Distributors between Fibrations

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Abstract

For a category $\mathbf{B}$ the fibration $\partial_1 : \widehat{\mathbf{B}}\downarrow\mathsf{Y}_{\mathbf{B}} \to \mathbf{B}$ is equivalent to the split fibration $\mathbf{Set}(\mathbf{B})$ over $\mathbf{B}$ where $\mathbf{Set}(\mathbf{B})(I) = \mathbf{Set}^{(\mathbf{B}/I)^{\mathsf{op}}}$. For split fibrations, *i.e.* categories C internal to $\widehat{\mathbf{B}}$, the split fibration $\mathbf{Sp}/\mathbf{B}(C, \mathbf{Set}(\mathbf{B}))$ amounts to the category of $\widehat{\mathbf{B}}$–valued functors on C (internal to $\widehat{\mathbf{B}}$).
Thus, for $C, D \in \mathbf{cat}(\widehat{\mathbf{B}})$ the split fibration $\mathbf{Sp}/\mathbf{B}(D^{\mathsf{op}}\times C, \mathbf{Set}(\mathbf{B}))$ amounts to the category of distributors from C to D (internal to $\widehat{\mathbf{B}}$). By analogy for fibrations $P, Q \in \mathbf{Fib}/\mathbf{B}$ the internal category of distributors from P to Q may be defined as

$$\mathsf{Dist}_{\mathbf{B}}(P, Q) = \mathsf{Func}_{\mathbf{B}}(Q^{\mathsf{op}}\times P, \mathbf{Set}(\mathbf{B}))$$

where Q^{op} is the fibration opposite to Q and $\mathsf{Func}_{\mathbf{B}}$ stands for the exponential in $\mathbf{Fib}/\mathbf{B}$.

1 Preliminaries

We first recall a result about fibrations generalising the well-known fact that presheaf toposes are closed under slicing.

Theorem 1.1 *Let* $\mathbf{B}$ *be a category and* $P : \mathbf{X} \to \mathbf{B}$ *be a discrete fibration over* $\mathbf{B}$*. Then* $(\mathbf{Fib}/\mathbf{B})/P$ *is* 2*-isomorphic to* $\mathbf{Fib}/\mathbf{X}$*. Moreover, this isomorphism preserves discreteness in the sense that it restricts to an isomorphism between* $(\mathbf{Disc}/\mathbf{B})/P$ *and* $\mathbf{Disc}/\mathbf{X}$*.*

Proof: First we show that a functor $F : \mathbf{Y} \to \mathbf{X}$ is a fibration iff $Q = P \circ F$ is a fibration and $F \in \mathsf{Cart}_{\mathbf{B}}(Q, P)$. The implication from left to right is easy.

For the reverse direction assume that $F \in \mathsf{Cart}_{\mathbf{B}}(Q, P)$. For showing that F is a fibration over $\mathbf{X}$ suppose that $X \in \mathbf{Y}$ and $f : A \to F(X)$ in $\mathbf{X}$. Let $\varphi : Y \to X$ be a Q-cartesian arrow over $P(f)$. We claim that φ is F-cartesian over f. As $F(\varphi)$ and f have the same codomain and are both above $F(f)$ it follows from discreteness of the fibration Q that $F(\varphi) = f$. Thus, the arrow φ is over f. For F-cartesianness of φ suppose that $\psi : Z \to X$ and $g : B \to A$ with $F(\psi) = f \circ g$. As f is Q-cartesian there exists a $\theta : Z \to Y$ over $P(g)$ with $\psi = \varphi \circ \theta$. As $P(g) = Q(\theta) = P(F(\theta))$ and both g and $F(\theta)$ have the same codomain A it follows by discreteness of P that $g = F(\theta)$. Uniqueness of θ is immediate from Q-cartesianness of φ as if $\theta' : Z \to Y$ with $\varphi \circ \theta'$ and $F(\theta') = g$ then $Q(\theta') = P(F(\theta')) = P(g)$ and thus $\theta = \theta'$.

Actually, we have shown that a morphism φ in $\mathbf{Y}$ is cartesian w.r.t. F iff it is cartesian w.r.t. $P \circ F$. Moreover, as P is a discrete fibration it reflects identities and, therefore, a morphism v in $\mathbf{Y}$ is vertical w.r.t. F iff it is vertical w.r.t. $P \circ F$.

Thus, if G is a fibration over $\mathbf{X}$ and $U : G \to F$ in $\mathbf{Cat}/\mathbf{X}$ then U is a cartesian functor from G to F over $\mathbf{X}$ iff U is a cartesian functor from $P \circ G$ to $P \circ F$. If U and U' are morphisms from G to F in $\mathbf{Cat}/\mathbf{X}$ and τ is a natural transformation from U to U' then τ is cartesian over $\mathbf{X}$ iff it is cartesian over $\mathbf{B}$.

As P reflects identities it follows that a fibration $F : \mathbf{Y} \to \mathbf{X}$ is discrete iff $P \circ F$ is discrete. $\square$

As for arbitrary $\mathbf{B}$ the presheaf category $\widehat{\mathbf{B}}$ is equivalent to $\mathbf{Disc}/\mathbf{B}$ we get as an immediate consequence of Theorem 1.1 the following corollary.

Corollary 1.1 *For every $X \in \widehat{\mathbf{B}}$ we have $\widehat{\mathbf{B}}/X \simeq \widehat{\mathbf{El}(X)}$ where $P_X = \partial_0 : \mathbf{El}(X) = \mathsf{Y}_{\mathbf{B}}{\downarrow}X \to \mathbf{B}$ is the discrete fibration obtained from X via the Grothendieck construction.*

Proof: By Theorem 1.1 we have $(\mathbf{Disc}/\mathbf{B})/P_X \cong \mathbf{Disc}/\mathbf{El}(X)$. The claim follows as $\widehat{\mathbf{B}}/X \cong (\mathbf{Disc}/\mathbf{B})/P_X$ and $\mathbf{Disc}/\mathbf{El}(X) \simeq \widehat{\mathbf{El}(X)}$. $\square$

In the following we will often tacitly use Corollary 1.1 for constructing maps to X by exhibiting the corresponding presheaf over $\mathbf{El}(X)$.

We have seen that *X-indexed families of fibrations* correspond to cartesian functors to X considered as a discrete fibration over $\mathbf{B}$. The dual concept of a cartesian functor from X to a fibration P over $\mathbf{B}$ can be understood as the notion of "X-indexed family of objects" of the category over $\mathbf{B}$ as given by P. Accordingly, we may associate with every $P \in \mathbf{Fib}/\mathbf{B}$ the functor $\widehat{Sp}(P) : \widehat{\mathbf{B}}^{\mathsf{op}} \to \mathbf{Cat}$ defined as

$$\widehat{Sp}(P)(X) = \mathbf{Fib}/\mathbf{B}(X, P) \qquad\qquad \widehat{Sp}(P)(f) = \mathbf{Fib}/\mathbf{B}(f, P)$$

where $X \in \widehat{\mathbf{B}}$ understood as a discrete fibration over $\mathbf{B}$ and $f : Y \to X$ is a morphism in $\widehat{\mathbf{B}}$ understood as a cartesian functor between discrete fibrations over $\mathbf{B}$. Now restricting $\widehat{Sp}(P)$ along the Yoneda functor $\mathsf{Y}_{\mathbf{B}} : \mathbf{B} \hookrightarrow \widehat{\mathbf{B}}$ gives rise to $Sp(P) = \widehat{Sp}(P) \circ \mathsf{Y}_{\mathbf{B}} : \mathbf{B}^{\mathsf{op}} \to \mathbf{Cat}$ canonically equivalent to P according to the fibred Yoneda lemma (see [B80]).[1]

Notice that $\mathbf{Sp}/\mathbf{B}$, the 2-category of split fibrations over $\mathbf{B}$, is isomorphic to $\mathbf{cat}(\widehat{\mathbf{B}})$, the 2-category of internal categories in $\widehat{\mathbf{B}}$. As for all $X \in \widehat{\mathbf{B}}$ there is a canonical isomorphism

$$\mathbf{Fib}/\mathbf{B}(X, P) \cong \mathbf{Sp}/\mathbf{B}(X, Sp(P))$$

natural in X we get that $\widehat{Sp}(P)$ is canonically isomorphic to the externalisation of $Sp(P)$ considered as a category internal to $\widehat{\mathbf{B}}$.

Due to the above mentioned identification of $\mathbf{Sp}/\mathbf{B}$ and $\mathbf{cat}(\widehat{\mathbf{B}})$ one may use the internal language of $\widehat{\mathbf{B}}$ for speaking about $\mathbf{Sp}/\mathbf{B}$. However, its expressivity is limited for the following reason. For an internal functor F the logic of $\widehat{\mathbf{B}}$ validates the proposition "F is an equivalence" if and only if $U(F)$ is an equivalence in $\mathbf{Fib}/\mathbf{B}$ (we call such F *weak equivalences*). Accordingly,

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[1]The full version of the fibred Yoneda lemma (see [Str] but due to J. Bénabou) says that the forgetful 2-functor $U : \mathbf{Sp}/\mathbf{B} \to \mathbf{Fib}/\mathbf{B}$ has a right 2-adjoint Sp, *i.e.*

$$\mathbf{Fib}/\mathbf{B}(U(S), P) \simeq \mathbf{Sp}/\mathbf{B}(S, Sp(P))$$

naturally in $S \in \mathbf{Sp}/\mathbf{B}$ and $P \in \mathbf{Fib}/\mathbf{B}$. For this 2-adjunction the components of the counit $E : U \circ Sp \to Id_{\mathbf{Fib}/\mathbf{B}}$ are all equivalences whereas the components of the unit $H : Id_{\mathbf{Sp}/\mathbf{B}} \to Sp \circ U$ in general are *not* equivalences in $\mathbf{Sp}/\mathbf{B}$ *although* all $U(H_S)$ are equivalences in $\mathbf{Fib}/\mathbf{B}$.

Thus, one may obtain $\mathbf{Fib}/\mathbf{B}$ from $\mathbf{Sp}/\mathbf{B}$ by freely quasi-inverting weak equivalences in $\mathbf{Sp}/\mathbf{B}$, *i.e.* those split cartesian functors in $\mathbf{Sp}/\mathbf{B}$ whose image under U gets an equivalence in $\mathbf{Fib}/\mathbf{B}$, where "quasi-inverting" means "sending to an equivalence". This, however, does not mean that $Sp_{P,Q} : \mathbf{Fib}/\mathbf{B}(P, Q) \to \mathbf{Sp}/\mathbf{B}(Sp(P), Sp(Q)) : F \mapsto Sp(F)$ is an isomorphism of categories, it just is an equivalence of categories.

in $\widehat{\mathbf{B}}$ one cannot express the notion of equivalence. Actually, this fails already for functors to discrete internal categories in $\widehat{\mathbf{B}}$ as in its internal logic one can express that a morphism is epic but not that it is split epic.[2]

¿ *Does that mean that* $\mathbf{Sp}/\mathbf{B}$ *looks like* $\mathbf{Fib}/\mathbf{B}$ *from point of view of* $\widehat{\mathbf{B}}$ *?*

In [B80] one finds the following characterisation of "fibred fibrations".

Theorem 1.2 *Let* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration. Then a* $F : \mathbf{Y} \to \mathbf{X}$ *is a fibration if and only if* $Q = P \circ F$ *is a fibration and* F *is a cartesian functor from* Q *to* P *such that*

(1) *every* $F_I : \mathbf{Y}_I \to \mathbf{X}_I$ *is a fibration and*

(2) *for every commuting square*

$$\begin{tikzcd} Y_1 \arrow[r, "\varphi_1"] \arrow[d, "g"'] & X_1 \arrow[d, "f"] \\ Y_2 \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

with φ_1 *and* φ_2 Q*-cartesian over* $u : J \to I$ *and* f *and* g Q*-vertical the arrow* g *is* F_J*-cartesian whenever* f *is* F_I*-cartesian.*

Definition 1.1 *Let* $\mathbf{B}$ *be a category. We write* $\mathbf{Fib}//\mathbf{B}$ *for the* 2*-fibration obtained from the* 2*-fibration* $\mathbf{Fib} \to \mathbf{Cat}$ *by change of base along* $\mathbf{B}/(-) : \mathbf{B} \to \mathbf{Cat}$. *Similarly, we write* $\mathbf{Disc}//\mathbf{B}$ *and* $\mathbf{Sp}//\mathbf{B}$ *for the* 2 *fibrations we get by change of base along* $\mathbf{B}/(-) : \mathbf{B} \to \mathbf{Cat}$ *from the* 2*-fibrations* $\mathbf{Disc} \to \mathbf{Cat}$ *and* $\mathbf{Sp} \to \mathbf{Cat}$, *respectively.* $\Diamond$

2 B as a Universe of Small Objects in $\widehat{\mathbf{B}}$

One wants to think of the representable objects in $\widehat{\mathbf{B}}$ as "small" objects in $\widehat{\mathbf{B}}$ whereas non-representable presheaves are thought of as "big" objects in $\widehat{\mathbf{B}}$. As usual for arbitrary $X \in \widehat{\mathbf{B}}$ a family of possibly big objects indexed by X is simply a morphism $f : Y \to X$ in $\widehat{\mathbf{B}}$. However, there arises the

[2] But if $\mathbf{B}$ has a terminal object then for a fixed morphism $f : X \to Y$ one can express in the internal language of $\widehat{\mathbf{B}}$ that it is a split epi by the formula $\exists s{:}X^Y.f \circ s = id$ since if it is forced at 1 then there does exist an element s of X^Y over 1 with $f \circ s = id$.

question of what is a family of "small" objects indexed by a possibly big object. An answer to this question is provided by A. Grothendieck's notion of *representable morphism.*

Definition 2.1 *A morphism $f : Y \to X$ in $\widehat{\mathbf{B}}$ is called* representable *or a* family of small objects *iff for all pullbacks*

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ I \arrow[r] & X \end{tikzcd}$$

the object J is representable whenever I is representable. $\Diamond$

Notice that $\mathsf{Y}_{\mathbf{B}}(\alpha)$ is representable iff all pullbacks of α exists in $\mathbf{B}$ (as $\mathsf{Y}_{\mathbf{B}}$ preserves and reflects finite limits). Thus, under the (reasonable) assumption that $\mathbf{B}$ has pullbacks families of small objects indexed by representable objects are precisely the maps in $\mathbf{B}$. A terminal projection $X \to 1$ is small iff $\mathsf{Y}_{\mathbf{B}}(I) \times X$ is representable for all $I \in \mathbf{B}$. Thus, even for representable presheaves X the terminal projection $X \to 1$ need not be a representable morphism unless $\mathbf{B}$ has binary products. Moreover, the terminal object in $\widehat{\mathbf{B}}$ need not be representable unless $\mathbf{B}$ has a terminal object.

We now investigate some closure properties of representable maps relevant when viewing them as families of small objects.

Lemma 2.1 *Let $\mathbf{B}$ be a category. Then for the collection $\mathcal{S}$ of representable maps in $\widehat{\mathbf{B}}$ it holds that*

(1) *$\mathcal{S}$ is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbf{B}}$.*

(2) *$\mathcal{S}$ is a subcategory of $\mathcal{S}$ containing all isomorphism of $\widehat{\mathbf{B}}$.*

(3) *There exists a* generic *family for $\mathcal{S}$, i.e. a map $el : E \to set(\mathbf{B})$ in $\mathcal{S}$ such that every map in $\mathcal{S}$ can be obtained as pullback of el.*

(4) *If $\mathbf{B}$ is locally cartesian closed then $\mathcal{S}$ is closed under dependent products, i.e. whenever $f : Y \to X$ and $g : Z \to Y$ are in $\mathcal{S}$ then $\Pi_f(g) \in \mathcal{S}$, too.*

Proof: It is obvious from Definition 2.1 that $f : Y \to X$ is in $\mathcal{S}$ iff for every $s : I \to X$ with I representable it holds that s^*f is a map between representable objects. From this it is clear that $\mathcal{S}$ is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbf{B}}$ and that $\mathcal{S}$ contains all isos. For closure under composition suppose $f : Y \to X$ and $g : Z \to Y$ are in $\mathcal{S}$. Let $s : I \to X$ with I representable. Consider the pullback

$$\begin{tikzcd} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ I \arrow[r, "s"'] & X \end{tikzcd}$$

As $f \in \mathcal{S}$ it follows that J is representable and, therefore, as $g \in \mathcal{S}$ it follows that K is representable, too. Thus $\mathcal{S}$ satisfies (1) and (2).
ad (3) : According to Corollary 1.1 morphisms in $\widehat{\mathbf{B}}$ to $I \in \mathbf{B}$ correspond to presheaves over $\mathbf{B}/I$, the category of elements of $\mathsf{Y}_{\mathbf{B}}(I)$. One easily sees that A is a representable presheaf over $\mathbf{B}/I$ iff the source of the corresponding morphism to I in $\widehat{\mathbf{B}}$ is a representable presheaf over $\mathbf{B}$. Thus, a presheaf $A : (\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}$ corresponds to a small map to I iff for all $\alpha : J \to I$ in $\mathbf{B}$ the presheaf $(\Sigma_\alpha)^*A = A \circ (\Sigma_\alpha)^{\mathsf{op}} : (\mathbf{B}/J)^{\mathsf{op}} \to \mathbf{Set}$ is representable. Such presheaves A we call "stably representable" and they organise into a presheaf $set(\mathbf{B}) : \mathbf{B}^{\mathsf{op}} \to \mathbf{Set}$ putting

$$set(\mathbf{B})(I) = \{A \in \widehat{\mathbf{B}/I} \mid A \text{ stably representable}\} \quad \text{and} \quad set(\mathbf{B})(\alpha) = (\Sigma_\alpha)^* \,.$$

Now we describe the generic map $el : E \to set(\mathbf{B})$ in terms of its corresponding presheaf (also denoted as E) on $\mathbf{El}(set(\mathbf{B}))$: if $A \in set(\mathbf{B})(I)$ then $E(I, A) = A(id_I)$ and for $\alpha : \alpha^*A \to A$ in $\mathbf{El}(set(\mathbf{B}))$ we define $E(\alpha) = A(\alpha \xrightarrow{\alpha} id_I)$. One readily checks that for $a : I \to set(\mathbf{B})$ the map $el(a) = a^*el$ is isomorphic to $a(id_I)$ via Theorem 1.1. Thus, the map el is a family of small objects. It is generic for families of small objects as every $f : Y \to X$ in $\mathcal{S}$ is isomorphic to χ_f^*el where $\chi_f : X \to set(\mathbf{B})$ is defined as follows: for $x \in X(I)$ the presheaf $(\chi_f)_I(x) : (\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}$ is given by

$$(\chi_f)_I(x)(\alpha) = \{y \mid fy = x\alpha\} \qquad \text{and} \qquad (\chi_f)_I(\beta : \alpha\beta \to \alpha)(y) = y\beta \,.$$

ad (4) : Suppose $f : Y \to X$ and $g : Z \to Y$ are in $\mathcal{S}$. We have to show that for $s : I \to X$ with I representable the map $s^*\Pi_f(g) \cong \Pi_\alpha(\beta)$ is representable, too, where

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Z \arrow[d, "g"] \\
J \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\
I \arrow[r] & Z
\end{tikzcd}$$

Now α and β are in $\mathbf{B}$ as f and g are in $\mathcal{S}$, respectively. The map $\Pi_\alpha(\beta)$ is in $\mathbf{B}$ as $\mathsf{Y}_{\mathbf{B}}$ preserves locally cartesian closed structure. $\square$

As representable objects are never initial for every $X \in \widehat{\mathbf{B}}$ the morphism $0 \to X$ is not small unless X is initial. Thus, if $f : Y \to X$ is in $\mathcal{S}$ and $m : Z \rightarrowtail Y$ is monic the composite $f \circ m$ in general will not be in $\mathcal{S}$. However, if $\mathbf{B}$ has finite (nonempty) limits then for $h_1, h_2 : g \to f$ with $f, g \in \mathcal{S}$ it holds that $g \circ e \in \mathcal{S}$ where e is the equalizer of h_1 and h_2 in $\widehat{\mathbf{B}}$.

In general $\Omega_{\widehat{\mathbf{B}}} \to 1$ will not be in $\mathcal{S}$. For example if $\mathbf{B}$ is a poset with a terminal object then $\Omega_{\widehat{\mathbf{B}}} \to 1$ is not in $\mathcal{S}$ as $\Omega_{\widehat{\mathbf{B}}}$ is not representable (because $\Omega_{\widehat{\mathbf{B}}}$ is not subterminal).

By Lemma 2.1(1) the collection $\mathcal{S}$ of representable morphisms in $\widehat{\mathbf{B}}$ determines a full subfibration $\mathcal{S}/\mathbf{B}$ of the fundamental fibration $P_{\widehat{\mathbf{B}}} = \partial_1 : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ for $\widehat{\mathbf{B}}$. That $\mathcal{S}/\mathbf{B}$ is definable (in the sense of [B80]) as a full subfibration of $P_{\widehat{\mathbf{B}}}$ follows from the next lemma.

Lemma 2.2 *For maps $f : Y \to X$ in $\widehat{\mathbf{B}}$ there exists a subobject $i : X_0 \hookrightarrow X$ such that for all $g : Z \to X$ it holds that $g^*f \in \mathcal{S}$ iff g factors through i.*

Proof: Define $X_0(I)$ as the set of all $s : I \to X$ such that s^*f is a representable morphism. Obviously, if $x \in X_0(I)$ and $\alpha : J \to I$ in $\mathbf{B}$ then $s\alpha \in X_0(J)$ as by Lemma 2.1 representable morphisms are stable under pullbacks in $\widehat{\mathbf{B}}$ and, therefore, X_0 is a subpresheaf of X. Let i be the corresponding inclusion. Suppose $g : Z \to X$ in $\widehat{\mathbf{B}}$. Then g^*f is a representable morphism iff for all generalised elements $s : I \to Z$ the map $s^*g^*f \cong (gs)^*f$

is a representable morphism, *i.e.* iff $gs \in X_0(I)$ for all generalised elements $s : I \to Z$, *i.e.* if g factors through i. $\square$

That representable morphisms capture the right notion of smallness is supported by the following Theorem we mention here just *pars pro toto*.

Theorem 2.1 *A split fibration S is locally small iff for the corresponding internal category C the "hom-family" $C_1 \to C_0 \times C_0$ is a representable morphism. Moreover, the split fibration S is small iff S is locally small and C_0 is representable.*

3 A Large Internal Category $\mathbf{Set}(\mathbf{B})$

Inspecting the proof of Lemma 2.1(3) one easily sees that if $\mathbf{B}$ has pullbacks then the fundamental fibration $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ is equivalent to the split fibration $\mathbf{set}(\mathbf{B})$ sending $I \in \mathbf{B}$ to the category $\mathbf{set}(\mathbf{B})(I)$ of *representable* presheaves over $\mathbf{B}/I$ and $\alpha : J \to I$ to Σ_α^*, *i.e.* change of base along Σ_α.[3]

The same constructions as in the proof of Lemma 2.1(3) can be performed when dropping the restriction to stably representable presheaves.

Definition 3.1 *For a category $\mathbf{B}$ let $\mathbf{Set}(\mathbf{B})$ be the presheaf of large categories over $\mathbf{B}$ with*

$$\mathbf{Set}(\mathbf{B})(I) = \mathbf{Set}^{(\mathbf{B}/I)^{\mathsf{op}}} \qquad \textit{and} \qquad \mathbf{Set}(\mathbf{B})(\alpha) = \mathbf{Set}^{(\Sigma_\alpha)^{\mathsf{op}}}$$

for objects I and morphisms α in $\mathbf{B}$.

We write $Set(\mathbf{B})$ for $|\mathbf{Set}(\mathbf{B})|$, i.e. $Set(\mathbf{B})(I) = |\mathbf{Set}(\mathbf{B})(I)|$, the class of objects of $\mathbf{Set}(\mathbf{B})(I) = \widehat{\mathbf{B}/I}$. $\Diamond$

Notice that the split fibration $\mathbf{Set}(\mathbf{B})$ is equivalent to $\mathsf{Y}_{\mathbf{B}}^* P_{\widehat{\mathbf{B}}}$ where $P_{\widehat{\mathbf{B}}} = \partial_1 : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ is the fundamental fibration for $\widehat{\mathbf{B}}$.[4]

[3] If $\mathbf{B}$ has pullbacks then for all $\alpha : J \to I$ in $\mathbf{B}$ change of base along Σ_α preserves representability of presheaves as $(\Sigma_\alpha)^* \mathsf{Y}_{\mathbf{B}/I}(\beta) = (\Sigma_\alpha)^* \mathbf{B}/I(-,\beta) \cong \mathbf{B}/J(-,\alpha^*\beta) = \mathsf{Y}_{\mathbf{B}/J}(\alpha^*\beta)$.

[4] Notice that $\mathsf{Y}_{\mathbf{B}}^* P_{\widehat{\mathbf{B}}}$ is the fibration of discrete fibrations over $\mathbf{B}$ in analogy to the fibration $\mathbf{Fib}/\mathbf{B} \to \mathbf{B}$ of fibrations over $\mathbf{B}$ as discussed at the end of the first chapter of [B80] which is constructed from the fibration $\mathbf{Fib} \to \mathbf{Cat}$ by change of base along $\mathsf{Y}_{\mathbf{B}}$. Obviously, the fibration $\mathsf{Y}_{\mathbf{B}}^* P_{\widehat{\mathbf{B}}}$ is a full subfibration of $\mathbf{Fib}/\mathbf{B} \to \mathbf{B}$.

Again as in the proof of Lemma 2.1(3) we can construct a presheaf E over $\mathbf{El}(Set(\mathbf{B}))$ sending $A \in Set(\mathbf{B})(I)$ to $E(A) = A(id_I)$ and $\alpha : \alpha^* A \to A$ to $E(\alpha) = A(\alpha \xrightarrow{\alpha} id_I)$. This presheaf E is generic in the sense that for every presheaf $A : \mathbf{El}(X)^{\mathsf{op}} \to \mathbf{Set}$ we have $A \cong E \circ \widehat{A}$ where $\widehat{A} : \mathbf{El}(X) \to Set(\mathbf{B})$ is the functor sending $x \in X(I)$ to $A \circ \widehat{x}^{\mathsf{op}}$ where $\widehat{x} : \mathbf{B}/I \to \mathbf{El}(X)$ is the cartesian functor (over $\mathbf{B}$) with $\widehat{x}(id_I) = x$. The discrete fibration corresponding to E is denoted as $El : E \to Set(\mathbf{B})$.

Notice that $set(\mathbf{B})$ (as constructed in the proof of Lemma 2.1(3)) is the greatest subpresheaf of $Set(\mathbf{B})$ such that the restriction of El to it gives rise to a small map.

4 Split Distributors between Split Fibrations

The theory of internal distributors between internal categories has been investigated in chapter 2 of [Joh]. As $\mathbf{Sp}/\mathbf{B}$ and $\mathbf{cat}(\widehat{\mathbf{B}})$ are strongly equivalent this gives rise to a notion of *split distributors between split fibrations*. From Section 3 it follows that for $A, B \in \mathbf{Sp}/\mathbf{B}$ an internal distributor from A to B is a split cartesian functor $\phi : B^{\mathsf{op}} \times A \to \mathbf{Set}(\mathbf{B})$.

In particular such a split distributor ϕ satisfies (1)

$$\begin{tikzcd}
B(J)^{\mathsf{op}} \times A(J) \arrow[d, "\phi_J"'] & B(I)^{\mathsf{op}} \times A(I) \arrow[l, "B(u)^{\mathsf{op}} \times A(u)"'] \arrow[d, "\phi_I"] \\
\widehat{\mathbf{B}/J} & \widehat{\mathbf{B}/I} \arrow[l, "(\Sigma_u)^*"]
\end{tikzcd}$$

for $u : J \to I$. Due to the adjoint correspondences

$$\frac{\dfrac{B(I)^{\mathsf{op}} \times A(I) \to \mathbf{Set}^{(\mathbf{B}/I)^{\mathsf{op}}}}{(\mathbf{B}/I)^{\mathsf{op}} \times B(I)^{\mathsf{op}} \times A(I) \to \mathbf{Set}}}{(\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}^{B(I)^{\mathsf{op}} \times A(I)}}$$

condition (1) can be formulated equivalently as (2)

$$
\begin{tikzcd}
(\mathbf{B}/I)^{\mathsf{op}} \times B(I)^{\mathsf{op}} \times A(I) \arrow[r, "\phi_I"] & \mathbf{Set} \\
(\mathbf{B}/J)^{\mathsf{op}} \times B(I)^{\mathsf{op}} \times A(I) \arrow[u, "(\Sigma_u)^{\mathsf{op}} \times B(I)^{\mathsf{op}} \times A(I)"] \arrow[r, "(\mathbf{B}/J)^{\mathsf{op}} \times u^* \times u^*"'] & (\mathbf{B}/J)^{\mathsf{op}} \times B(J)^{\mathsf{op}} \times A(J) \arrow[u, "\phi_J"']
\end{tikzcd}
$$

and as (3)

$$
\begin{tikzcd}
(\mathbf{B}/J)^{\mathsf{op}} \arrow[r, "(\Sigma_u)^{\mathsf{op}}"] \arrow[d, "\phi_J"'] & (\mathbf{B}/I)^{\mathsf{op}} \arrow[d, "\phi_I"] \\
\mathsf{Dist}(A(J), B(J)) \arrow[r, "(B(u)^{\mathsf{op}} \times A(u))^*"'] & \mathsf{Dist}(A(I), B(I))
\end{tikzcd}
$$

for $u : J \to I$. Notice that (3) says that

$$\phi_{uv} = (B(u)^{\mathsf{op}} \times A(u))^* \phi_v$$

where we write ϕ_u as an abbreviation for $\phi_I(u)$. Using ϕ_I also as a shorthand for $\phi_I(id_I)$ we get as an instance that

$$\phi_u = (B(u)^{\mathsf{op}} \times A(u))^* \phi_J$$

for $u : J \to I$.

Using this latter view as presheaves of distributors we can formulate composition of split distributors $\phi : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ and $\psi : B \longrightarrow\!\!\!\!\!\!\!+\;\; C$ as follows. For $I \in \mathbf{B}$ we put

$$(\psi\phi)_I = \psi_I \phi_I$$

and for $u : J \to I$ in $\mathbf{B}$ we put

$$(\psi\phi)_u = (C(u)^{\mathsf{op}} \times A(u))^* (\psi\phi)_J$$

as we are forced to do by condition (3). Notice, however, that in general it does *not* hold that reindexing of distributors preserves composition and, accordingly, in general we do *not* have $(\psi\phi)_u \cong \psi_u\phi_u$ as one might expect. For $u : J \to I$ we define $(\psi\phi)_{u\overset{u}{\to}id_I} : (\psi\phi)_I \to (\psi\phi)_u$ as

$$\begin{tikzcd}
\psi_I\phi_I \arrow[r, "(\psi_{u\overset{u}{\to}id_I})(\phi_{u\overset{u}{\to}id_I})"] \arrow[dr, "(\psi\phi)_{u\overset{u}{\to}id_I}"'] & (u^*\psi_J)(u^*\psi_J) \arrow[d, "\mu"] \\
 & u^*(\psi\phi)_J
\end{tikzcd}$$

where u^* is a shorthand for $(B(u)^{\mathsf{op}} \times A(u))^*$ and μ is the obvious natural transformation. For $u : J \to I$ and $v : K \to J$ in $\mathbf{B}$ we define $(\psi\phi)_{uv\overset{v}{\to}u}$ as $u^*(\psi\phi)_{v\overset{v}{\to}id_J}$ in order to make condition (3) hold for $\psi\phi$. It is a tedious, but straightforward exercise to verify that $\psi\phi$ defined this way is actually a split distributor from A to C.

Next we discuss the relation between ordinary distributors between ordinary categories and split distributors between the associated split family fibrations. If $\phi : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B}$ then the associated split distributor $\mathit{Fam}(\phi) : \mathit{Fam}(\mathbb{A}) \to \mathit{Fam}(\mathbb{B})$ is given by

$$\mathit{Fam}(\phi)_I(J\overset{u}{\to}I)(Y,X) = \prod_{j\in J} \phi(Y_{u(j)}, X_{u(j)})$$

in accordance with the usual[5] definition of $\mathit{Fam} : \mathbf{Cat} \to \mathbf{Sp}/\mathbf{Set}$. Whereas for functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{C}$ it holds that $\mathit{Fam}(G)\mathit{Fam}(F) = \mathit{Fam}(GF)$ this does *not* hold for $\mathit{Fam} : \mathsf{Dist} \to \mathbf{SpDist}/\mathbf{Set}$. The reason is that for ordinary distributors $\phi : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B}$ and $\psi : \mathbb{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$ it will not hold in general that $\mathit{Fam}(\psi)\mathit{Fam}(\phi) \cong \mathit{Fam}(\psi\phi)$ because—as already remarked above—change of base for distributors does not commute with composition (see Appendix A for details).

Extending the observations of Section 3 we get a 1-1-correspondence between morphism $C^{\mathsf{op}} \to \mathbf{Set}(\mathbf{B})$ in $\mathbf{Sp}/\mathbf{B}$ and discrete fibrations over $\int C$,

[5] Notice that according to the usual definition of *Fam* we have $\mathit{Fam}(\phi) : \mathit{Fam}(\mathbb{B}^{\mathsf{op}}) \times \mathit{Fam}(\mathbb{A}) \to \mathit{Fam}(\mathbf{Set})$ where $\mathit{Fam}(\phi)_I(Y,X) = (\phi(Y_i,X_i))_{i\in I}$. Moreover, we have $\mathit{Fam}(\mathbf{Set}) \simeq \mathbf{set}(\mathbf{Set})$ where $(S_i)_{i\in I}$ in $\mathit{Fam}(\mathbf{Set})(I)$ corresponds to the the presheaf $A : (\mathbf{Set}/I)^{\mathsf{op}} \to \mathbf{Set}$ with $A(J\overset{u}{\to}I) = \prod_{j\in J} S_{u(j)}$ and $A(uv\overset{v}{\to}u)(s) = s \circ v$. This explains why we have defined $\mathit{Fam}(\phi)_I(J \overset{u}{\to} I)(Y,X)$ as $\prod_{j\in J}\phi(Y_{u(j)}, X_{u(j)})$.

the total category of the split fibration C. Thus, in particular, one may consider $\phi : B^{\mathsf{op}} \times A \to \mathbf{Set}(\mathbf{B})$ as a discrete fibration over $\int(B \times A^{\mathsf{op}})$, *i.e.* as a(n ordinary) presheaf $\Phi : \left(\int(B \times A^{\mathsf{op}})\right)^{\mathsf{op}} \to \mathbf{Set}$, which seems much easier to handle than the ϕ.

Such a Φ is given by a distributor $\Phi_I : B(I)^{\mathsf{op}} \times A(I) \to \mathbf{Set}$ for all I in $\mathbf{B}$ (Φ_I corresponds to $\varphi_I(id_i)$ above) and for every $u : J \to I$, $X \in A(I)$ and $Y \in B(I)$ a map $\Phi_{u,Y,X} : \Phi_I(Y, X) \to \Phi(J)(u^*Y, u^*X)$ which data are related by the law

$$\Phi_J(u^*\beta, u^*\alpha) \circ \Phi_{u,Y,X} = \Phi_{u,Y'X'} \circ \Phi_I(\beta, \alpha)$$

for $\alpha : X \to X'$ in $A(I)$ and $\beta : Y' \to Y$ in $B(I)$. Diagramatically this amounts to the commutation of the square

$$\begin{tikzcd}
\Phi_J(u^*Y, u^*X) \arrow[d, "\Phi_J(u^*\beta, u^*\alpha)"'] & \Phi_I(Y, X) \arrow[l, "\Phi_{u,Y,X}"'] \arrow[d, "\Phi_I(\beta,\alpha)"] \\
\Phi_J(u^*Y', u^*X') & \Phi_I(Y', X') \arrow[l, "\Phi_{u,Y',X'}"]
\end{tikzcd}$$

i.e. that $\Phi_u : \Phi_I \Rightarrow \Phi_J \circ (B(u)^{\mathsf{op}} \times A(u))$. Moreover, these natural transformations satisfy the coherence conditions $\Phi_{id_I} = id_{\Phi_I}$ and $\Phi_{uv} = (\Phi_v)_{u^*} \circ \Phi_u$.

Next we will argue why also from this point of view composition of distributors between split fibrations is fibrewise. Let us recall this construction from Ch. 2 of [Joh] where he discusses distributors between internal categories and their composition.

Let A, B and C be split fibrations over $\mathbf{B}$ and $\Phi : A \longrightarrow\!\!\!\!\!\!\!\mid\;\; B$ and $\Psi : B \longrightarrow\!\!\!\!\!\!\!\mid\;\; C$. For defining the composite $\Psi\Phi$ first consider the presheaf D over $\mathbf{B}$ with

$$\mathbf{D}(I) = \coprod_{Z\in C(I)} \coprod_{X\in A(I)} \coprod_{Y\in B(I)} \Psi_I(Z, Y) \times \Phi_I(Y, X)$$

the presheaf E over $\mathbf{B}$ with

$$E(I) = \coprod_{Z\in C(I)} \coprod_{X\in A(I)} \coprod_{Y_1,Y_2\in B(I)} \Psi_I(Z, Y_2) \times Y(I)(Y_2, Y_1) \times \Phi_I(Y_1, X)$$

and the natural transformations τ_1 and τ_2 from E to D defined as

$$\tau_1(g, \beta, f) = (\Psi_I(Z, \beta)(g), f) \qquad \text{and} \qquad \tau_2(g, \beta, f) = (g, \Phi_I(\beta, X)(f)$$

respectively. Let $\pi = \langle \pi_1, \pi_2 \rangle : D \to |C| \times |A|$ be the obvious projection and notice that π coequalizes τ_1 and τ_2. We define $|\Psi\Phi|$ as the coequalizer of τ_1 and τ_2 giving rise to a unique morphism $|\Psi\Phi|$ making the diagram

$$\begin{tikzcd} E \arrow[r, shift left, "\tau_1"] \arrow[r, shift right, "\tau_2"'] & D \arrow[r, two heads, "\text{coeq.}"] \arrow[dr, "\pi"'] & \bullet \arrow[d, "|\Psi\Phi|"] \\ & & |C| \times |A| \end{tikzcd}$$

commute. Since coequalizers are computed fibrewise in $\widehat{\mathbf{B}}$ it follows that $(\Psi\Phi)_u([g, f]) = [\Psi_u(g), \Phi_u(f)]$. Moreover, for $f \in \Phi_I(Y, X)$, $g \in \Psi_I(Y, X)$, $\alpha : X \to X'$ and $\gamma : Z' \to Z$ defining $(\Psi\Phi)_I(\gamma, \alpha)([g, f]) = [g\gamma, \alpha f]$ makes $\Psi\Phi$ into a presheaf over $C \times A^{\mathsf{op}}$ as desired.

5 Distributors between Fibrations

For a category C internal to $\mathbf{B}$ the analogue of the category of set valued presheaves over C is given by the fibration $P_{\mathbf{B}}{}^{P_C} \simeq \mathbf{set}(\mathbf{B})^{P_C}$ over $\mathbf{B}$ where P_C is the externalisation of C. Now as split fibrations P over $\mathbf{B}$ appear as categories internal to $\widehat{\mathbf{B}}$ the fibrational version of category of presheaves over P is given by $P_{\widehat{\mathbf{B}}}{}^{P} \simeq \mathbf{Set}(\mathbf{B})^{P}$.

In [B73, Joh] it has been defined and investigated what are distributors between internal categories. For internal categories A, B a distributor from A to B is given by a family $\Phi_0 : F \to B_0 \times A_0$ together with an action Φ_1 of the morphisms of A and B. As arbitrary fibrations over $\mathbf{B}$ are equivalent to split fibrations and, therefore, to categories internal to $\widehat{\mathbf{B}}$ it is clear what is a distributor between split fibrations A and B, namely a family $\Phi_0 : F \to B_0 \times A_0$ together with an action Φ_1. Such a distributor Φ will be called locally small iff the map Φ_0 is a family of small objects (in the sense of Def. 2.1).

From section 3 it is clear that distributors from split fibration A to split fibration B are just split cartesian functors from $B^{\mathsf{op}} \times A$ to $\mathbf{Set}(\mathbf{B})$ which are locally small iff they factor through $\mathbf{set}(\mathbf{B})$. Moreover, distributors from A to B themselves organise into the split fibration as given by the exponential $\mathbf{Set}(\mathbf{B})^{B^{\mathsf{op}} \times A}$ in $\mathbf{Sp}/\mathbf{B}$.

For distributors between ordinary categories we know that $\mathsf{Dist}(\mathbf{C}, \mathbf{D}) \simeq$

$\mathsf{Dist}(\mathbf{C}', \mathbf{D}')$ whenever $\mathbf{C} \simeq \mathbf{C}'$ and $\mathbf{D} \simeq \mathbf{D}'$.[6] Analogously, as by the fibred Yoneda lemma every fibration P is equivalent to the split fibration $Sp(P)$ any reasonable notion of distributors between fibrations should satisfy

$$\mathsf{Dist}_{\mathbf{B}}(P, Q) \simeq \mathsf{Dist}_{\mathbf{B}}(Sp(P), Sp(Q)) = \mathbf{Set}(\mathbf{B})^{Sp(Q)^{\mathsf{op}} \times Sp(P)}$$

for all $P, Q \in \mathbf{Fib}/\mathbf{B}$.

At first one might be inclined to define $\mathsf{Dist}_{\mathbf{B}}(P, Q)$ as the exponential $\mathbf{Set}(\mathbf{B})^{Q^{\mathsf{op}} \times P}$ which, however, has to be taken in $\mathbf{Fib}/\mathbf{B}$ and, therefore, is given by $\mathsf{Func}_{\mathbf{B}}(Q^{\mathsf{op}} \times P, \mathbf{Set}(\mathbf{B}))$. Alas, this does not seem to work as for split fibrations P and Q it does *not* hold that

$$\mathbf{Fib}/\mathbf{B}(Q^{\mathsf{op}} \times P, \mathbf{Set}(\mathbf{B})) \simeq \mathbf{Sp}/\mathbf{B}(Q^{\mathsf{op}} \times P, \mathbf{Set}(\mathbf{B}))$$

as $H_{\mathbf{Set}(\mathbf{B})} : \mathbf{Set}(\mathbf{B}) \to Sp(U(\mathbf{Set}(\mathbf{B})))$ is not an equivalence.

6 Coherence Conditions for Cartesian Functors from P to $\mathbf{Set}(\mathbf{B})$

Usually for fibrations $P : \mathbf{X} \to \mathbf{B}$ the analogue of a (covariant) "set-valued" presheaf (over the category as given by P) is a cartesian functor from P to $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$. Accordingly, the analogue of a (covariant) "class-valued" presheaf (over the category as given by P) is a cartesian functor from P to $\mathsf{Y}_{\mathbf{B}}{}^{*} P_{\widehat{\mathbf{B}}}$. Of course, up to equivalence one may replace $P_{\mathbf{B}}$ and $\mathsf{Y}_{\mathbf{B}}{}^{*} P_{\widehat{\mathbf{B}}}$ by the equivalent split fibrations $\mathbf{set}(\mathbf{B})$ and $\mathbf{Set}(\mathbf{B})$, respectively.

For sake of concreteness we explicitly state the coherence conditions for the indexed functors corresponding to cartesian functors from a fibration P to a split fibration $U(S)$. Let $F : P \to U(S)$ be cartesian. Then the corresponding indexed functor is given by the family of functors $F_I : \mathbf{X}_I \to \mathbf{Y}_I$ together with the family of natural isomorphisms $\theta_u : F_J \circ u^* \Rightarrow u^* \circ F_I$ for $u : J \to I$ in $\mathbf{B}$ where $\theta_{u,X}$, the component of θ_I at $X \in \mathbf{X}_I$, is given by

[6] If $e : \mathbf{C} \xrightarrow{\sim} \mathbf{C}'$ and $f : \mathbf{D} \xrightarrow{\sim} \mathbf{D}'$ are equivalences then $f^* \circ (-) \circ \phi_e : \mathsf{Dist}(\mathbf{C}', \mathbf{D}') \xrightarrow{\sim} \mathsf{Dist}(\mathbf{C}, \mathbf{D})$ where $\phi_f \dashv f^*$. Notice that $f^* \circ \phi' \circ \phi_e(C, D) = \phi'(fD, eC)$.

$$
\begin{tikzcd}
F(u^*X) \arrow[d, "\theta_{u,X}"'] \arrow[dr, "F(u_X)"] & \\
u^*F(X) \arrow[r, "u_{F(X)}"'] & F(X)
\end{tikzcd}
$$

where $\theta_{u,X}$ is the unique vertical arrow making the diagram commute.

Let $K \overset{v}{\to} J \overset{u}{\to} I$ and $X \in \mathbf{X}_I$. Then we have

$$
\begin{tikzcd}
F(v^*u^*X) \arrow[d, "\theta_{v,u^*X}"'] \arrow[dr, "F(v_{u^*X})"] & & \\
v^*F(u^*X) \arrow[r, "v_{F(u^*X)}"'] \arrow[d, "v^*\theta_{u,X}"'] & F(u^*X) \arrow[d, "\theta_{u,X}"] \arrow[dr, "F(u_X)"] & \\
v^*u^*F(X) \arrow[r, "v_{u^*F(X)}"'] & u^*F(X) \arrow[r, "u_{F(X)}"'] & F(X)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F(v^*u^*X) \arrow[d, "F(c_{u,v,X})"'] \arrow[ddr, "F(u_X v_{u^*X})"] & \\
F((uv)^*X) \arrow[d, "\theta_{uv,X}"'] \arrow[dr, "F((uv)_X)"] & \\
(uv)^*F(X) \arrow[r, "(uv)_{F(X)}"'] & F(X)
\end{tikzcd}
$$

where $c_{u,v,X}$ is the unique vertical arrow with $u_X \circ v_{u^*X} = (uv)_X \circ c_{u,v,X}$. As $U(S)$ is split we have $u_{F(X)} \circ v_{u^*X} = (uv)_{F(X)}$ from which it follows that $v^*\theta_{u,X} \circ \theta_{v,u^*X} = \theta_{uv,X} \circ F(c_{u,v,X})$.

Thus, the coherence condition for θ is

$$v^*\theta_u \circ \theta_v u^* = \theta_{uv} \circ F c_{u,v}$$

i.e.

$$\begin{tikzcd}
F_K v^* u^* \arrow[r, "\theta_v u^*"] \arrow[d, "F c_{u,v}"'] & v^* F_J u^* \arrow[d, "v^*\theta_u"] \\
F_K (uv)^* \arrow[r, "\theta_{uv}"'] & (uv)^* F_I = v^* u^* F_I
\end{tikzcd}$$

for all $u : J \to I$ and $v : K \to I$.

Of course, for split cartesian functors the θ's are identities and one simply has

$$\begin{tikzcd}
\mathbf{X}_J \arrow[d, "F_J"'] & \mathbf{X}_I \arrow[l, "u^*"'] \arrow[d, "F_I"] \\
\mathbf{Y}_J & \mathbf{Y}_I \arrow[l, "u^*"]
\end{tikzcd}$$

for all $u : J \to I$.

7 Distributors between Fibrations as Fibred Discrete Fibrations

For fibrations $P : \mathbf{X} \to \mathbf{B}$ and $Q : \mathbf{Y} \to \mathbf{B}$ a distributor from P to Q is given by a discrete fibration $F : \mathbf{F} \to \mathbf{X}^{\mathsf{op}} \times_{\mathbf{B}} \mathbf{Y}$ as in

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "F"] \arrow[dr] & \mathbf{X}^{\mathsf{op}} \times_{\mathbf{B}} \mathbf{Y} \arrow[d, "P^{\mathsf{op}} \times_{\mathbf{B}} Q"] \\
& \mathbf{B}
\end{tikzcd}$$

$i.e.$ a fibred discrete fibration to $P^{\mathsf{op}} \times_{\mathbf{B}} Q$ (see Theorem 1.2). Every $F_I : \mathbf{F}_I \to \mathbf{X}_I^{\mathsf{op}} \times \mathbf{Y}_I$ is a discrete fibration over $\mathbf{X}_I^{\mathsf{op}} \times \mathbf{Y}_I$ corresponding to a functor from $\mathbf{Y}_I^{\mathsf{op}} \times \mathbf{X}_I$ to $\mathbf{Set}$, $i.e.$ a distributor from $\mathbf{X}_I$ to $\mathbf{Y}_I$, and for every $u : J \to I$ the diagram

$$\begin{tikzcd}
\mathbf{F}_J \arrow[d, "F_J"'] & \mathbf{F}_I \arrow[l, "u^*"'] \arrow[d, "F_I"] \arrow[dl, phantom, "\cong"] \\
\mathbf{X}_J^{\mathsf{op}} \times \mathbf{Y}_J & \mathbf{X}_J^{\mathsf{op}} \times \mathbf{Y}_J \arrow[l, "u^*"]
\end{tikzcd}$$

commutes up to isomorphism.

Choosing (normalized) cleavages for P and Q we can define a cartesian functor $\widetilde{F} : Q^{\mathsf{op}} \times_{\mathbf{B}} P \to \mathbf{Set}(\mathbf{B}) = \mathbf{Disc}//\mathbf{B}$. For $X \in \mathbf{X}_I$ and $Y \in \mathbf{Y}_I$ the presheaf $\widetilde{F}(Y, X) : (\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}$ is defined as follows. For $u : J \to I$ we define $\widetilde{F}(Y, X)(u)$ as the (underlying) set (of the discrete category) $\mathbf{F}_{(u^*Y, u^*X)}$. For $v : K \to J$ we define $\widetilde{F}(Y, X)(uv \xrightarrow{v} u) : \widetilde{F}(Y, X)(uv) \to \widetilde{F}(Y, X)(u)$ as the reindexing map $\varphi^* : \mathbf{F}_{(u^*Y, u^*X)} \to \mathbf{F}_{((uv)^*Y, (uv)^*X)}$ where $\varphi = (\varphi_2, \varphi_1)$ is the unique cartesian arrow over v such that the following diagrams commute

$$\begin{tikzcd}
(uv)^*X \arrow[d, "\varphi_1"'] \arrow[dr, "\mathsf{Cart}(uv, X)"] & \\
u^*X \arrow[r, "\mathsf{Cart}(u, X)"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(uv)^*Y \arrow[d, "\varphi_2"'] \arrow[dr, "\mathsf{Cart}(uv, Y)"] & \\
u^*Y \arrow[r, "\mathsf{Cart}(u, Y)"'] & Y
\end{tikzcd}$$

with φ_1 and φ_2 over v. For every $u : J \to I$ one can construct a canonical isomorphism between $\widetilde{F}(u^*Y, u^*X)$ and $(\Sigma_u)^*\widetilde{F}(Y, X) = \widetilde{F}(Y, X) \circ (\Sigma_u)^{\mathsf{op}}$. Using this canonical isomorphism we can define the morphism part of the cartesian functor $\widetilde{F} : Q^{\mathsf{op}} \times_{\mathbf{B}} P \to \mathbf{Set}(\mathbf{B})$. [7]

Whereas single distributors from P to Q correspond to discrete fibrations over $\mathbf{X}^{\mathsf{op}} \times \mathbf{Y}$ the collection of all distributors from P to Q organises onto the fibration $\mathsf{Dist}_{\mathbf{B}}(P, Q)$ over $\mathbf{B}$ whose fibre over I is given by

[7] This generalizes to fibrations $F : \mathbf{F} \to \mathbf{X}^{\mathsf{op}} \times \mathbf{Y}$ corresponding to fibrations over $P^{\mathsf{op}} \times_{\mathbf{B}} Q$ fibred over $\mathbf{B}$. These correspond to *cartesian pseudo-functors* from $P \times_{\mathbf{B}} Q^{\mathsf{op}}$ to $\mathbf{Fib}//\mathbf{B}$ which is obtained from the 2-fibration $\mathbf{Fib} \to \mathbf{Cat}$ by change of base along $\mathbf{B}/(-) : \mathbf{B}^{\mathsf{op}} \to \mathbf{Cat}$.

$\mathbf{Disc}/\big(\mathbf{B}/I{\times}\mathbf{X}_I^{\mathsf{op}}{\times}\mathbf{Y}_I\big)$ and whose morphisms over $u : J \to I$ in $\mathbf{B}$ are given by squares

$$\begin{tikzcd}
\mathbf{F} \arrow[rr, "K"] \arrow[d, "F"'] & & \mathbf{E} \arrow[d, "E"] \\
\mathbf{B}/J{\times_{\mathbf{B}}}\mathbf{X}^{\mathsf{op}}{\times_{\mathbf{B}}}\mathbf{Y} \arrow[rr, "\mathbf{B}/u{\times_{\mathbf{B}}}\mathbf{X}^{\mathsf{op}}{\times_{\mathbf{B}}}\mathbf{Y}"] \arrow[dr, "P_J{\times_{\mathbf{B}}}P^{\mathsf{op}}{\times_{\mathbf{B}}}Q"'] & & \mathbf{B}/I{\times_{\mathbf{B}}}\mathbf{X}^{\mathsf{op}}{\times_{\mathbf{B}}}\mathbf{Y} \arrow[dl, "P_I{\times_{\mathbf{B}}}P^{\mathsf{op}}{\times_{\mathbf{B}}}Q"] \\
& \mathbf{B} &
\end{tikzcd}$$

with K cartesian over $\mathbf{B}/u{\times_{\mathbf{B}}}\mathbf{X}^{\mathsf{op}}{\times_{\mathbf{B}}}\mathbf{Y}$.

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A Counterexample for $\mathit{Fam}(\psi\phi) = \mathit{Fam}(\psi)\mathit{Fam}(\phi)$

The reason for this failure is that the operation er sending a relation $R \subseteq X \times X$ to the least equivalence relation $\mathsf{er}(R)$ on X containing R does not commute with infinite products. Let I be an infinite set and $R_i \subseteq X_i \times X_i$ be an I-indexed family of relations. Then in general the inclusion $\mathsf{er}(\prod_{i\in I} R_i) \subseteq \prod_{i\in I} \mathsf{er}(R_i)$ will be *proper*.

The following counterexample makes use of this observation. Let $\mathbb{A}$ and $\mathbb{C}$ be the terminal category $\mathbf{1}$ whose single object we denote by $*$ and let $\mathbb{B}$ be the category whose shape is a zig-zag, *i.e.* whose objects are $\{X_n \mid n \in \mathbb{N}\} \cup \{Y_n \mid n \in \mathbb{N}\}$ and whose nontrivial morphisms are $f_n : X_n \to Y_n$ and $g_n : X_{n+1} \to Y_n$. Obviously, in $\mathbb{B}$ there are no nontrivial compositions. The distributors $\phi : \mathbb{A} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathbb{B}$ and $\psi : \mathbb{B} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathbb{C}$ are both given as constant functors to $\mathbf{Set}$ with value $1 = \{*\}$. Writing a_n and b_n for the unique elements of $\phi(*, X_n)$ and $\psi(Y_n, *)$, respectively, we have

$\mathbf{D}(\phi)$ $\qquad\qquad$ $\mathbf{D}(\psi)$

$$\begin{tikzcd}
 & X_n \arrow[dr, "f_n"] & & \\
* \arrow[ur, "a_n"] \arrow[dr, "a_{n+1}"'] & & Y_n \arrow[dr, "b_n"] & \\
 & X_{n+1} \arrow[ur, "g_n"] \arrow[dr, "f_{n+1}"'] & & * \\
 & & Y_{n+1} \arrow[ur, "b_{n+1}"'] &
\end{tikzcd}$$

where $\mathbf{D}(\phi)$ and $\mathbf{D}(\psi)$ are the display categories of ϕ and ψ, respectively. Obviously, the composition $\psi\phi$ is isomorphic to the $\mathbf{Set}$-valued functor from $\mathbb{C}^{\mathsf{op}} \times \mathbb{A}$ with value 1. Notice, however, that for generating the equivalence relation $\sim$ with

$$(\psi\phi)(*, *) = \Big(\coprod_Z \psi(*, Z) \times \psi(Z, *)\Big)_{/\sim}$$

from the relation $\sim_0 = \{\langle (b_n, f_n a_n), (b_n f_n, a_n)\rangle \mid n \in \mathbb{N}\}$ requires ω steps. For this reason $(\mathit{Fam}(\psi)_{\mathbb{N}} \mathit{Fam}(\phi)_{\mathbb{N}})(*, *)$ will contain more than one element as the families $(b_0 f_0, a_0)_{n\in\mathbb{N}}$ and $(b_n f_n, a_n)_{n\in\mathbb{N}}$ are not related by a finite path w.r.t. the relation $\sim_0^{\mathbb{N}}$.