

In this appendix we will have a closer look at the category of assemblies $\mathcal{A} = \mathsf{Sep}_{\neg\neg}(\mathbf{DN})$ and the discrete objects in there. They will turn out as closely related to assemblies and modest sets in $\mathcal{E}\!f\!f$.

Let $U \in \mathbf{DN}$ be the subterminal object whose underlying set is a singleton $\{*\}$ and where $[\![* =_U *]\!] = (\{0\}, \{0\}, \emptyset)$. One can show that $\Gamma_U = \mathbf{DN}(U, -) : \mathbf{DN} \to \mathbf{Set}$ is the sheafification functor for the $\neg\neg$-topology on $\mathbf{DN}$ whose right adjoint is the inclusion $\nabla_U : \mathbf{Set} \to \mathbf{DN}$ sending S to (S, eq_S).

The category $\mathcal{A} = \mathsf{Sep}_{\neg\neg}(\mathbf{DN})$ of *assemblies*, i.e. $\neg\neg$-separated objects in $\mathbf{DN}$ can be described up to equivalence as follows. An assembly is given by a pair $X = (|X|, ||\cdot||_X)$ where $|X|$ is a set and $||x||_X^+ \neq \emptyset$ for all $x \in |X|$. Often we write simply $||x||$ instead of $||x||_X$ when X is clear from the context. A morphism (of assemblies) from X to Y is an ordinary set-theoretic function $f : |X| \to |Y|$ such that there exist

$$e^+ \in \bigcap_{x\in|X|} ||x||^+ \to ||f(x)||^+ \qquad e^- \in \bigcap_{x\in|X|} ||x||^+ \times ||f(x)||^- \to \mathsf{P_f}(||x||^-)$$

satisfying

$$\forall x{\in}|X|.\, \forall a{\in}||x||^+, b{\in}||f(x)||^-.\ [\forall c \in e^-\langle a,b\rangle . ||x||(a,c)] \Rightarrow ||f(x)||(e^+a, b)\,.$$

When restricting Γ_U to $\mathcal{A}$ then it looks particularly simple, namely $\Gamma_U(X) = |X|$ and similarly for morphisms. The inclusion ∇_U sends S to $(S, \lambda x{:}\, S.\top)$. Therefore, unlike in case of $\mathcal{E}\!f\!f$ the category of assemblies is *not well-pointed* anymore but, of course, $U = (\{*\}, * \mapsto (\{0\}, \{0\}, \emptyset))$ is a separating object.

For this reason for $X, Y \in \mathcal{A}$ their exponential $F = Y^X$ looks somewhat different from what one might expect (from the case of assemblies in $\mathcal{E}\!f\!f$). The underlying set of F is not $\mathcal{A}(X, Y)$ but

$$|F| = \{f : |X| \to |Y| \mid \exists e \in \bigcap_{x\in|X|} ||x||^+ \to ||f(x)||^+\}$$

i.e. the underlying set of F consists of all morphisms from $\mathcal{U}(X)$ to $\mathcal{U}(Y)$ in the category of assemblies in $\mathcal{E}\!f\!f$ (where $\mathcal{U}$ is the forgetful functor sending

$X \in \mathcal{A}$ to the assembly $(|X|, \lambda x{:}\,|X|.||x||^+)$ in $\mathcal{E}\!f\!f$). The existence predicate $||\cdot||_F$ is defined for $f \in |F|$ as follows: $||f||_F^+$ consists of all $\langle e^+, e^- \rangle$ with

$$e^+ \in \bigcap_{x\in|X|} ||x||^+ \to ||f(x)||^+ \qquad e^- \in \bigcap_{x\in|X|} ||x||^+ \times ||f(x)||^- \to \mathsf{P_f}(||x||^-)$$

$||f||^- = \bigcup_{x\in|X|} ||x||^+ \times ||f(x)||^-$ and $||f||_F(\langle e^+, e^- \rangle, \langle a, b\rangle)$ holds iff

$$\forall x{\in}|X|.\, a{\in}||x||^+ \wedge b{\in}||f(x)||^- \Rightarrow [\forall c \in e^-\langle a, b\rangle.||x||(a, c)] \Rightarrow ||f(x)||(e^+a, b)\,.$$

Recall that an object X in $\mathcal{A}$ is called *discrete* iff

$$X^{\nabla_U(!_2)} : X^{\nabla_U(1)} \to X^{\nabla_U(2)}$$

is an isomorphism. Using the explicit construction of exponentials above one show that X is discrete iff $\mathcal{U}(X)$ is a modest set in $\mathsf{Sep}_{\neg\neg}(\mathcal{E}\!f\!f)$, the category of assemblies in $\mathcal{E}\!f\!f$. Thus X is discrete iff $n \in ||x_1||^+ \cap ||x_2||^+$ implies $x_1 = x_2$.

We conjecture that the "Sierpinski object"

$$S = \{p \in \Omega \mid \exists f \in N^N.\, p \leftrightarrow \exists n \in N. f(n) = 0\}$$

stays within ω–**Set**, too. Provided this is correct SDT in **DN** shouldn't look too differently from SDT in $\mathcal{E}\!f\!f$.